

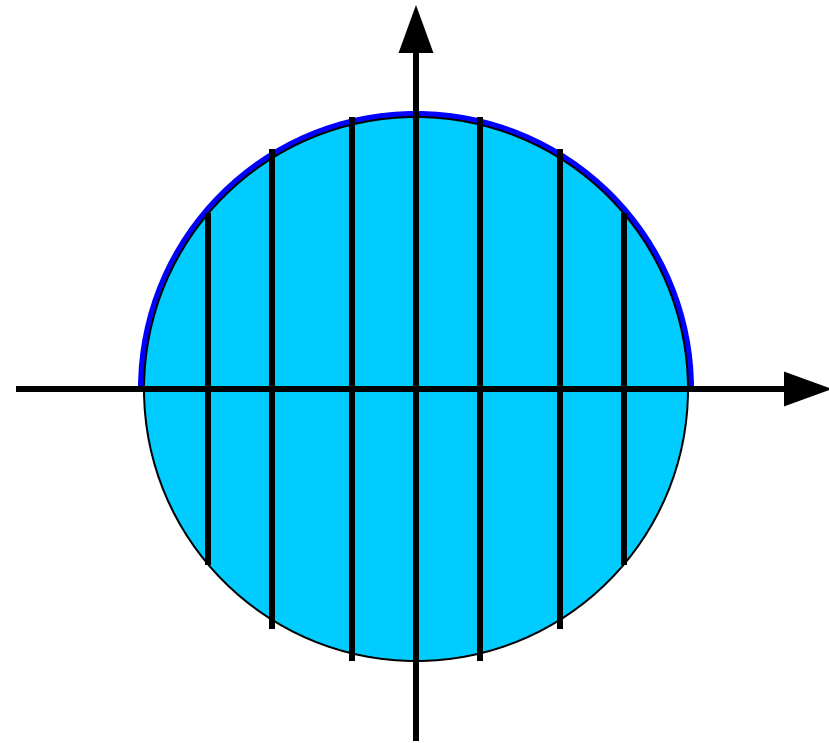
Volume

Section 7.3a

A decorative graphic consisting of several horizontal lines of varying lengths and colors (teal, light blue, white) extending from the right side of the slide towards the center.

Recall a problem we did way back in Section 5.1...

Estimate the volume of a solid sphere of radius 4.



$$f(x) = \sqrt{16 - x^2}$$

Each slice can be approximated by a cylinder:

$$V = \pi r^2 h$$

Height 1 Radius $\sqrt{16 - x^2}$
: :
: :

Volume of each cylinder: $\pi \left(\sqrt{16 - x^2} \right)^2 (1) = \pi (16 - x^2)$

By letting the height of each cylinder approach zero, we could find the exact volume using a definite integral!!!

Volume as an Integral

Now, we will use similar techniques to calculate volumes of many different types of solids □ Let's talk through Figure 7.16 on p.383

The volume of this cylinder is given by

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x$$

And the following sum approximates the volume of the entire solid:

$$\sum V_k = \sum A(x_k) \times \Delta x$$

This is a Riemann sum for $A(x)$ on $[a, b]$. We get better approximations as the partitions get smaller □ Their limiting integral can be defined as the *volume of the solid*.

Definition: Volume of a Solid

The **volume** of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

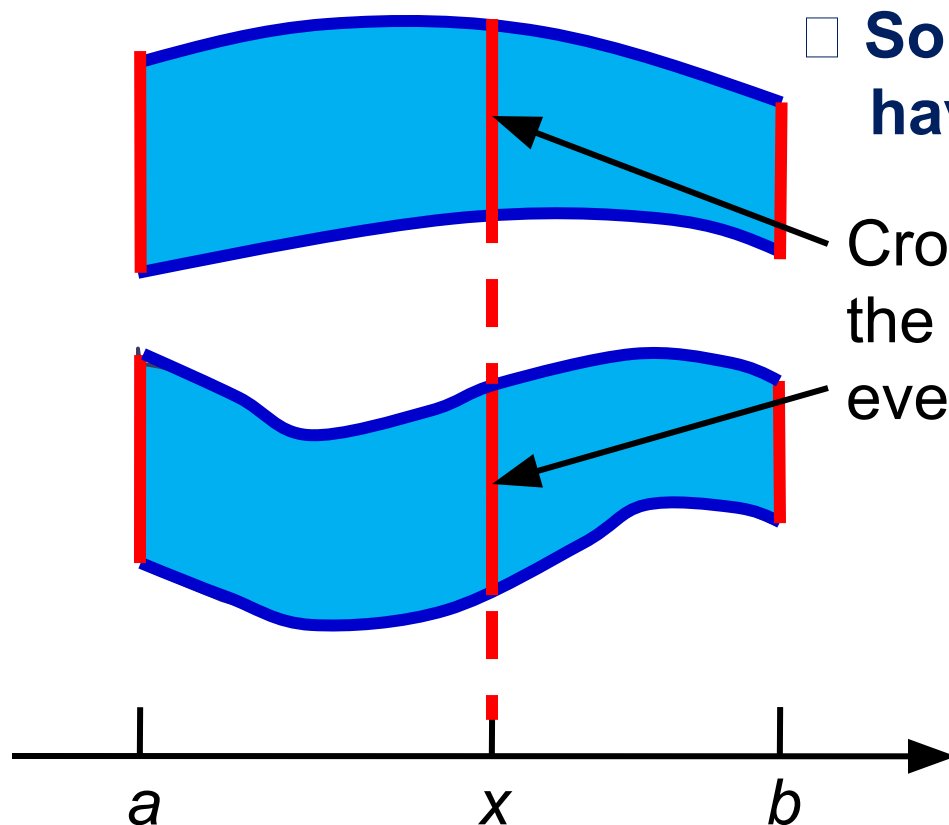
$$V = \int_a^b A(x) dx$$

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

A Note: Cavalieri's Theorem

If two plane regions can be arranged to lie over the same interval of the x -axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area.



□ **So these blue shaded regions have the exact same area!!!**

Cross sections have the same length at every point in $[a, b]$

This idea can be extended to *volume* as well.....take a look at Figure 7.17 on p.384.

Our First Practice Problem

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

Let's follow our four-step process:

1. Sketch. Draw the pyramid with its vertex at the origin and its altitude along the interval $0 \leq x \leq 3$. Sketch a typical cross section at a point x between 0 and 3.

2. Find a formula for $A(x)$. The cross section at x is a square x meters on a side, so

$$A(x) = x^2$$

Our First Practice Problem

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

Let's follow our four-step process:

3. Find the limits of integration. The squares go from $x = 0$ to $x = 3$.

4. Integrate to find the volume.

$$V = \int_0^3 A(x) dx = \int_0^3 x^2 dx = \frac{x^3}{3} \Big|_0^3 = 9 \text{ m}^3$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x -axis are circular discs whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

How about a diagram of this solid?

Width of each cross section: $w = (2 - x^2) - x^2 = 2 - 2x^2$

Area of each cross section: $A(x) = \pi r^2 = \pi \left(\frac{w}{2}\right)^2$
 $= \pi (1 - x^2)^2$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x -axis are circular discs whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

To find volume, integrate these areas with respect to x :

$$\begin{aligned} V &= \int_{-1}^1 \pi (1 - x^2)^2 dx = \pi \int_{-1}^1 (x^4 - 2x^2 + 1) dx \\ &= \pi \left[\frac{1}{5} x^5 - \frac{2}{3} x^3 + x \right]_{-1}^1 = \frac{16}{15} \pi \end{aligned}$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x -axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

How about a diagram of this solid?

$$\text{Cross section width: } w = 2\sqrt{1 - x^2}$$

$$\begin{aligned} \text{Cross section area: } A(x) = s^2 &= \left(\frac{w}{\sqrt{2}} \right)^2 = \left(\frac{2\sqrt{1 - x^2}}{\sqrt{2}} \right)^2 \\ &= 2(1 - x^2) \end{aligned}$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x -axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

$$\begin{aligned}\text{Volume: } V &= \int_{-1}^1 2(1 - x^2) dx = 2 \left[x - \frac{1}{3} x^3 \right]_{-1}^1 \\ &= 2 \left(\frac{2}{3} - \left(-\frac{2}{3} \right) \right) = \frac{8}{3}\end{aligned}$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x -axis are circular discs with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

The diagram?

Cross section width: $w = \sec x - \tan x$

$$\begin{aligned} \text{Cross section area: } A(x) &= \pi r^2 = \pi \left(\frac{w}{2} \right)^2 \\ &= \frac{\pi}{4} (\sec x - \tan x)^2 \end{aligned}$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x -axis are circular discs with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

$$\begin{aligned}\text{Volume: } V &= \int_{-\pi/3}^{\pi/3} \frac{\pi}{4} (\sec x - \tan x)^2 dx \\ &= \frac{\pi}{4} \int_{-\pi/3}^{\pi/3} (\sec^2 x - 2 \sec x \tan x + \tan^2 x) dx \\ &= \frac{\pi}{4} \int_{-\pi/3}^{\pi/3} (\sec^2 x - 2 \sec x \tan x + \sec^2 x - 1) dx \\ &= \frac{\pi}{4} \left[\tan x - 2 \sec x + \tan x - x \right]_{-\pi/3}^{\pi/3}\end{aligned}$$

Guided Practice

The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x -axis are circular discs with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

$$\begin{aligned} &= \frac{\pi}{4} \left[\tan x - 2 \sec x + \tan x - x \right]_{-\pi/3}^{\pi/3} \\ &= \frac{\pi}{2} \left[\tan x - \sec x - \frac{1}{2} x \right]_{-\pi/3}^{\pi/3} \\ &= \frac{\pi}{2} \left[\left(\sqrt{3} - 2 - \frac{\pi}{6} \right) - \left(-\sqrt{3} - 2 + \frac{\pi}{6} \right) \right] = \pi \sqrt{3} - \frac{\pi^2}{6} \end{aligned}$$