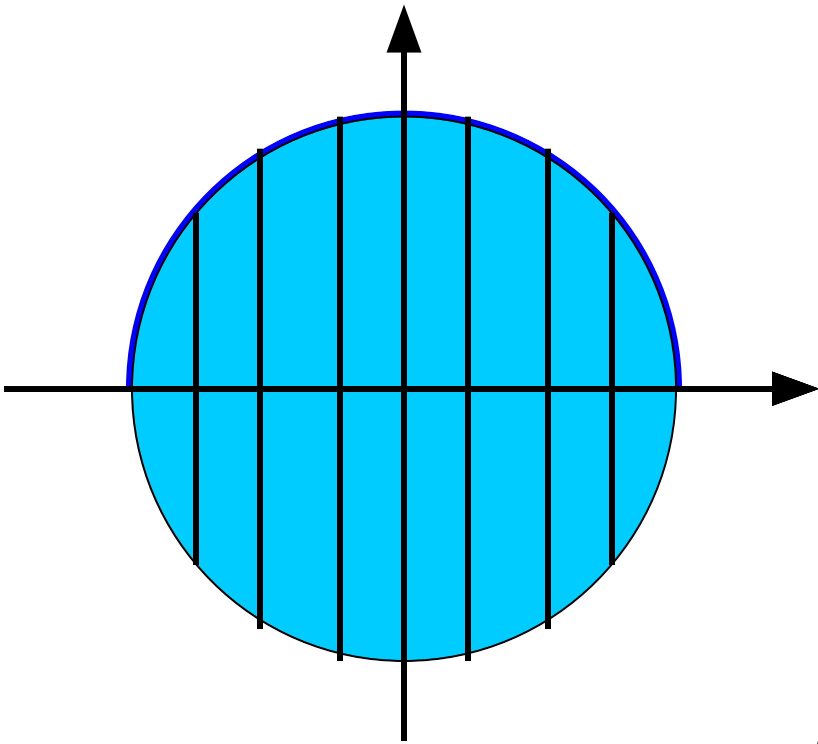
Volume

Section 7.3a

Recall a problem we did way back in Section 5.1…

Estimate the volume of a solid sphere of radius 4.



$$f(x)=\sqrt{16-x^2}$$

Each slice can be approximated by a cylinder:

$$V=\pi r^2 h$$

Height 1 Radius $\sqrt{16-x^2}$

: :

Volume of each cylinder: $\pi\left(\sqrt{16-x^2}\right)^2(1)=\pi\left(16-x^2\right)$

By letting the height of each cylinder approach zero, we could find the exact volume using a definite integral!!!

Volume as an Integral

Now, we will use similar techniques to calculate volumes of many different types of solids Let's talk through Figure 7.16 on p.383

The volume of this cylinder is given by

$$V_k = \text{base area x height} = A\left(x_k\right)\times\Delta x$$

And the following sum approximates the volume of the entire solid:

$$\sum V_k = \sum A\left(x_k\right)\times\Delta x$$

This is a Riemann sum for $A(x)$ on $[a, b]$. We get better approximations as the partitions get smaller Their limiting integral can be defined as the *volume of the solid*.

Definition: Volume of a Solid

The **volume** of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

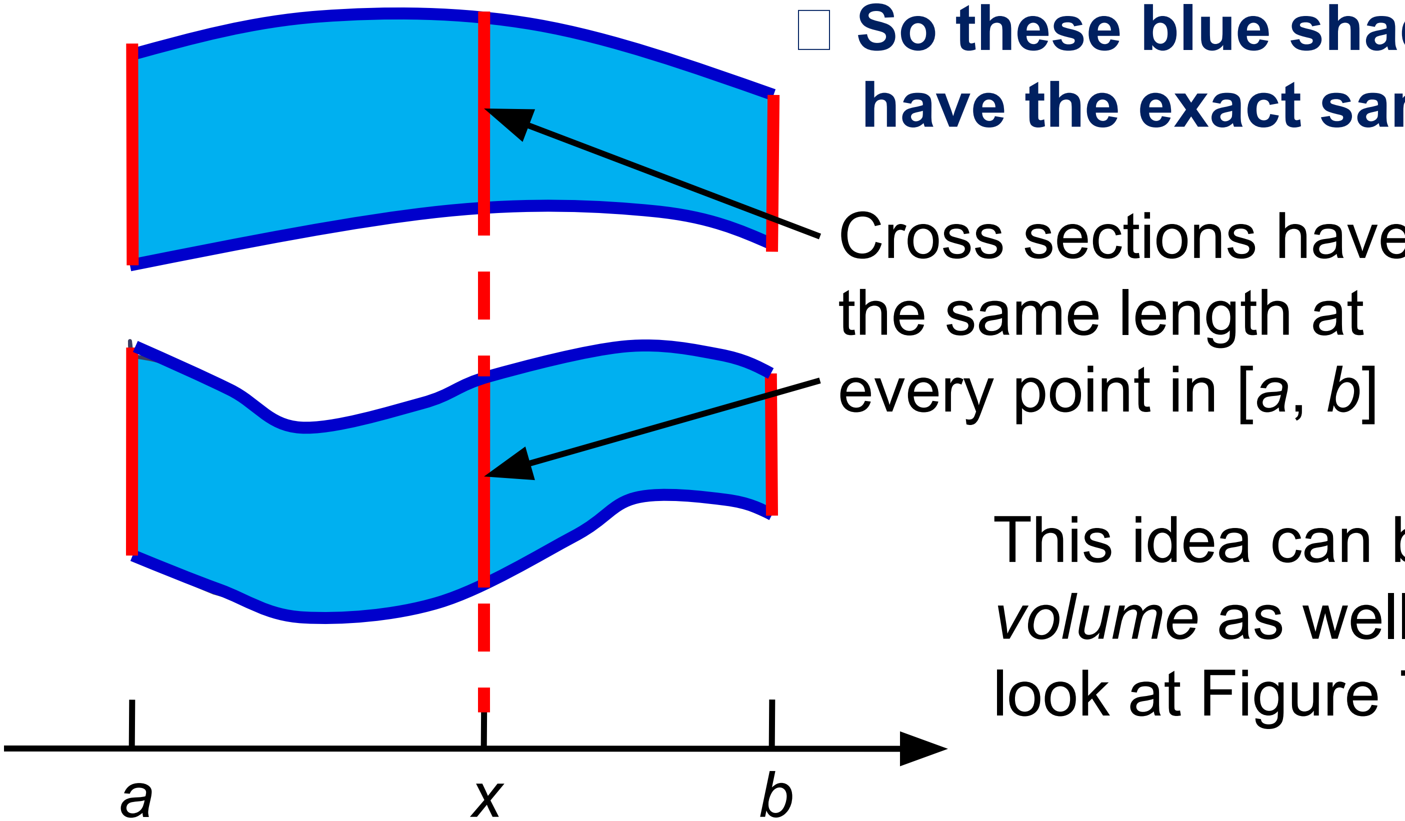
$$V = \int_a^b A(x)\,dx$$

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

A Note: Cavalieri's Theorem

If two plane regions can be arranged to lie over the same interval of the *x*-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area.



- **So these blue shaded regions have the exact same area!!!**

This idea can be extended to *volume* as well……take a look at Figure 7.17 on p.384.

Our First Practice Problem

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

Let's follow our four-step process:

1. Sketch. Draw the pyramid with its vertex at the origin and its altitude along the interval $0 \le x \le 3$. Sketch a typical cross section at a point *x* between 0 and 3.

2. Find a formula for *A*(*x*). The cross section at *x* is a square *x* meters on a side, so

$$A(x) = x^2$$

Our First Practice Problem

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

Let's follow our four-step process:

3. Find the limits of integration. The squares go from $x = 0$ to $x = 3$.

4. Integrate to find the volume.

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right]_0^3 = 9\text{ m}^3$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at *x* = –1 and *x* = 1. The cross sections perpendicular to the *x*-axis are circular discs whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

How about a diagram of this solid?

Width of each cross section: $w = \left(2 - x^2\right) - x^2 = 2 - 2x^2$

Area of each cross section: $A\left(x\right) = \pi r^2 = \pi \left(\dfrac{w}{2}\right)^2$

$$= \pi \left(1 - x^2\right)^2$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at *x* = –1 and *x* = 1. The cross sections perpendicular to the *x*-axis are circular discs whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

To find volume, integrate these areas with respect to *x*:

$$V = \int_{-1}^{1} \pi \left(1 - x^2\right)^2 dx = \pi \int_{-1}^{1} \left(x^4 - 2x^2 + 1\right) dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1} = \frac{16}{15}\pi$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at *x* = –1 and *x* = 1. The cross sections perpendicular to the *x*-axis between these planes are squares whose diagonals run from the semi-circle $y=-\sqrt{1-x^2}$ to the semicircle $y=\sqrt{1-x^2}$.

How about a diagram of this solid?

Cross section width: $w=2\sqrt{1-x^2}$

Cross section area: $A(x)=s^2=\left(\frac{w}{\sqrt{2}}\right)^2=\left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$

$$=2\left(1-x^2\right)$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at *x* = –1 and *x* = 1. The cross sections perpendicular to the *x*-axis between these planes are squares whose diagonals run from the semi-circle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

Volume: $$V = \int_{-1}^{1} 2\left(1-x^2\right)dx = 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= 2\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = \frac{8}{3}$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at $x=-\pi/3$ and $x=\pi/3$. The cross sections perpendicular to the *x*-axis are circular discs with diameters running from the curve $y=\tan x$ to the curve $y=\sec x$.

The diagram?

Cross section width: $w=\sec x-\tan x$

Cross section area: $A(x)=\pi r^2=\pi\left(\dfrac{w}{2}\right)^2$

$$=\frac{\pi}{4}(\sec x-\tan x)^2$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at $x=-\pi/3$ and $x=\pi/3$. The cross sections perpendicular to the *x*-axis are circular discs with diameters running from the curve $y=\tan x$ to the curve $y=\sec x$.

Volume: $V=\int_{-\pi/3}^{\pi/3}\frac{\pi}{4}\left(\sec x-\tan x\right)^2\,dx$

$$=\frac{\pi}{4}\int_{-\pi/3}^{\pi/3}\left(\sec^2 x-2\sec x\tan x+\tan^2 x\right)dx$$

$$=\frac{\pi}{4}\int_{-\pi/3}^{\pi/3}\left(\sec^2 x-2\sec x\tan x+\sec^2 x-1\right)dx$$

$$=\frac{\pi}{4}\Big[\tan x-2\sec x+\tan x-x\Big]_{-\pi/3}^{\pi/3}$$

Guided Practice

The solid lies between planes perpendicular to the *x*-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the *x*-axis are circular discs with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

$$= \frac{\pi}{4}\left[\tan x - 2\sec x + \tan x - x\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{\pi}{2}\left[\tan x - \sec x - \frac{1}{2}x\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{\pi}{2}\left[\left(\sqrt{3} - 2 - \frac{\pi}{6}\right) - \left(-\sqrt{3} - 2 + \frac{\pi}{6}\right)\right] = \pi\sqrt{3} - \frac{\pi^2}{6}$$