Lecture 2.
Conditional statements.
Converse and inverse theorem.
Types of proofs.
Mathematical induction.
Sequences.

Conditional statements.

Phrases such as *if... then...*, and ... *if and only if* ... are frequently used to connect **simple statements** that can be described as either **true** or **false**. For the sake of typographical convenience, there are conventional logical symbols for representing such phrases.

Suppose P and Q are two different statements. The compound statements

$$\text{if } P \text{ then } Q$$

and

$$P \text{ implies } Q$$

mean that if P is true then Q is true. This is written symbolically as

$$P \Rightarrow Q. \tag{1.1}$$

We say that

$$P \text{ is a } \textbf{sufficient condition} \text{ for } Q$$

or

$$Q \text{ is a } \textbf{necessary condition} \text{ for } P.$$

In the above context, P stands for the hypothesis or assumption, and Q is the conclusion.

Let’s train the brain!

Compound statement:

If student pass exams then he will receive the scholarships.

Formulate by yourselves:

…. Is sufficient condition for ….

…. Is necessary condition for ….

…. implies ….

Theorem. Inverse and converse theorem.

Theorem. If P then Q. $P \Rightarrow Q$

Converse Theorem. If Q then P. $Q \Rightarrow P$

Inverse Theorem. If P then not Q. $P \Rightarrow \overline{Q}$

Example 2.1

Theorem. If student pass exams then he will receive the scholarships.

Converse Theorem. If student receive the scholarships then he will pass exams.

Inverse Theorem. If student pass exams then he will not receive the scholarships.

Let's train the brain!

1) Theorem. If it rains then the ground is wet. Formulate
   a) converse theorem
   b) inverse theorem
   c) converse of inverse theorem
   d) inverse of converse theorem

2) Theorem. If $a=0$ or $b=0$ then $ab=0$.
Formulate
   a) converse theorem
   b) inverse theorem
   c) converse of inverse theorem
   d) inverse of converse theorem

2) Theorem. If two sides of a triangle are equal, then it is isosceles
Formulate a) converse theorem b) inverse theorem c) converse of inverse theorem d) inverse of converse theorem

Types of proofs.

Theorem. If P then Q. $P \Rightarrow Q$

1. Assume that P is true and prove that Q is true (**direct proof**).
2. Assume that Q is false and prove that P is false (**contrapositive proof**).
3. Assume that P is true and Q is false, and then prove that this leads to a contradiction (**proof by contradiction**).

Theorem. If a=0 or b=0 then ab=0.

Theorem. If a=0 or b=0 then ab=0.

Direct proof. Let a=0 or b=0 then its' product ab will be equal 0 by rules of multiplication.
Contrapositive proof. Assume that $ab \neq 0$. Then by rules of multiplication **both** $a \neq 0$ **and** $b \neq 0$. It is true, because that source theorem is true.
Proof by contradiction. Assume that a=0 or b=0 and $ab \neq 0$. But if a=0 or b=0 then $ab = 0$ by rules of multiplication. Our first assumption leads to contradiction. Because that source theorem is true.

Mathematical induction

The main idea of MI is – let we have some statement connected with natural numbers (sometimes with integers).

Examples:

1) for any natural n the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is true
2) for any natural n number $2^{3^n}+1$ is divisible by 3^{n+1}
3) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}, n \geq 2$)
4) $F_1, F_2, \ldots, F_{m-1}, F_m \vdash G \Rightarrow F_1, F_2, \ldots, F_{m-1} \vdash F_m \rightarrow G$ - deduction theorem (in axiomatic system of mathematical logic)
5) For lists xs of any length map f (take k xs) = take k (map f xs) – functions of Haskell (programming language)

A proof by induction consists of three steps.

The first, the base step (BS) (or basis), proves the statement for n = 0 or for n=1.

The second step is the assumption of induction (AS). We assume that the statement holds for any given $n = k$ or $n \le k$.

And third step is the induction step (IS), it proves that if the statement holds for any given case n = k, then it must also hold for the next case n = k + 1.

These steps establish that the statement holds for every natural number n. So if statement holds for some initial number n_0 (by BS), then it holds for $n_0 + 1$(by AS and IS). If it holds for $n_0 + 1$, then it holds for $n_0 + 1 + 1 = n_0 + 2$ and so on.

Example 2.4 Prove that any sum equal or greater than 8 cents may be collected using coins of 3 and 5 cents

BS. Can we collect sum = 8 cents?

AS. Assume that we can collect sum of n cents.

IS. So we have a sum of n cents. Can you offer the way how we can get n+1 cents? Which coins we need to remove and which coins we need to add?

2.1 Sequences of real numbers

We deal with the fundamental properties of sequences and series of real numbers. We place particular emphasis on the concept of "convergence," a thorough understanding of which is important for the study of the various branches of mathematical physics that we are concerned with subsequent chapters.

At first let's imagine we have a natural sequence:

$$1,2,3,\ldots,n\ldots,n',\ldots \quad (1)$$

where numbers stand in increasing order, so that a larger number n' follows after smaller number n (or a smaller number precedes a larger one).

Now let's change every number in sequence (1) by some real number x_n using some rule. Then we have got a **real sequence**:

$$x_1, x_2, x_3, \ldots, x_n, \ldots, x_{n'}, \ldots (2)$$

Briefly, we shall write real sequence (or simply - sequence) as (x_n).

Note! Notation (x_n) and $\{x_n\}$ are not equal! If $x_n = 1$ for any $n \in N$ then (x_n) means infinite sequence 1,1,1,1….. whereas $\{x_n\}$ means a set consisted of single number 1.

Examples of sequences

1) Arithmetic progression

$$\underset{1}{a},\ \underset{2}{a+d},\ \underset{3}{a+2d},\ \ldots,\ \underset{n}{a+(n-1)d},\ \ldots$$

2) Geometric progression

$$\underset{1}{a},\ \underset{2}{aq},\ \underset{3}{aq^2},\ \ldots,\ \underset{n}{aq^{n-1}},\ \ldots$$

3) Approximation to π

$$(x_n) = (3.1,\ 3.14,\ 3.142,\ \cdots,\ x_n,\ \cdots),$$

$x_n = 1;$	$1,$	$1,$	$1,$	$1,$	$1,$	$1,$	$\ldots$
	1	2	3	4	5	6	
$x_n = (-1)^{n+1};$	$1,$	$-1,$	$1,$	$-1,$	$1,$	$-1,$	$\ldots$
	1	2	3	4	5	6	
$x_n = \dfrac{1+(-1)^n}{n},$	$0,$	$1,$	$0,$	$\frac{1}{2},$	$0,$	$\frac{1}{3},$	$\ldots$
	1	2	3	4	5	6	

We start with a precise definition of the convergence of a real sequence, which is an initial and crucial step for various branches of mathematics.

♠ **Convergence of a real sequence:**

A real sequence (x_n) is said to be **convergent** if there exists a real number x with the following property: For every $\varepsilon > 0$, there is an integer N such that

$$n \geq N \;\Rightarrow\; |x_n - x| < \varepsilon. \tag{2.1}$$

We must emphasize that the magnitude of ε is arbitrary. No matter how small an ε we choose, it must always be possible to find a number N that will increase as ε decreases.

Remark. In the language of **neighborhoods**, the above definition is stated as follows: *The sequence (x_n) converges to x if every neighborhood of x contains all but a finite number of elements of the sequence.*

When (x_n) is convergent, the number x specified in this definition is called a **limit** of the sequence (x_n), and we say that x_n converges to x. This is expressed symbolically by writing

$$\lim_{n\to\infty} x_n = x,$$

or simply by

$$x_n \to x.$$

If (x_n) is not convergent, it is called **divergent**.

Remark. The limit x may or may not belong to (x_n); this situation is similar to the case of the **limit point** of a set of real numbers discussed in Sect. 1.1.5.

Example 2.1. Let sequence is defined be $x_n = \frac{1}{n}$. Is that sequence is convergent?

Solution. Sequence consist of numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots$$

With increasing the index the elements are closer to zero. Suppose that 0 is the limit of this sequence. To prove this fact we must find integer index N for every $\varepsilon > 0$ such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

n is natural number and we can rewrite absolute expression as

$$\frac{1}{n} < \varepsilon$$

Then we can multiply both part by n and divide by ε:

$$1 < n\varepsilon$$

$$\frac{1}{\varepsilon} < n$$

$$n > \frac{1}{\varepsilon}$$

So, inequality (2.1) will be satisfied when n is greater than $1/\varepsilon$

ε	5	0.1	0.08	0.0016
N(ε)	1	11	13	?