

# Lecture 2.

Conditional statements.

Converse and inverse theorem.

Types of proofs.

Mathematical induction.

Sequences.

# Conditional statements.

Phrases such as *if... then...*, and *... if and only if ...* are frequently used to connect **simple statements** that can be described as either **true** or **false**. For the sake of typographical convenience, there are conventional logical symbols for representing such phrases.

Suppose  $P$  and  $Q$  are two different statements. The compound statements

if  $P$  then  $Q$

and

$P$  implies  $Q$

mean that if  $P$  is true then  $Q$  is true. This is written symbolically as

$$P \Rightarrow Q. \quad (1.1)$$

We say that

$P$  is a **sufficient condition** for  $Q$

or

$Q$  is a **necessary condition** for  $P$ .

In the above context,  $P$  stands for the hypothesis or assumption, and  $Q$  is the conclusion.

# Let's train the brain!

*Compound statement:*

If student pass exams then he will receive the scholarships.

Formulate by yourselves:

.... Is sufficient condition for ....

.... Is necessary condition for ....

.... implies ....

# Theorem. Inverse and converse theorem.

**Theorem.** If P then Q.  $P \Rightarrow Q$

**Converse Theorem.** If Q then P.  $Q \Rightarrow P$

**Inverse Theorem.** If P then not Q.  $P \Rightarrow \bar{Q}$

## Example 2.1

**Theorem.** If student pass exams then he will receive the scholarships.

**Converse Theorem.** If student receive the scholarships then he will pass exams.

**Inverse Theorem.** If student pass exams then he will not receive the scholarships.

# Let's train the brain!

1) Theorem. If it rains then the ground is wet. Formulate

- a) converse theorem
- b) inverse theorem
- c) converse of inverse theorem
- d) inverse of converse theorem

2) Theorem. If  $a=0$  or  $b=0$  then  $ab=0$ .

Formulate

- a) converse theorem
- b) inverse theorem
- c) converse of inverse theorem
- d) inverse of converse theorem

2) Theorem. If two sides of a triangle are equal, then it is isosceles

Formulate    a) converse theorem    b) inverse theorem    c) converse of  
inverse theorem    d) inverse of converse theorem

# Types of proofs.

**Theorem.** If  $P$  then  $Q$ .  $P \Rightarrow Q$

1. Assume that  $P$  is true and prove that  $Q$  is true (direct proof).
2. Assume that  $Q$  is false and prove that  $P$  is false (contrapositive proof).
3. Assume that  $P$  is true and  $Q$  is false, and then prove that this leads to a contradiction (proof by contradiction).

**Theorem.** If  $a=0$  or  $b=0$  then  $ab=0$ .

**Theorem.** If  $a=0$  or  $b=0$  then  $ab=0$ .

**Direct proof.** Let  $a=0$  or  $b=0$  then its' product  $ab$  will be equal 0 by rules of multiplication.

**Contrapositive proof.** Assume that  $ab \neq 0$ . Then by rules of multiplication **both**  $a \neq 0$  **and**  $b \neq 0$ . It is true, because that source theorem is true.

**Proof by contradiction.** Assume that  $a=0$  or  $b=0$  and  $ab \neq 0$ . But if  $a=0$  or  $b=0$  then  $ab = 0$  by rules of multiplication. Our first assumption leads to contradiction. Because that source theorem is true.

# Mathematical induction

The main idea of MI is – let we have some statement connected with natural numbers (sometimes with integers).

Examples:

1) for any natural  $n$  the formula  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$  is true

2) for any natural  $n$  number  $2^{3^n} + 1$  is divisible by  $3^{n+1}$

3)  $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}, n \geq 2$  )

4)  $F_1, F_2, \dots, F_{m-1}, F_m \vdash G \Rightarrow F_1, F_2, \dots, F_{m-1} \vdash F_m \rightarrow G$  - deduction theorem (in axiomatic system of mathematical logic)

5) For lists xs of any length  $\text{map } f (\text{take } k \text{ xs}) = \text{take } k (\text{map } f \text{ xs})$  – functions of Haskell (programming language)

A proof by induction consists of three steps.

The first, the base step (BS) (or basis), proves the statement for  $n = 0$  or for  $n=1$ .

The second step is the assumption of induction (AS). We assume that the statement holds for any given  $n = k$  or  $n \leq k$ .

And third step is the induction step (IS), it proves that if the statement holds for any given case  $n = k$ , then it must also hold for the next case  $n = k + 1$ .

These steps establish that the statement holds for every natural number  $n$ . So if statement holds for some initial number  $n_0$  (by BS), then it holds for  $n_0 + 1$  (by AS and IS). If it holds for  $n_0 + 1$ , then it holds for  $n_0 + 1 + 1 = n_0 + 2$  and so on.



**Example 2.4 Prove that any sum equal or greater than 8 cents may be collected using coins of 3 and 5 cents**

BS. Can we collect sum = 8 cents?

AS. Assume that we can collect sum of  $n$  cents.

IS. So we have a sum of  $n$  cents. Can you offer the way how we can get  $n+1$  cents? Which coins we need to remove and which coins we need to add?

# 2.1 Sequences of real numbers

We deal with the fundamental properties of sequences and series of real numbers. We place particular emphasis on the concept of “convergence,” a thorough understanding of which is important for the study of the various branches of mathematical physics that we are concerned with subsequent chapters.

At first let's imagine we have a natural sequence:

$$1, 2, 3, \dots, n, \dots, n', \dots \quad (1)$$

where numbers stand in increasing order, so that a larger number  $n'$  follows after smaller number  $n$  (or a smaller number precedes a larger one).

Now let's change every number in sequence (1) by some real number  $x_n$  using some rule. Then we have got a **real sequence**:

$$x_1, x_2, x_3, \dots, x_n, \dots, x_{n'}, \dots \quad (2)$$

Briefly, we shall write real sequence (or simply - sequence) as  $(x_n)$ .

**Note! Notation  $(x_n)$  and  $\{x_n\}$  are not equal! If  $x_n = 1$  for any  $n \in N$  then  $(x_n)$  means infinite sequence 1,1,1,1..... whereas  $\{x_n\}$  means a set consisted of single number 1.**

# Examples of sequences

## 1) Arithmetic progression

$$a_1, a_2, a_3, \dots, a_n, \dots$$

## 2) Geometric progression

$$a_1, a_2, a_3, \dots, a_n, \dots$$

## 3) Approximation to $\pi$

$$(x_n) = (3.1, 3.14, 3.142, \dots, x_n, \dots),$$

$$x_n = 1;$$

$$\begin{array}{cccccc} 1, & 1, & 1, & 1, & 1, & 1, \dots \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$x_n = (-1)^{n+1};$$

$$\begin{array}{cccccc} 1, & -1, & 1, & -1, & 1, & -1, \dots \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$x_n = \frac{1 + (-1)^n}{n},$$

$$\begin{array}{cccccc} 0, & 1, & 0, & \frac{1}{2}, & 0, & \frac{1}{3}, \dots \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

We start with a precise definition of the convergence of a real sequence, which is an initial and crucial step for various branches of mathematics.

♠ **Convergence of a real sequence:**

A real sequence  $(x_n)$  is said to be **convergent** if there exists a real number  $x$  with the following property: For every  $\varepsilon > 0$ , there is an integer  $N$  such that

$$n \geq N \Rightarrow |x_n - x| < \varepsilon. \quad (2.1)$$

We must emphasize that the magnitude of  $\varepsilon$  is arbitrary. No matter how small an  $\varepsilon$  we choose, it must always be possible to find a number  $N$  that will increase as  $\varepsilon$  decreases.

*Remark.* In the language of neighborhoods, the above definition is stated as follows: *The sequence  $(x_n)$  converges to  $x$  if every neighborhood of  $x$  contains all but a finite number of elements of the sequence.*

When  $(x_n)$  is convergent, the number  $x$  specified in this definition is called a limit of the sequence  $(x_n)$ , and we say that  $x_n$  converges to  $x$ . This is expressed symbolically by writing

$$\lim_{n \rightarrow \infty} x_n = x,$$

or simply by

$$x_n \rightarrow x.$$

If  $(x_n)$  is not convergent, it is called divergent.

*Remark.* The limit  $x$  may or may not belong to  $(x_n)$ ; this situation is similar to the case of the limit point of a set of real numbers discussed in Sect. 1.1.5.

Example 2.1. Let sequence is defined be  $x_n = \frac{1}{n}$ . Is that sequence is convergent?

Solution. Sequence consist of numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

With increasing the index the elements are closer to zero. Suppose that 0 is the limit of this sequence. To prove this fact we must find integer index  $N$  for every  $\varepsilon > 0$  such that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon$$

$$\left| \frac{1}{n} \right| < \varepsilon$$

$n$  is natural number and we can rewrite absolute expression as

$$\frac{1}{n} < \varepsilon$$

Then we can multiply both part by  $n$  and divide by  $\varepsilon$ :

$$1 < n\varepsilon$$

$$\frac{1}{\varepsilon} < n$$

$$n > \frac{1}{\varepsilon}$$

So, inequality (2.1) will be satisfied when  $n$  is greater than  $1/\varepsilon$

$\varepsilon$	5	0.1	0.08	0.0016
$N(\varepsilon)$	1	11	13	?