Macroeconomics

Class 8.

More about Solow model

Sinking into memories: A general production function in the Solow growth model

- Consider a general production function

$$Y = F(L, K)$$

- This is a "neoclassical" production function if there are positive and diminishing returns to K and L; if there are constant returns to scale (CRS); and if it obeys the Inada conditions:

$$f(0) = 0;\ f'(0) = \infty;\ \lim_{k \to \infty} f'(k) = 0$$

- with CRS, we have output per worker of

$$Y / L = F(1, K / L)$$

If we write K/L as k and Y/L as y, then in *intensive form*:

$$y = f(k)$$

Sinking into memories: The Cobb-Douglas production function

- One simple production function that provides – as many economists believe – a reasonable description of actual economies is the ***Cobb-Douglas***:

$$Y = AK^{\alpha}L^{1-\alpha}$$

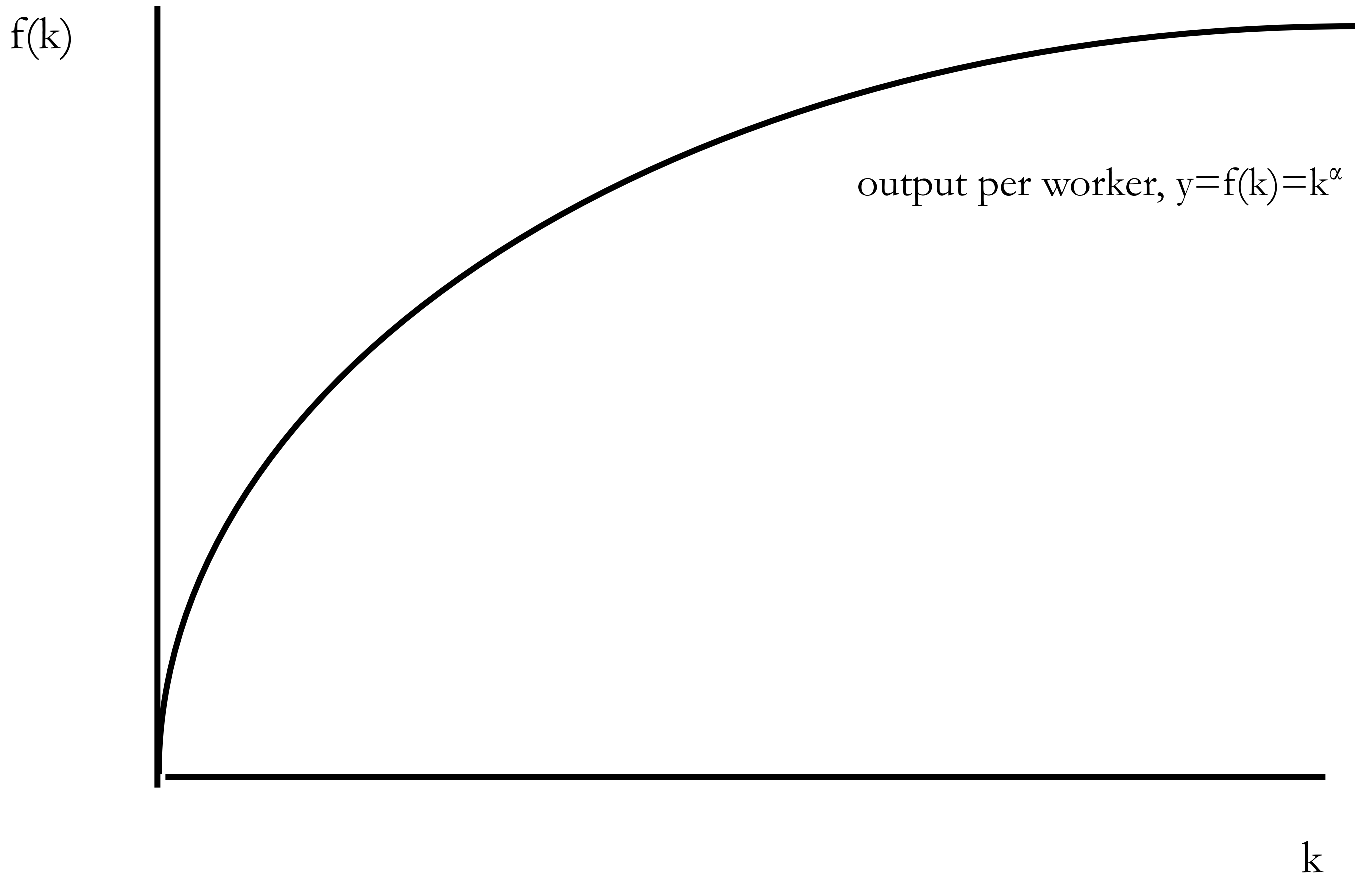
where A>0 is the level of technology and α is a constant with 0<α<1. The CD production function can be written in *intensive form* as

$$y = Ak^{\alpha}$$

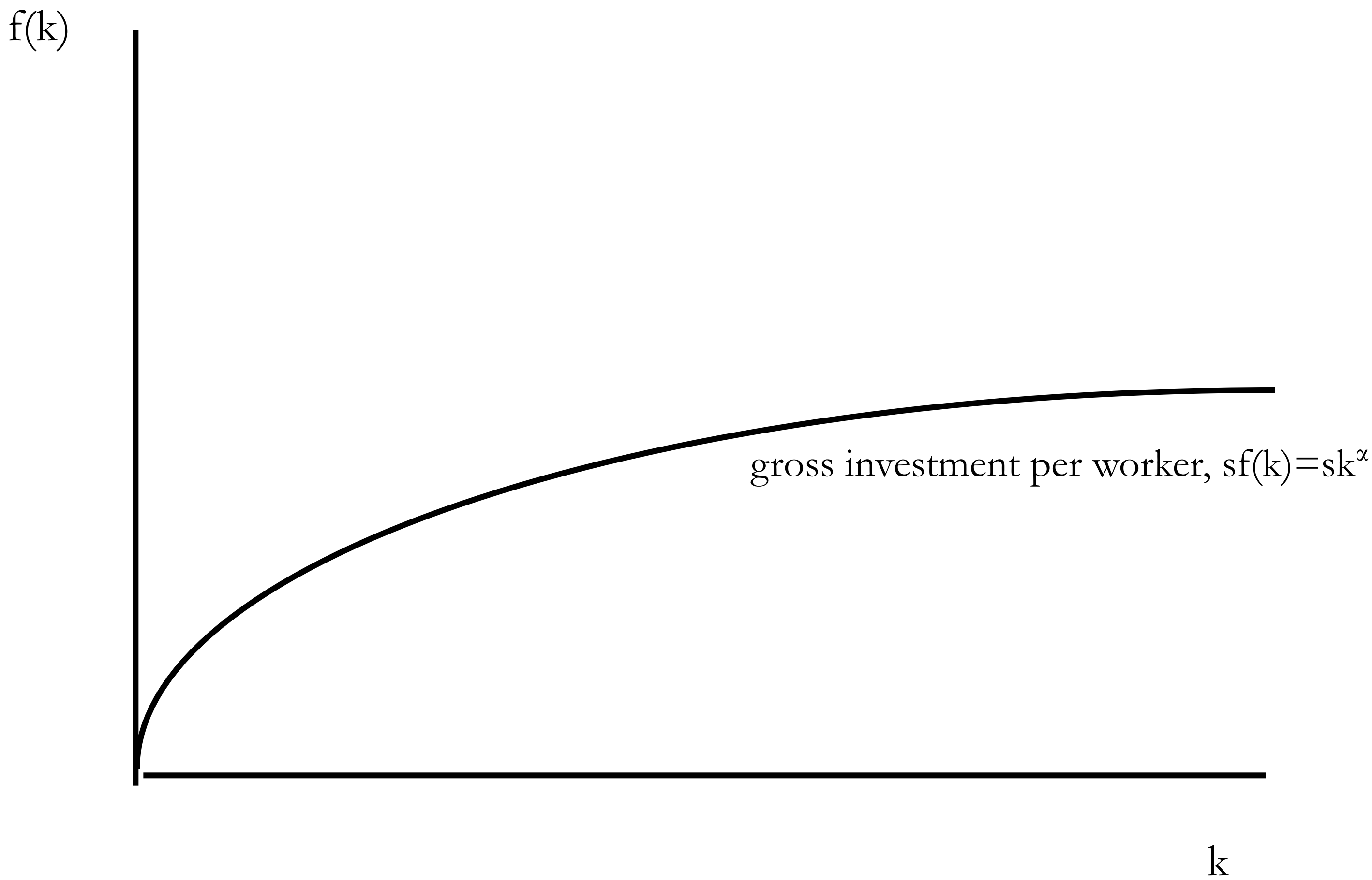
The marginal product can be found from the derivative:

$$MPK = \frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha} = \alpha \frac{AK^{\alpha}L^{1-\alpha}}{K} = \alpha \frac{Y}{K} = \alpha APK$$

Sinking into memories: Diminishing returns to capital



Sinking into memories: The economy is saving and investing a constant fraction of income…



Sinking into memories: What is "labor-augmenting technical progress"?

- This is technical progress that increases contribution of labor into output!

Sinking into memories: If we take into account “labor-augmenting technical progress” that

- We now write the production function as:

$$Y = F(K, L \times E)$$

- where $L \times E$ = the number of effective workers.
  - Hence, increases in labor efficiency have the same effect on output as increases in the labor force.

Sinking into memories: Production function with technical progress in the intensive form

- Notation:

  $y = Y/LE$ = output per effective worker

  $k = K/LE$ = capital per effective worker

- Production function per effective worker:

  $y = f(k)$

- Saving and investment per effective worker:

  $s\,y = s\,f(k)$

Sinking into memories: What is break-even investment?

$(\delta + n + g)k$ = break-even investment: the amount of investment necessary to keep k constant.

Consists of:

- δk to replace depreciating capital
- nk to provide capital for new workers
- gk to provide capital for the new "effective" workers created by technological progress

Sinking into memories: Derivation of equilibrium capital per effective worker

$$\because \Delta K \text{ (net investment)} = I \text{ (gross investment)} - \delta \cdot K \text{ (depreciation}$$

$$\Rightarrow \frac{\Delta K}{L \cdot E} = \frac{I}{L \cdot E} - \frac{\delta K}{L \cdot E} \equiv i - \delta \cdot k = s \cdot f(k) - \delta \cdot k$$

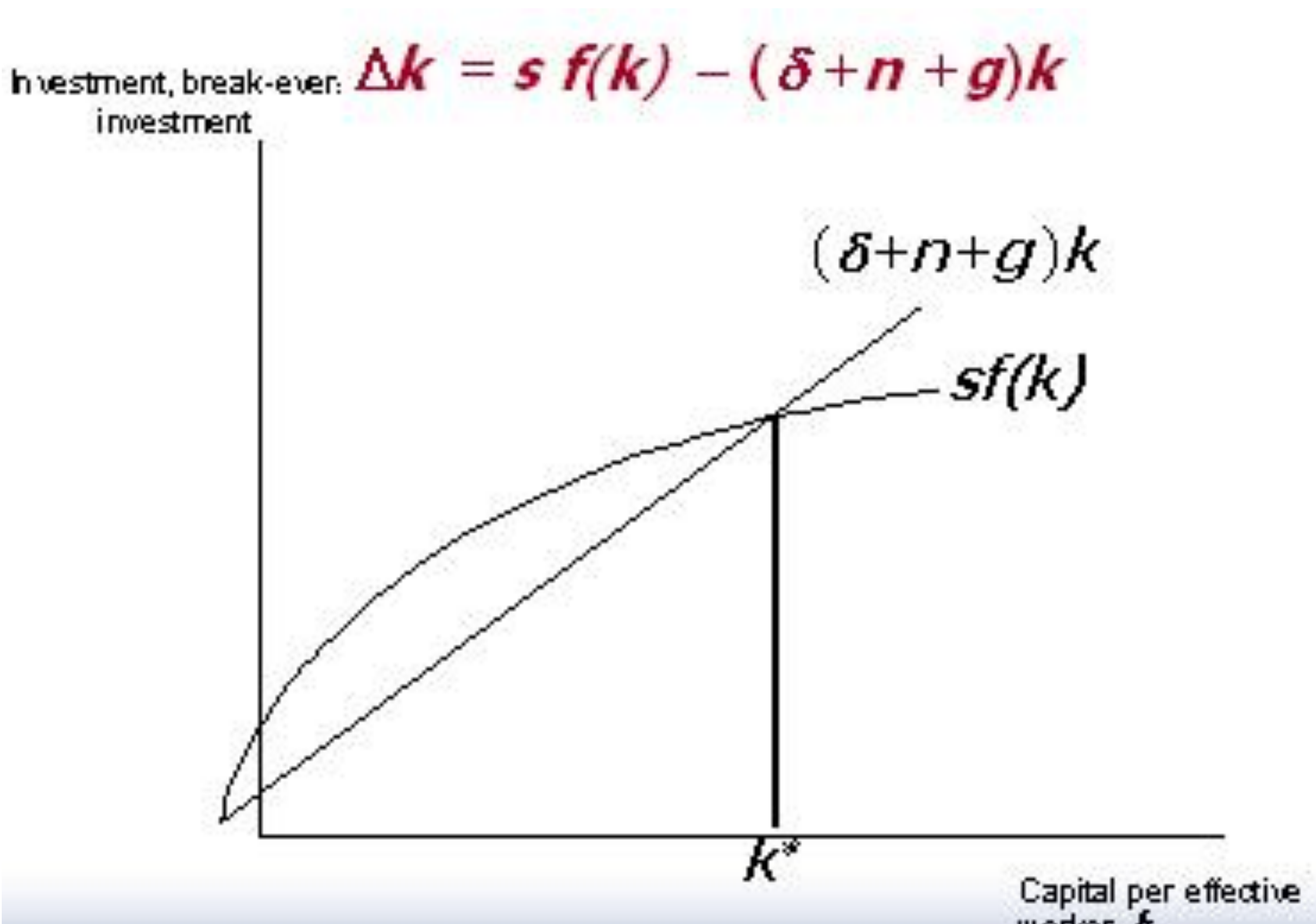
$$\text{Since } \Delta k = \Delta\left(\frac{K}{L \cdot E}\right) = \frac{(L \cdot E)\Delta K - K(L \cdot \Delta E + E \cdot \Delta L)}{(L \cdot E)^2}$$

$$= \frac{\Delta K}{L \cdot E} - \frac{K}{L \cdot E}\left(\frac{\Delta L}{L} + \frac{\Delta E}{E}\right)$$

$$= s \cdot f(k) - \delta \cdot k - k(n + g), \text{ where } \frac{\Delta L}{L} \equiv n, \text{ and } \frac{\Delta E}{E} \equiv g$$

$$\Rightarrow \Delta k = s \cdot f(k) - (\delta + n + g)k$$

Sinking into memories: Equilibrium as a situation of steady-state growth



Sinking into memories: Dynamics of parameters on the steady-state

variable	symbol	Steady-state growth rate
Capital per effective worker	$k = K/(L \times E)$	0
Output per effective worker	$y = Y/(L \times E)$	0
Capital per worker	$(K/L) = k \times E$	g
Output per worker	$(Y/L) = y \times E$	g
Total output	$Y = y \times E \times L$	$n + g$

Sinking into memories: Balanced growth

Solow model's steady state exhibits **balanced growth** - many variables grow at the same rate.

- Solow model predicts ***Y/L*** and ***K/L*** grow at same rate (***g***), so that ***K/Y*** should be constant.

  This is true in the real world.

- Solow model predicts real wage grows at same rate as ***Y/L***, while real rental price is constant. Also true in the real world.

Sinking into memories: Growth in steady state and outside steady state

- ***In the steady state –*** when actual investment per “effective worker” = break-even investment ***- the rate of economic growth will be equal to the sum of rate of population growth and rate of technical progress = n+g.***
- If “initial” capital stock is less than steady state capital stock, then the rate of economic growth will be more than *n+g*.

Sinking into memories: Unconditional convergence

- Solow model predicts that, other things equal, "poor" countries (with lower Y/L and K/L) should grow faster than "rich" ones.
- If true, then the income gap between rich & poor countries would shrink over time, and living standards "converge."
- In real world, many poor countries do NOT grow faster than rich ones. Does this mean the Solow model fails?

Sinking into memories: Conditional convergence

- No, because "other things" aren't equal.
  - In samples of countries with similar savings & pop. growth rates, income gaps shrink about 2%/year.
  - In larger samples, if one controls for differences in saving, population growth, and human capital, incomes converge by about 2%/year.
- What the Solow model *really* predicts is **conditional convergence** - countries converge to their own steady states, which are determined by saving, population growth, and education. And this prediction comes true in the real world.

Sinking into memories: The concept of the Golden Rule

To find the Golden Rule capital stock, express c^* in terms of k^*:

$$
\begin{aligned}
c^* &= y^* - i^* \\
&= f(k^*) - (\delta + n + g)\,k^*
\end{aligned}
$$

c^* is maximized when

$$\text{MPK} = \delta + n + g$$

or equivalently,

$$\boxed{\text{MPK} - \delta = n + g}$$

In the Golden Rule Steady State, the marginal product of capital net of depreciation equals the pop. growth rate plus the rate of tech progress.

Sinking into memories: The Golden Rule – for what?

- Use the Golden Rule to determine whether our saving rate and capital stock are too high, too low, or about right.
- To do this, we need to compare $(MPK - \delta)$ to $(n + g)$.
- If $(MPK - \delta) > (n + g)$, then we are below the Golden Rule steady state and should increase s.
- If $(MPK - \delta) < (n + g)$, then we are above the Golden Rule steady state and should reduce s.

Sinking into memories: Accounting of growth in Solow model (Part 1)

- Production Function (CRTS)

$$Y = F(K, L)$$

- Increases in Capital and Labor

$$\Delta Y = (MPK \times \Delta K) + (MPL \times L)$$

$$\Rightarrow \quad \frac{\Delta Y}{Y} = \alpha \frac{\Delta K}{K} + (1 - \alpha) \frac{\Delta L}{L}$$

where α is capital's share and $(1 - \alpha)$ is labor's share.

Sinking into memories: Accounting of growth in Solow model (Part 2)

- Production Function with Technology

$$Y = AF(K, L)$$

- Technological Progress

$$\frac{\Delta Y}{Y} = \alpha \frac{\Delta K}{K} + (1-\alpha)\frac{\Delta L}{L} + \frac{\Delta A}{A}$$

$$\begin{array}{c}\text{Growth in}\\\text{output}\end{array} = \begin{array}{c}\text{Contribution}\\\text{of capital}\end{array} + \begin{array}{c}\text{Contribution}\\\text{of labor}\end{array} + \begin{array}{c}\text{Growth in Total}\\\text{Factor Productivity}\end{array}$$

The above is the growth-accounting equation.

Sinking into memories: Accounting of growth in Solow model (Part 3)

- **Solow Residual**

$$\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - \alpha \frac{\Delta K}{K} - (1-\alpha)\frac{\Delta L}{L}$$

- ***$\Delta A/A$*** is the change in output that cannot be explained by changes in inputs. Thus, the growth in total factor productivity is computed as a residual – that is, as the amount of output growth that remains after we have accounted for the determinants of growth that we can measure. Indeed, **$\Delta A/A$ is sometimes called the *Solow residual***, after Robert Solow, who first showed how to compute it.

Accounting of growth in the U.S. economy In the end of the XX century

(average percentage, %, increase per year) Years	$\begin{pmatrix}\text{Output Growth}\\ \Delta Y/Y\end{pmatrix}$ =	SOURCE OF GROWTH $\begin{pmatrix}\text{Capital}\\ \alpha\Delta K/K\end{pmatrix}$ +	$\begin{pmatrix}\text{Labor}\\ (1-\alpha)\Delta L/L\end{pmatrix}$ +	$\begin{pmatrix}\text{Total Factor Productivity}\\ \Delta A/A\end{pmatrix}$
1950-1960	3.5	1.1	0.8	1.6
1960-1970	4.1	1.2	1.3	1.7
1970-1980	3.1	0.9	1.6	0.5
1980-1990	2.9	0.8	1.3	0.8
1990-1996	2.2	0.6	0.8	0.8
1950-1996	3.2	0.9	1.2	1.1

Accounting of growth among “Asian Tigers” In the end of the XX century

%	Hong Kong (1966-1991)	Singapore (1966-1990)	South Korea (1966-1990)	Taiwan (1966-1990)
GDP per capita growth	5.7	6.8	6.8	6.7
TFP* growth	2.3	0.2	1.7	2.6
Δ% labor force participation	38→49	27→51	27→36	28→37
Δ% secondary education or higher	27.2→71.4	15.8→66.3	26.5→75.0	25.8→67.6

*TFP: total factor productivity

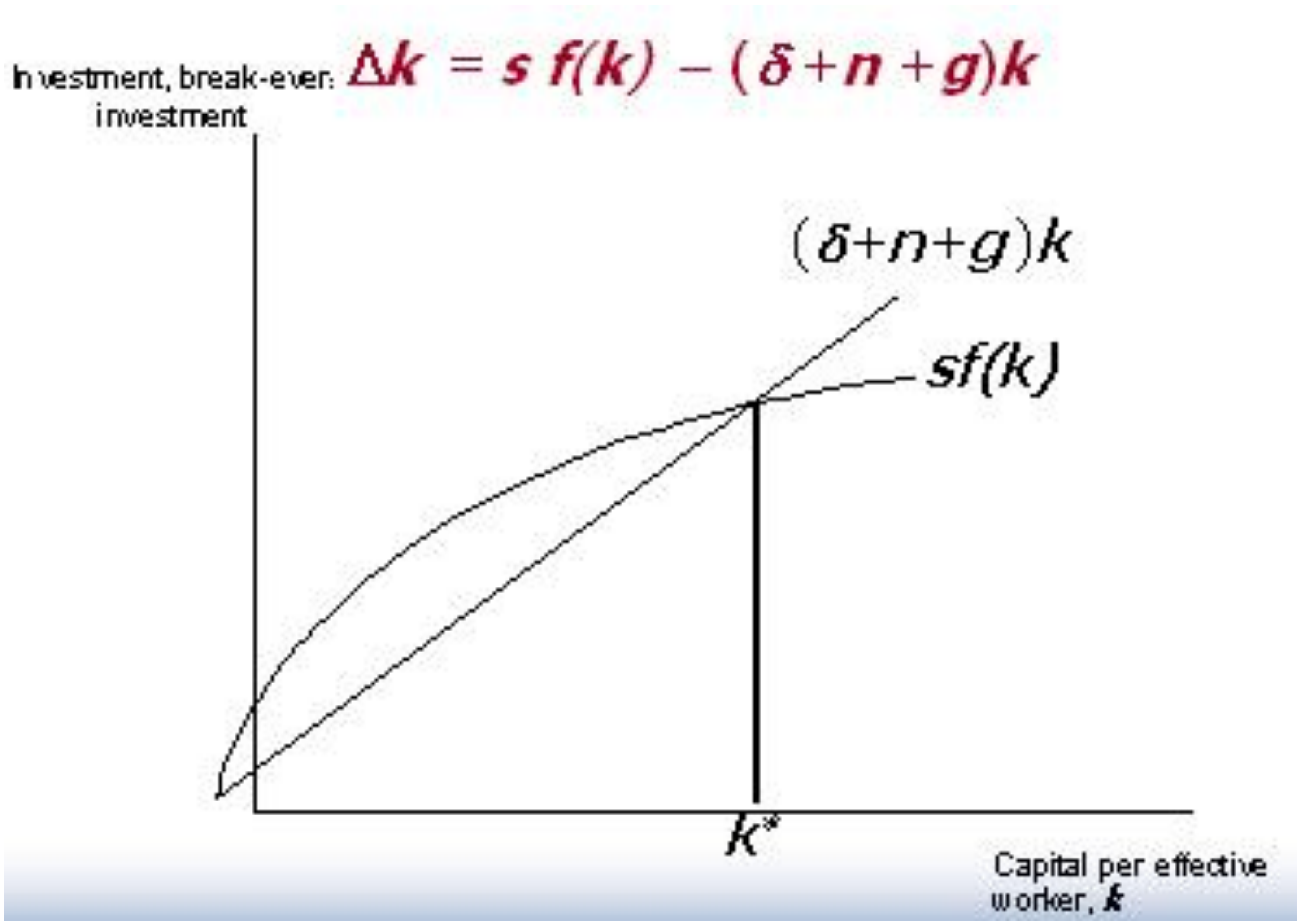
Source: Alwyn Young, “The Tyranny of Numbers: Confronting the Statistical Realities of the East Asian Growth Experience,” *Quarterly Journal of Economics*, August 1995.

Exercise #1: the condition

The savings rate = 0.3; the rate of population growth = 0.03; the rate of technical progress = 0.02; the depreciation rate = 0.1. The production function is the Cobb-Douglas function with labor-augmenting technical progress, that is: $Y = K^{0.5}(LE)^{0.5}$

Calculate: Calculate equilibrium capital per effective worker ratio, amount of actual investment and amount of actual consumption.

Exercise #1: the solution: the graph



Exercise #1: the solution: the figures

1) If $Y = K^{0.5}(LE)^{0.5}$

Then $y = k^{0.5}$

2) $sy = sk^{0.5} = (n + g + d)k$

$0.3k^{0.5} = (0.03 + 0.02 + 0.1)k$

$0.3k^{0.5} = 0.15k$; $2k^{0.5} = k$

$k = 4$; $y = 2$

3) actual investment = savings = $s*y = 0.3*2 = 0.6$.

4) actual consumption = $y – s = 2 – 0.6 = 1.4$.

Exercise #2: the condition

The rate of population growth = 0.04; the rate of technical progress = 0.06; the depreciation rate = 0.08, capital per effective worker ratio = 4. The production function is the Cobb-Douglas function with labor-augmenting technical progress, that is: $Y = K^{0.5}(LE)^{0.5}$

Calculate: Calculate equilibrium savings rate, amount of actual investment and amount of actual consumption

Exercise #2: the solution:

1) If $Y = K^{0.5}(LE)^{0.5}$

Then $y = k^{0.5}$

2) $sy = sk^{0.5} = (n + g + d)k$

$s*4^{0.5} = (0.04 + 0.06 + 0.08)*4$

$s = 0.18*4 : 2 = 0.36 = 36\%$

3) actual investment = savings = $s*y = 0.36*2 = 0.72$.

4) actual consumption = $y – s = 2 – 0.72 = 1.28$.

Exercise #2: the additional question

Is this saving rate – 36% - consistent with the golden rule?

Exercise #2: reply to the additional question

Max $c = (1 - s)y$

…

If we take $\partial c/\partial s$ and make it equal to zero that it implies that $s = \alpha$ or $s = 0.5$

Exercise #3: the condition

The savings rate = 0.48; the rate of population growth = 0.04; the rate of technical progress = 0.03; the depreciation rate = 0.05. The production function is the Cobb-Douglas function with labor-augmenting technical progress, that is: $Y = K^{0.5}(LE)^{0.5}$

Calculate: Calculate equilibrium capital per effective worker ratio, amount of actual investment and amount of actual consumption.

Exercise #4: the condition

The rate of population growth = 0.03; the rate of technical progress = 0.02; the depreciation rate = 0.07, capital per effective worker ratio = 36. The production function is the Cobb-Douglas function with labor-augmenting technical progress, that is: $Y = K^{0.5}(LE)^{0.5}$

Calculate: Calculate equilibrium savings rate, amount of actual investment and amount of actual consumption

Exercise #5: the condition

The production function is: $Y = AK^{0.4}L^{0.6}$

The rate of economic growth = 3.9%, the rate of capital accumulation = 3%, the rate of population growth = 2%.

Calculate Solow residual