Algebraic constructions generated by causal structure of space-times

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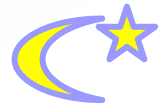
Algebraic Quantum Field Theory (AQFT). Tools



-algebras

C^*-algebra is an involutive Banach algebra $\mathfrak{A}$ satisfying $\|A^* A\| = \|A\|^2 \ \forall A \in \mathfrak{A}$.

***-homomorphism**

$\rho: C^* \to C^*$

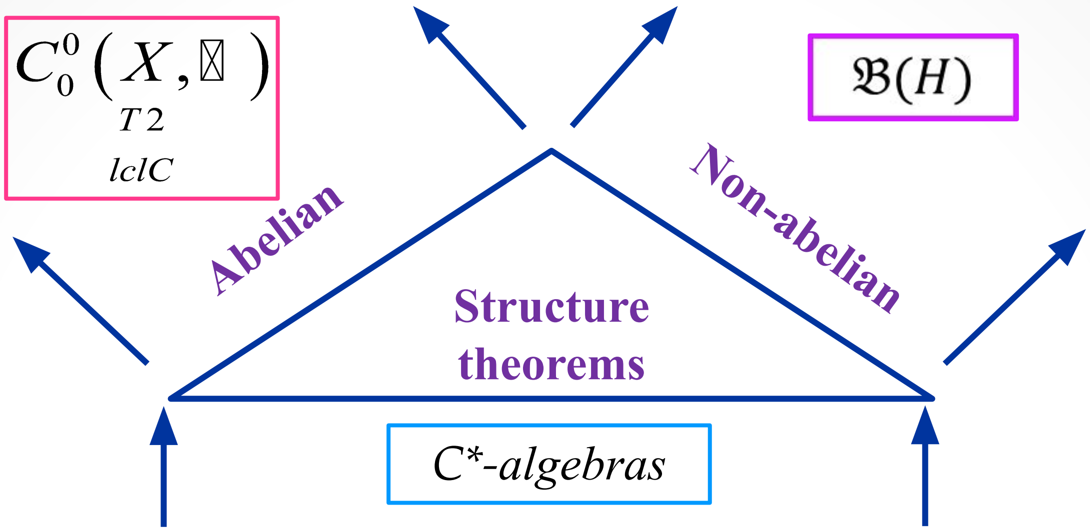
$A \mapsto \rho(A)$,

- $\rho(\lambda(A_1 + A_2)) = \lambda\, \rho(A_1) + \rho(A_2)$ — *is ℂ-linear*
- $\rho(A_1 A_2) = \rho(A_2) \cdot \rho(A_1)$ — *respects multiplication*
- $\rho(A^*) = \left(\rho(A)\right)^*$ — *respects involution*

Representation of a C^*-algebra π: $C^* \to \mathfrak{B}(\mathcal{H})$ *linear bounded operators on a ℂ-Hilbert space $\mathcal{H}$*

**-homomorphism*

Structure theorems for C^* - algebras



Gelfand-Naimark Theorem

$$\boxed{\underset{lclC}{HTop} \simeq \underset{abl}{C^*Alg}^{\bowtie}}$$

Trivial examples of C*-algebras

abelian $\mathbb{C}$ $\quad$ $Mat(n,\mathbb{C})$ *non-abelian*

Algebraic Quantum Field Theory (AQFT)

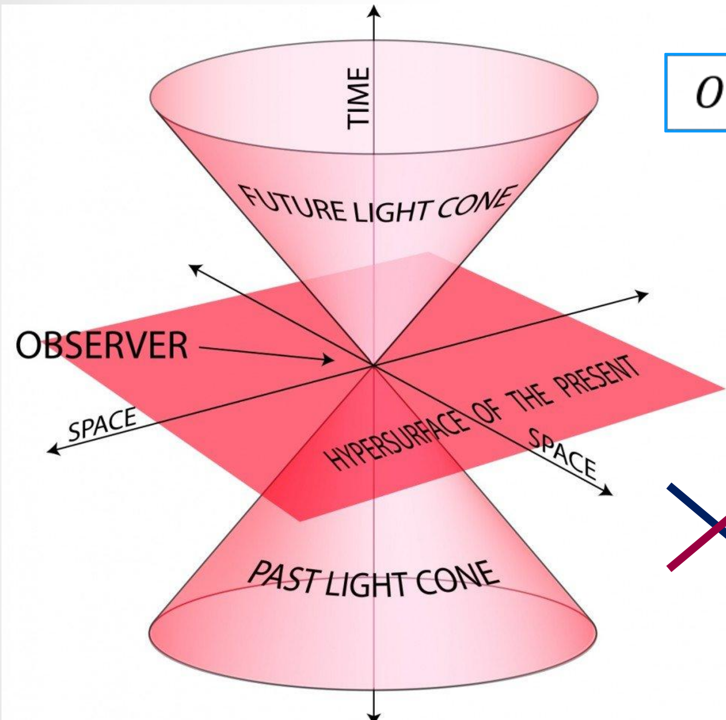
Net of observable algebras $\{\mathfrak{A}(O): O \in K\}$ [1,2]

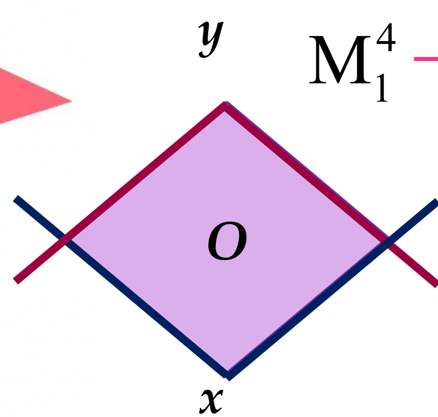
$$\boxed{\overline{V_+} \equiv \left\{ x \in M_1^4 \mid x \cdot x \in \mathbb{R}_{\geq 0}, x^0 \in \mathbb{R}_{>0} \right\}}$$

Haag-Kastler axioms

- ***Isotony***
- ***Causality***
- ***Covariance***
- ***Time slice axiom***
- ***Spectrum condition***

$$\boxed{O = I_+(x) \cap I_-(y)}$$





Finite contractible region

$$M_1^4 \longrightarrow \mathfrak{A} \xrightarrow{\text{GNS-theorem}} \pi(\mathfrak{A})$$

$\mathfrak{A}(O)$ — *Abstract C*-algebra*

$\pi(\mathfrak{A}(O))$ — *Operator algebra on $\mathcal{H}$*

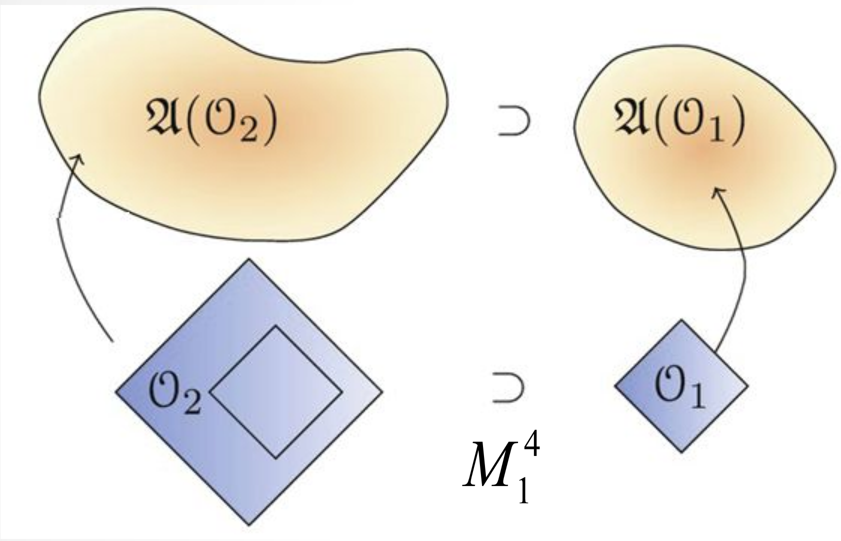
1 *Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. 5(7), 848–861 (1964)*

2 *Araki, H.: Mathematical Theory of Quantum Fields, vol. 101. Oxford UniversityPress, Oxford (1999)*

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

Isotony



$$\boxed{O_1 \subseteq O_2 \ \Rightarrow \exists \alpha_{12}\colon \mathfrak{A}(O_1) \to \mathfrak{A}(O_2)}$$

- For $''U \subseteq {'U}$ $\exists$ an **embedding**

$$in_{''U}^{'U} : \quad \mathfrak{A}(''U) \to \mathfrak{A}('U)$$

- For $'''U \subseteq {''U} \subseteq {'U}$, the embedding satisfy **the consistency conditions**

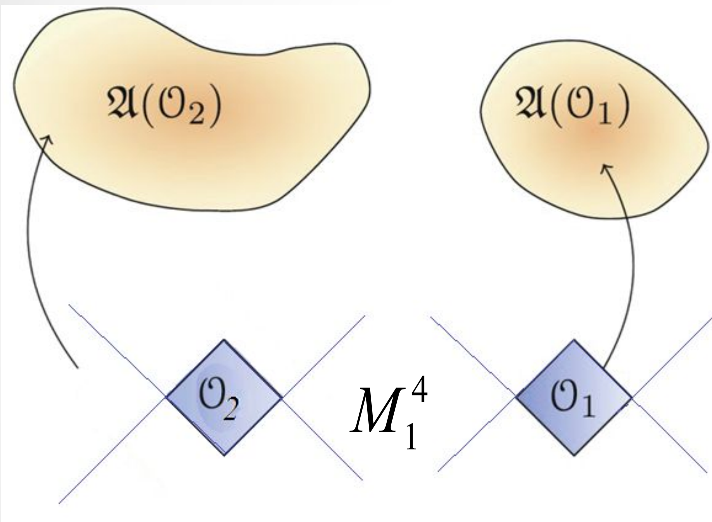
$$in_{''U}^{'U} \circ in_{'''U}^{''U} = in_{'''U}^{'U}$$

$$\begin{tikzcd}
\mathfrak{A}('''U) \arrow[r, "in_{'''U}^{''U}"] \arrow[rr, bend left] & \mathfrak{A}(''U) \arrow[r, "in_{''U}^{'U}"] & \mathfrak{A}('U)
\end{tikzcd}$$

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

Microcausality (locality)



$$[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = 0$$
O_1, O_2 – spatially separated diamonds

Connected component of the Poincare group identity $\Pi_+^\uparrow$,
is represented by authomorphisms $\rho_{(\Lambda,a)}$:

$$\mathfrak{A} \to \mathfrak{A}$$

$$A \mapsto \rho_{(\Lambda,a)}(A)$$

such that

$$\begin{cases} \rho_{(\Lambda,a)} \circ in_{''U}^{'U} = in_{(\Lambda,a)''U}^{(\Lambda,a)'U} \circ \rho_{(\Lambda,a)} \\ \rho_{(\tilde{\Lambda},\tilde{a})} \circ \rho_{(\Lambda,a)} = \rho_{(\tilde{\Lambda},\tilde{a})\cdot(\Lambda,a)} \end{cases}$$

$$\forall A \in \mathfrak{A}(''U)$$
$$''U \subseteq {}'U$$

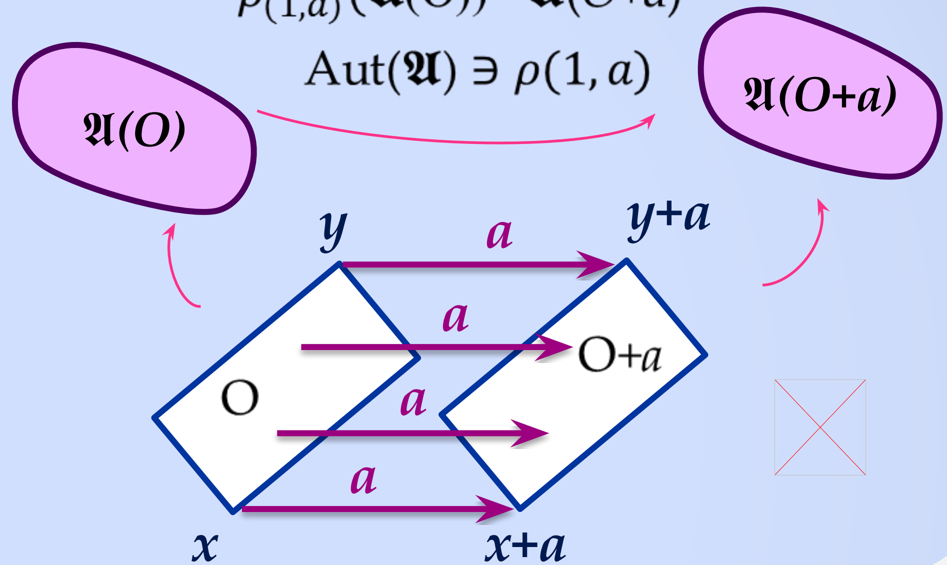
Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

- ***Covariance***
- $\rho_{(\Lambda,a)}(\mathfrak{A}(O))=\mathfrak{A}(\Lambda O+a)$
  $\forall(\Lambda,a)\in\Pi_+^\uparrow$
- $\rho_{(\Lambda,a)}\in Aut(\mathfrak{A})$

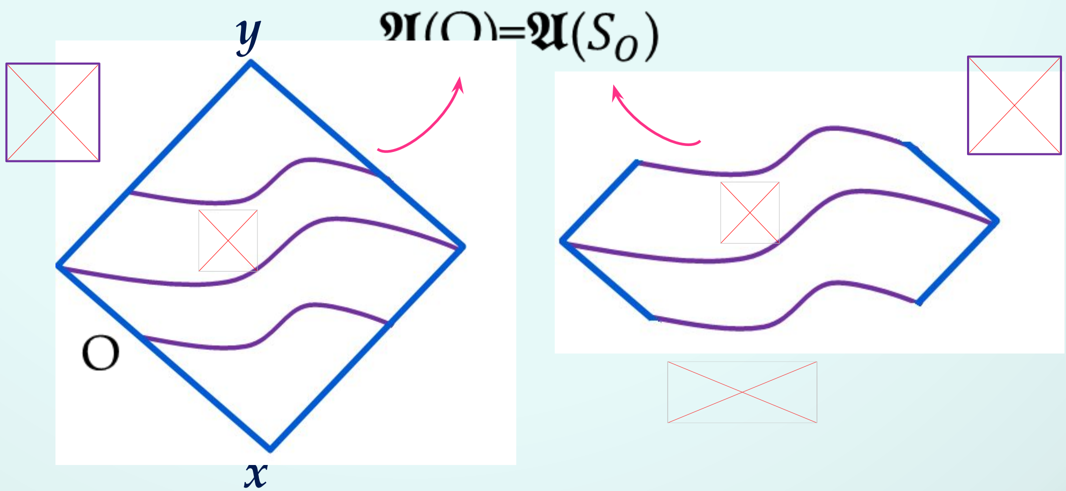
- ***Translational covariance***

$\rho_{(1,a)}(\mathfrak{A}(O))=\mathfrak{A}(O+a)$



- ***Time slice axiom***

$\mathfrak{A}(O)=\mathfrak{A}(S_O)$



Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

- ***Spectrum condition***

- $spec(P_\beta) \subset \overline{V_+}$
- P_β – *generators of translations*

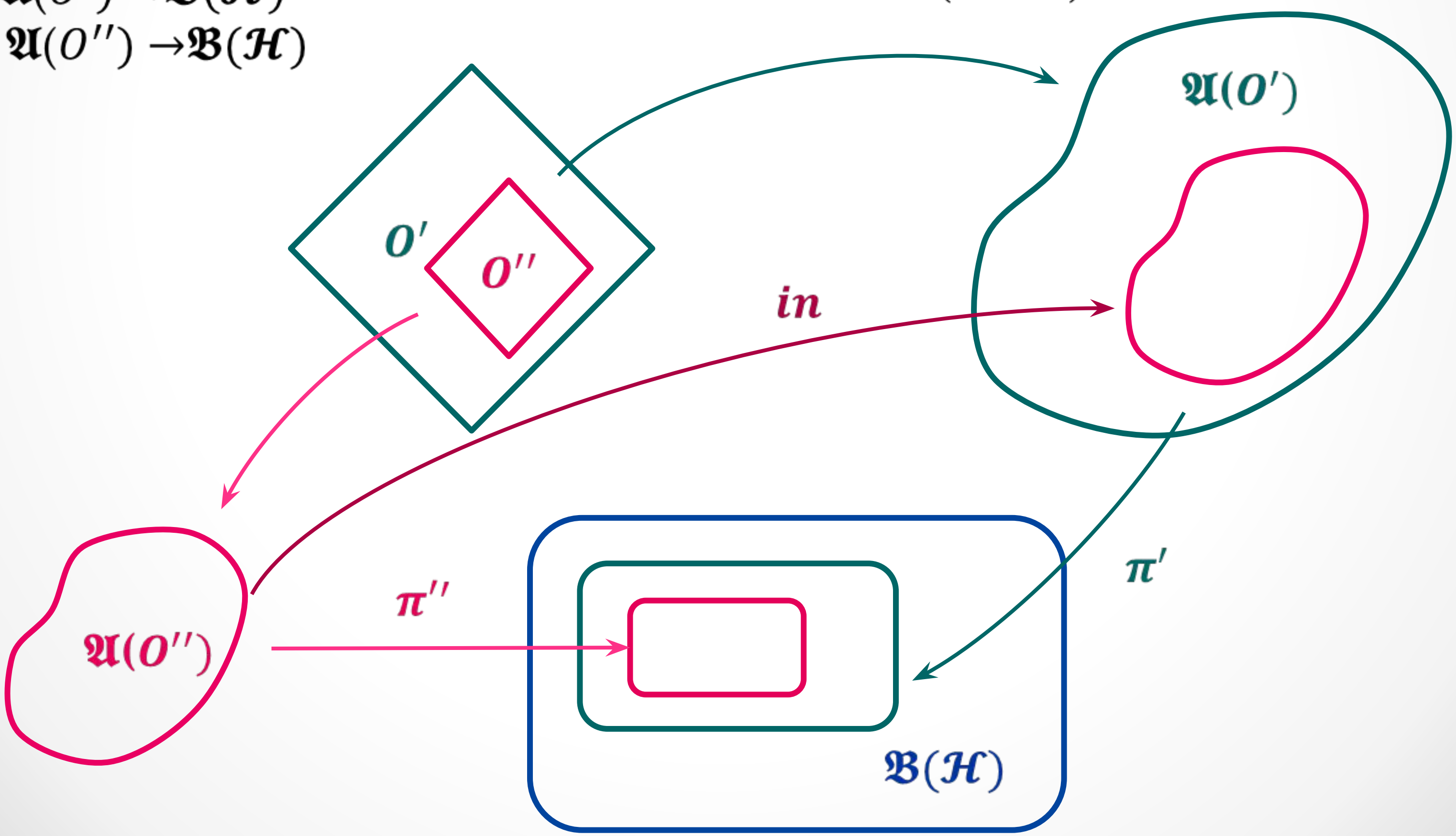
Consistent family of representations

$O' \supseteq O''$

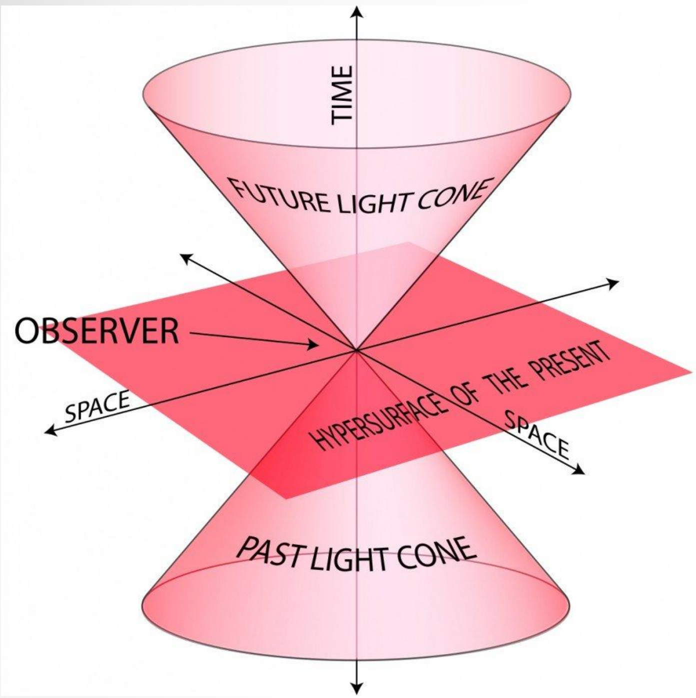
$\pi' : \mathfrak{A}(O') \to \mathfrak{B}(\mathcal{H})$

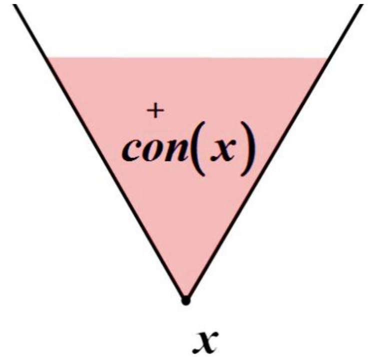
$\pi'' : \mathfrak{A}(O'') \to \mathfrak{B}(\mathcal{H})$

$\pi'|_{in(\mathfrak{A}(O''))} = \pi''$



Minkowski space-time

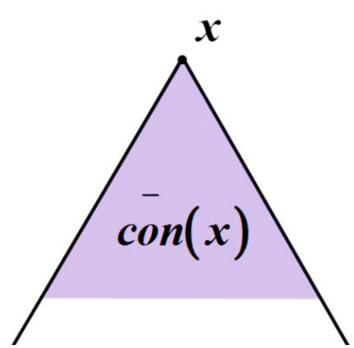




the family of
upper cones

$$\overset{+}{Con}\left(M_1^D\right) \equiv \left\{ \varnothing, \overset{+}{\underset{x}{con}} \,\middle|\, x \in M_1^D, x \cdot x \geq 0, x^0 \geq 0 \right\}$$

$$x \cdot x \equiv \eta_{ab} x^a x^b$$
$$a, b = \overline{0, D-1}$$
$$\left(\eta_{ab}\right) \equiv diag(+,-,-,\ldots,-)$$



the family of
lower cones

$$\overset{-}{Con}\left(M_1^D\right) \equiv \left\{ \varnothing, \overset{-}{\underset{x}{con}} \,\middle|\, x \in M_1^D, x \cdot x \geq 0, x^0 \leq 0 \right\}$$

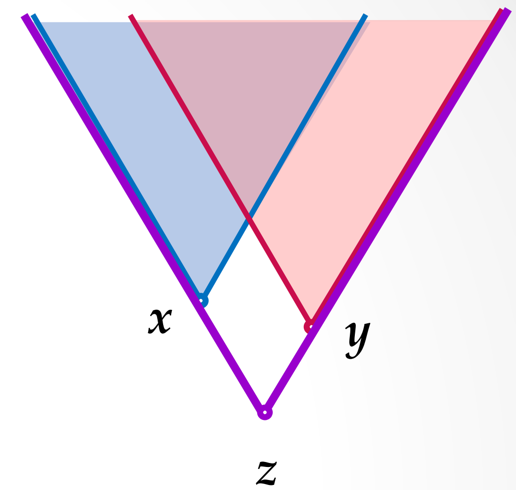
Operations on upper cones

addition

$$\overset{+}{\vee}:\ \overset{+}{Con}\left(M_1^D\right)\times\overset{+}{Con}\left(M_1^D\right)\to\overset{+}{Con}\left(M_1^D\right)$$

$$\left(\overset{+}{con}_x,\ \overset{+}{con}_y\right)\mapsto\ \overset{+}{con}_x\ \overset{+}{\vee}\ \overset{+}{con}_y\ \equiv$$

$$\equiv\ \begin{array}{l}\textit{smallest upper cone,}\\ \textit{containing } \overset{+}{con}_x \text{ and } \overset{+}{con}_y\end{array} = \overset{+}{con}_z$$

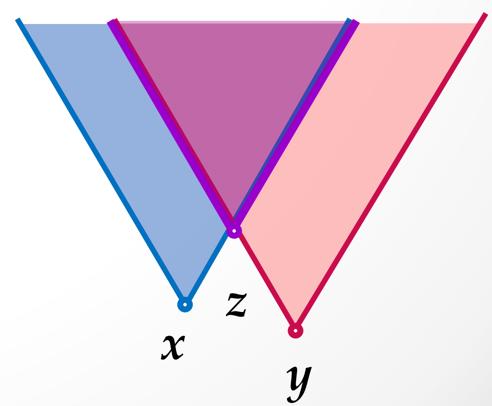


multiplication

$$\overset{+}{\wedge}:\ \overset{+}{Con}\left(M_1^D\right)\times\overset{+}{Con}\left(M_1^D\right)\to\overset{+}{Con}\left(M_1^D\right)$$

$$\left(\overset{+}{con}_x,\ \overset{+}{con}_y\right)\mapsto\ \overset{+}{con}_x\ \overset{+}{\wedge}\ \overset{+}{con}_y\ \equiv$$

$$\equiv\ \overset{+}{con}_x\ \cap\ \overset{+}{con}_y = \overset{+}{con}_z$$



Addition and multiplication on upper cones. Properties

Notation $\{\overset{+}{\vee}, \overset{+}{\wedge}\} = \overset{+}{\oplus}$

$\forall\ x, y \in M_1^D$

- ***idempotency***

$$\overset{+}{con}_x \overset{+}{\oplus} \overset{+}{con}_x \equiv \overset{+}{con}_x$$

- ***commutativity***

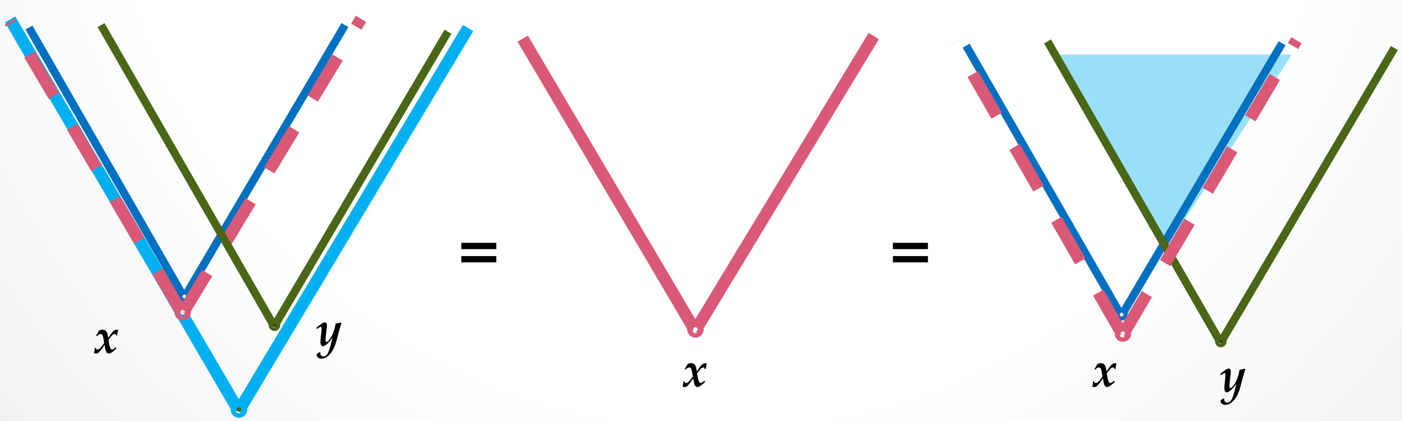
$$\overset{+}{con}_x \overset{+}{\oplus} \overset{+}{con}_y \equiv \overset{+}{con}_y \overset{+}{\oplus} \overset{+}{con}_x$$

- ***associativity***

$$\left(\overset{+}{con}_x \overset{+}{\oplus} \overset{+}{con}_y\right) \overset{+}{\oplus} \overset{+}{con}_z \equiv \overset{+}{con}_x \overset{+}{\oplus} \left(\overset{+}{con}_y \overset{+}{\oplus} \overset{+}{con}_z\right)$$

- ***absorption identity***

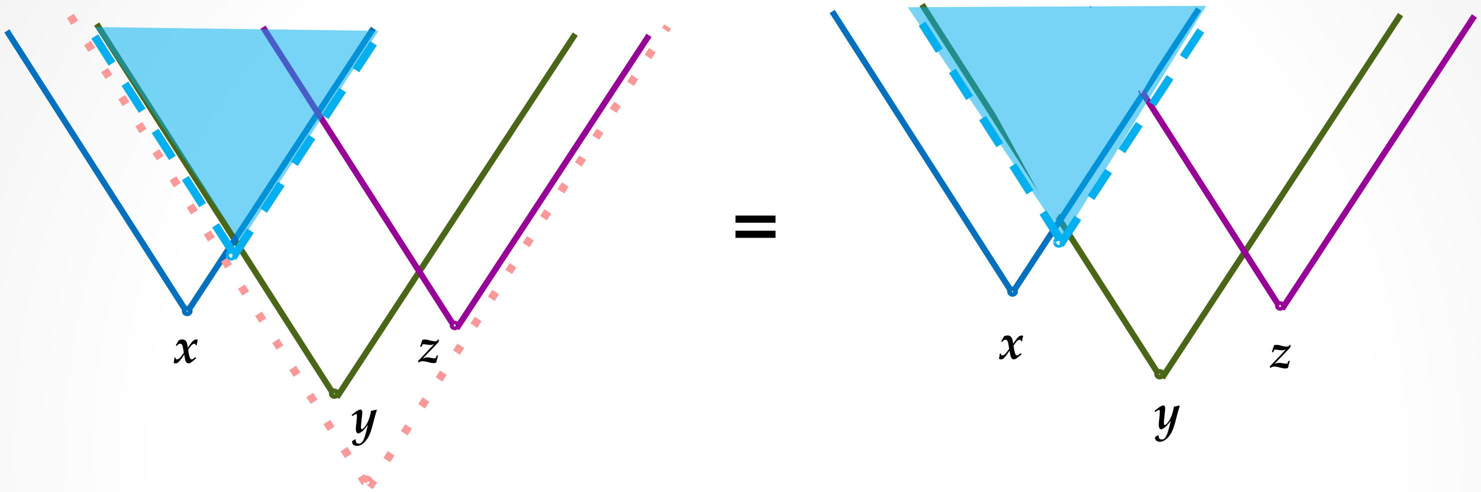
$$\overset{+}{con}_x \overset{+}{\vee} \left(\overset{+}{con}_x \overset{+}{\wedge} \overset{+}{con}_y\right) \equiv \overset{+}{con}_x \equiv \overset{+}{con}_x \overset{+}{\wedge} \left(\overset{+}{con}_x \overset{+}{\vee} \overset{+}{con}_y\right)$$



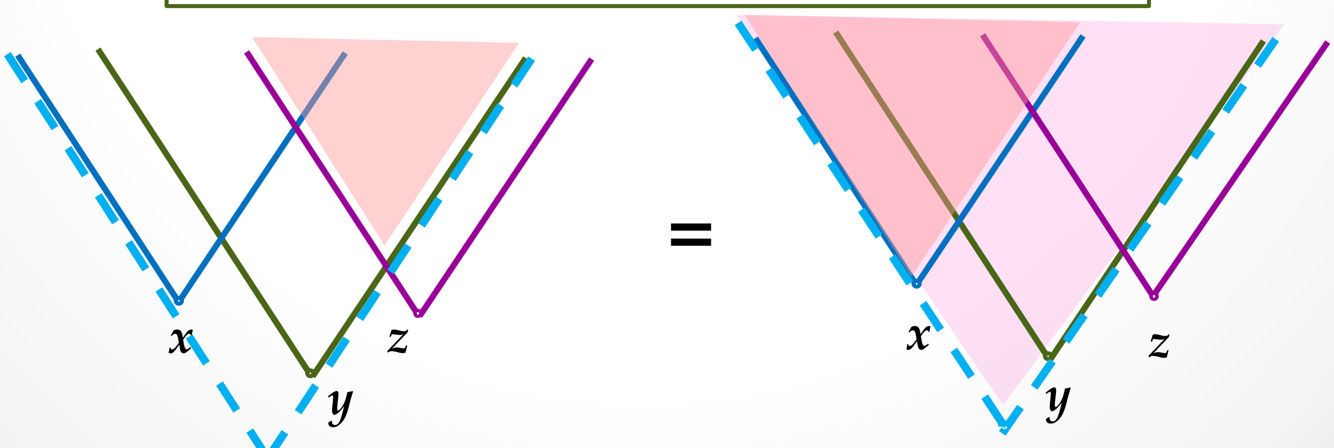
Distributivity of the obtained lattice

$$\left(Con^+(M_1^D), \wedge, \vee\right)$$

$$con_x^+ \wedge \left(con_y^+ \vee con_z^+\right) = \left(con_y^+ \wedge con_z^+\right) \vee \left(con_y^+ \wedge con_z^+\right)$$



$$con_x^+ \vee \left(con_y^+ \wedge con_z^+\right) = \left(con_y^+ \vee con_z^+\right) \wedge \left(con_y^+ \vee con_z^+\right)$$



Bijection of distributive lattices $\left(Con\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right)$ *and* $\left(Con\left(M_1^D\right), \overset{-}{\wedge}, \overset{-}{\vee}\right)$

Bijection T transforms one cone into another without changing the vertex

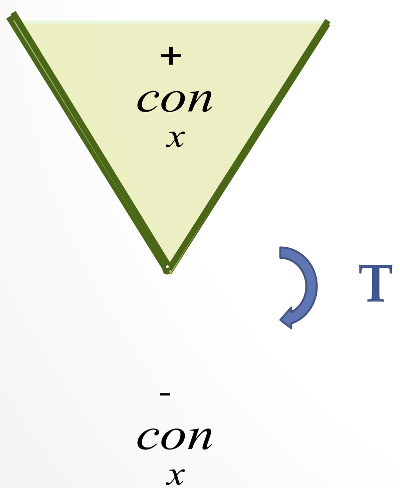
$$T : \left(Con\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right) \to \left(Con\left(M_1^D\right), \overset{-}{\vee}, \overset{-}{\wedge}\right)$$

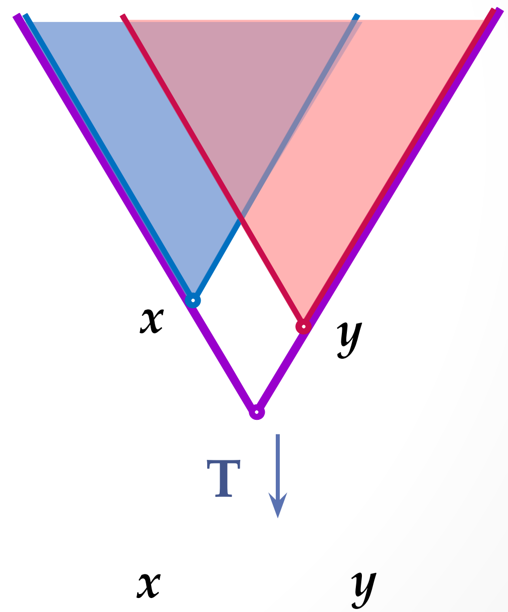
$$\overset{+}{con}_x \mapsto T\left(\overset{+}{con}_x\right) \equiv \overset{-}{con}_x$$

$$T\left(\overset{+}{con}_x \overset{+}{\vee} \overset{+}{con}_y\right) = T\left(\overset{+}{con}_x\right) \overset{-}{\wedge} T\left(\overset{+}{con}_y\right)$$

$$T\left(\overset{+}{con}_x \overset{+}{\wedge} \overset{+}{con}_y\right) = T\left(\overset{+}{con}_x\right) \overset{-}{\vee} T\left(\overset{+}{con}_y\right)$$

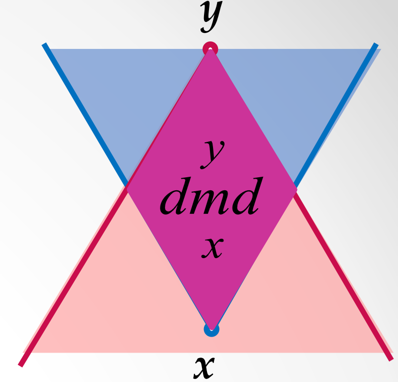
$$\overset{+}{con}_x \subseteq \overset{+}{con}_y \Rightarrow T\left(\overset{+}{con}_x\right) \supseteq T\left(\overset{+}{con}_y\right)$$





Operations on diamonds

$$Dmd\left(M_1^D\right) \equiv \left\{ \begin{array}{l} dmd_x^y = con_x^+ \cap con_y^- = con_x^+ \cap T\left(con_y^+\right) = \\ = T^{-1}\left(con_x^-\right) \cap con_y^- \end{array} \middle| \; x, y \in M_1^D \right\}$$

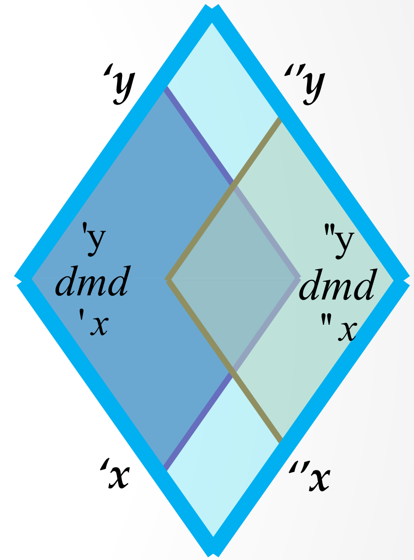


addition

$$\vee : Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$

$$\left(dmd_{'x}^{'y}, dmd_{''x}^{''y} \right) \mapsto dmd_{'x}^{'y} \vee dmd_{''x}^{''y} \equiv$$

$$\equiv \left(con_{'x}^{+} \vee con_{''x}^{+} \right) \cap \left(con_{'y}^{-} \vee con_{''y}^{-} \right)$$

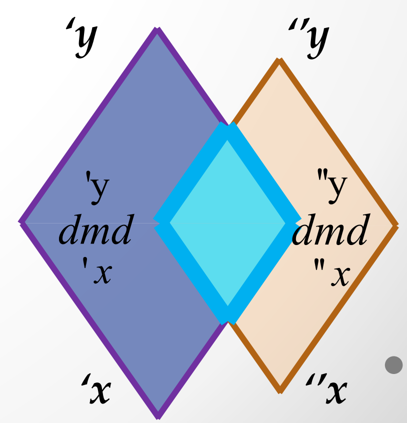


multiplication

$$\wedge : Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$

$$\left(dmd_{'x}^{'y}, dmd_{''x}^{''y} \right) \mapsto dmd_{'x}^{'y} \wedge dmd_{''x}^{''y} \equiv$$

$$\equiv \left(con_{'x}^{+} \wedge con_{''x}^{+} \right) \cap \left(con_{'y}^{-} \wedge con_{''y}^{-} \right)$$



Addition and multiplication on diamonds. Properties

- *idempotency* $\quad {}_{x}\mathrm{dmd}^{y} \; C \; {}_{x}\mathrm{dmd}^{y} = {}_{x}\mathrm{dmd}^{y}$

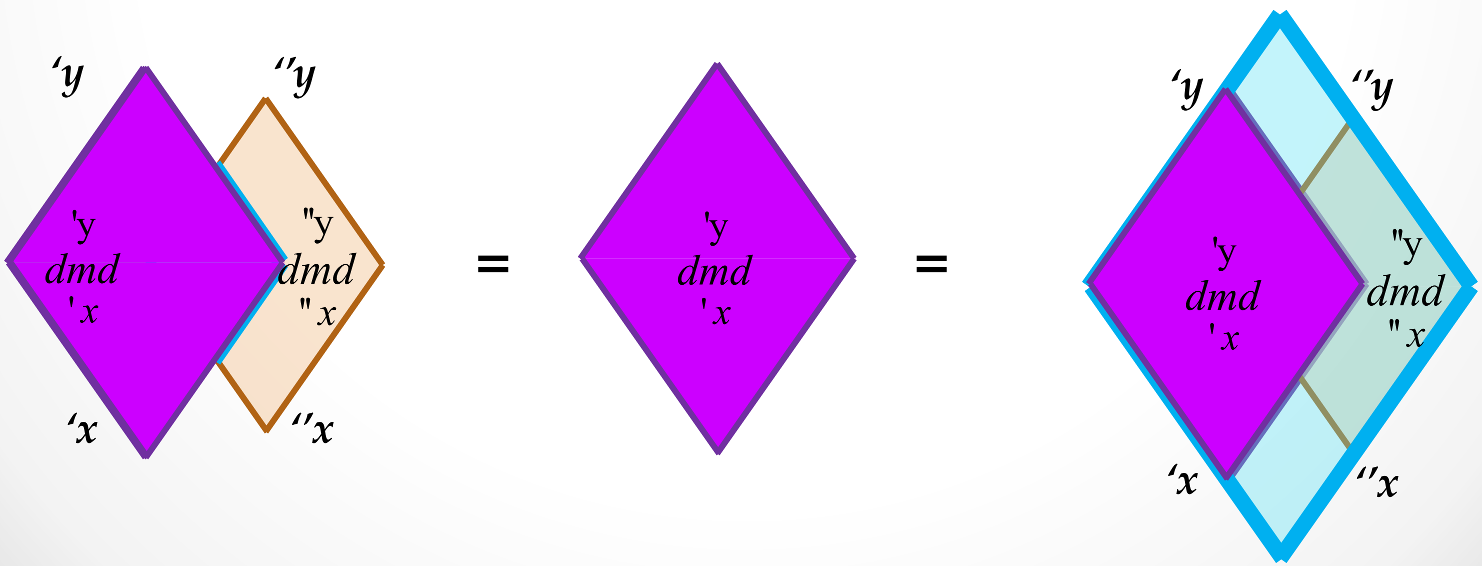
Notation $\{\vee, \wedge\} = C$

- *commutativity* $\quad {}_{'x}\mathrm{dmd}^{'y} \; C \; {}_{''x}\mathrm{dmd}^{''y} = {}_{''x}\mathrm{dmd}^{''y} \; C \; {}_{'x}\mathrm{dmd}^{'y}$

$\forall \; x, y \in M_1^D$

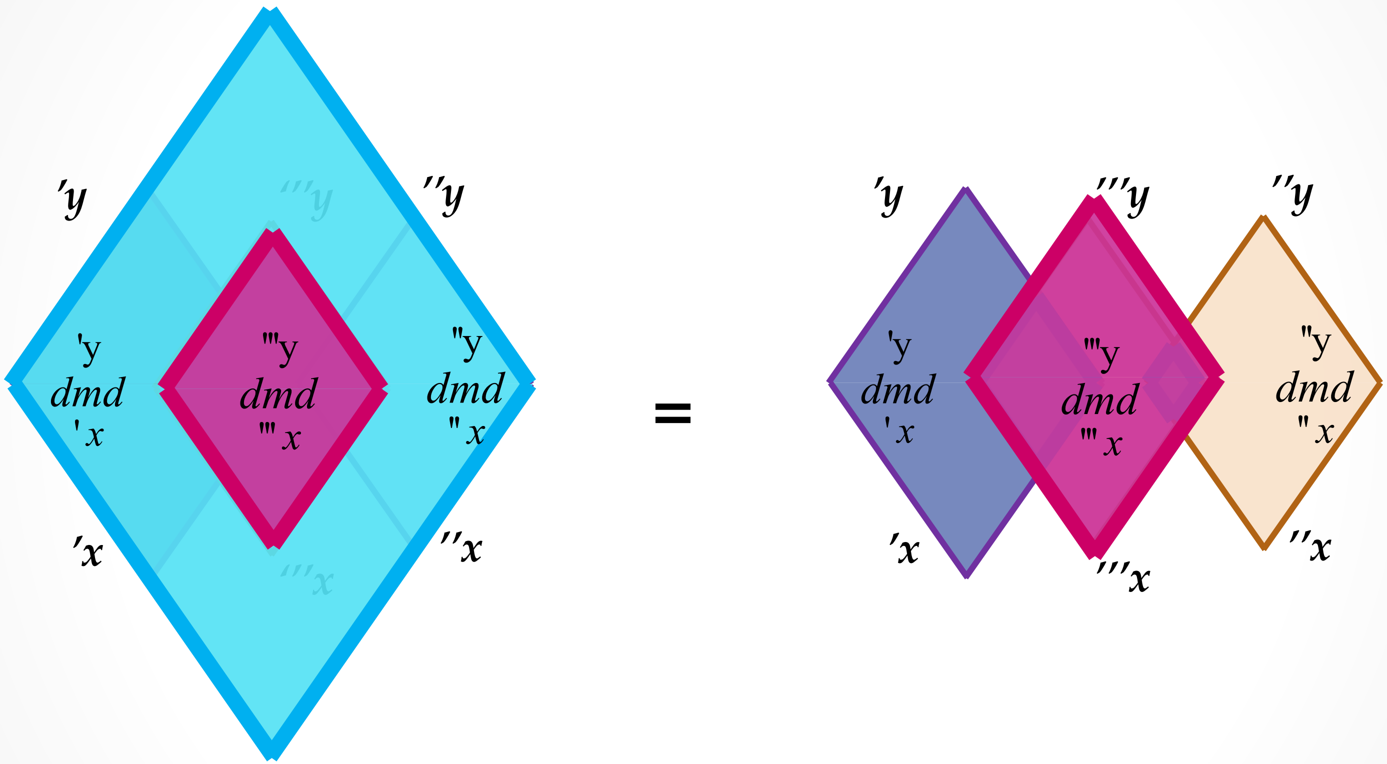
- *associativity* $\quad \left({}_{'x}\mathrm{dmd}^{'y} \; C \; {}_{''x}\mathrm{dmd}^{''y} \right) C \; {}_{'''x}\mathrm{dmd}^{'''y} = {}_{'x}\mathrm{dmd}^{'y} \; C \left({}_{''x}\mathrm{dmd}^{''y} \; C \; {}_{'''x}\mathrm{dmd}^{'''y} \right)$

- *absorption identity* $\quad {}_{'x}\mathrm{dmd}^{'y} \vee \left({}_{'x}\mathrm{dmd}^{'y} \wedge {}_{''x}\mathrm{dmd}^{''y} \right) = {}_{'x}\mathrm{dmd}^{'y} = {}_{'x}\mathrm{dmd}^{'y} \wedge \left({}_{'x}\mathrm{dmd}^{'y} \vee {}_{''x}\mathrm{dmd}^{''y} \right)$



Distributivity of the obtained lattice $\left(Dmd\left(M_1^D\right), \wedge, \vee\right)$

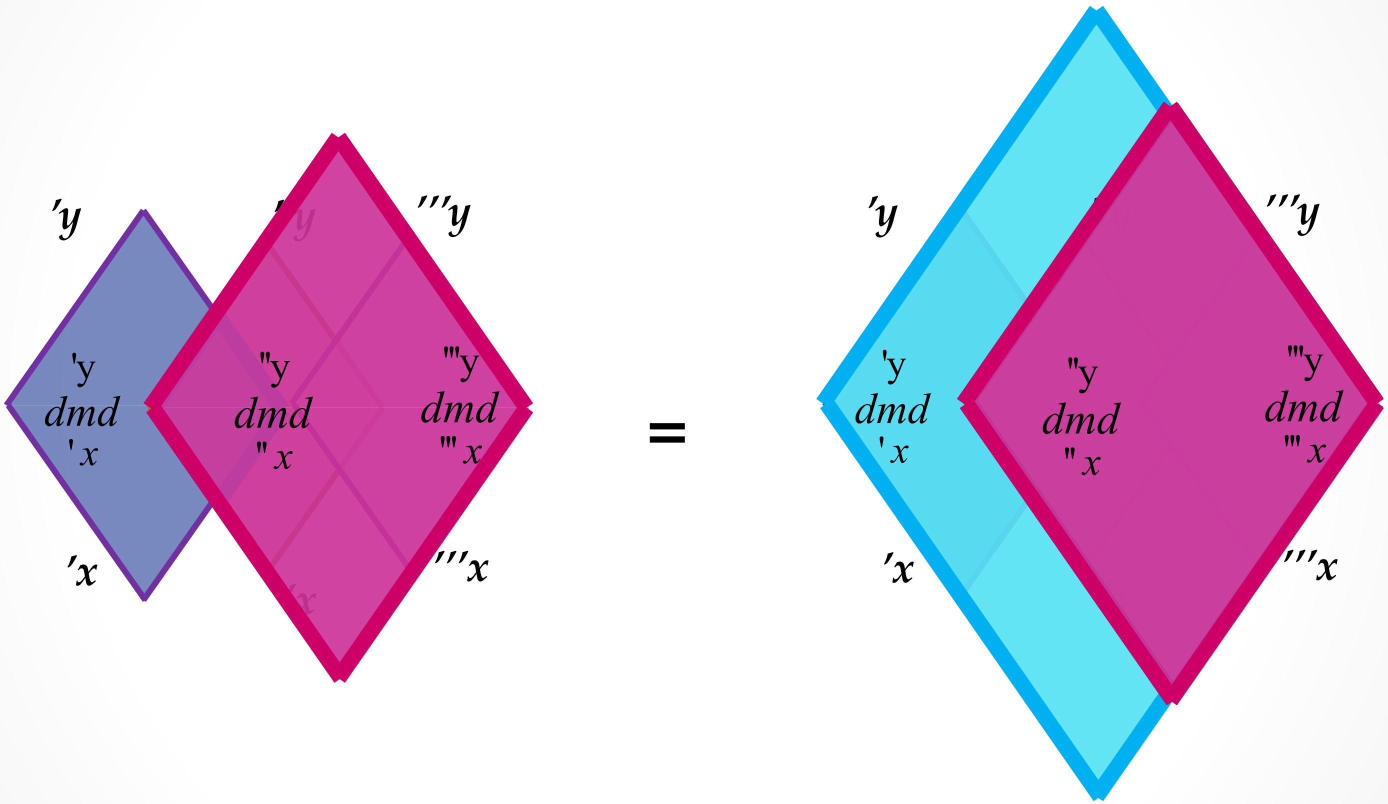
$$\left(dmd_{'x}^{'y} \vee dmd_{''x}^{''y}\right) \wedge dmd_{'''x}^{'''y} = \left(dmd_{'x}^{'y} \wedge dmd_{'''x}^{'''y}\right) \vee \left(dmd_{''x}^{''y} \wedge dmd_{'''x}^{'''y}\right)$$



*The **distributive identities** for the introduced operations:*
***addition** = "**union**" and **multiplication**=" **intersection**"*
*hold only when the involved **intersections of diamonds are nonempty***

Distributivity of the obtained lattice $\left(Dmd\left(M_1^D\right),\ \text{Ù},\ \text{Ú}\right)$

$$\left(dmd_{\,'x}^{\,'y}\ \text{Ù}\ dmd_{\,''x}^{\,''y} \right) \text{Ú}\ dmd_{\,'''x}^{\,'''y} = \left(dmd_{\,'x}^{\,'y}\ \text{Ú}\ dmd_{\,'''x}^{\,'''y} \right) \text{Ù} \left(dmd_{\,''x}^{\,''y}\ \text{Ú}\ dmd_{\,'''x}^{\,'''y} \right)$$



*The **distributive identities** for the introduced operations:*
addition *= "**union**" and **multiplication**="**intersection**"*
*hold only when the involved **intersections of diamonds are nonempty***

Thank you
for your attention!