Algebraic constructions generated by causal structure of space-times

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Algebraic Quantum Field Theory (AQFT). Tools

C^*-algebras

C^*-algebra is an involutive Banach algebra $\mathfrak{A}$ satisfying $\|A^* A\| = \|A\|^2 \ \forall A \in \mathfrak{A}$.

***-homomorphism**

$\rho: C^* \to C^*$

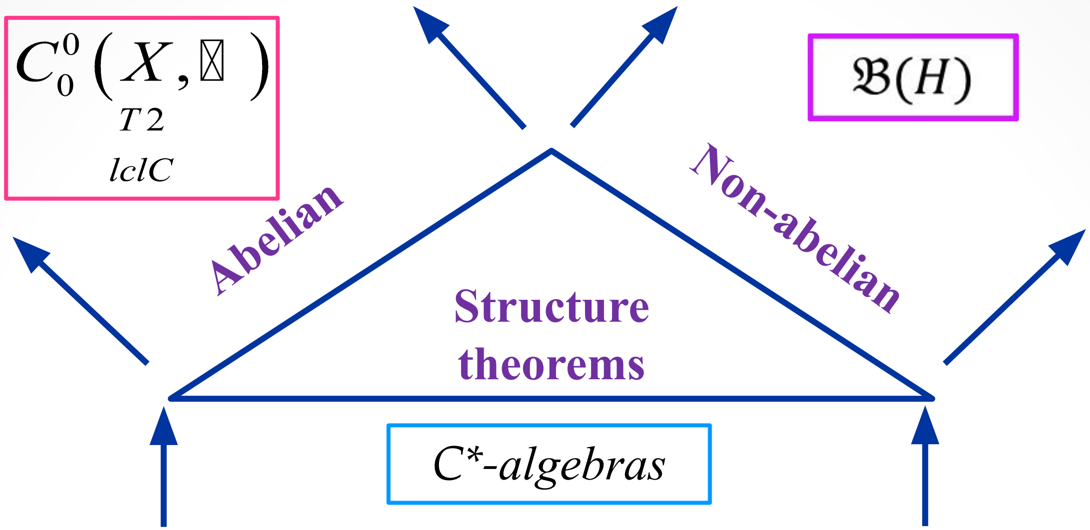
$A \mapsto \rho(A)$,

- $\rho(\lambda(A_1 + A_2)) = \lambda\, \rho(A_1) + \rho(A_2)$ — *is $\mathbb{C}$-linear*
- $\rho(A_1 A_2) = \rho(A_2) \cdot \rho(A_1)$ — *respects multiplication*
- $\rho(A^*) = \left(\rho(A)\right)^*$ — *respects involution*

Representation of a **C^*-algebra** π: $C^* \to \mathfrak{B}(\mathcal{H})$ *linear bounded operators on a $\mathbb{C}$-Hilbert space $\mathcal{H}$*

**-homomorphism*

Structure theorems for C^* - algebras



Gelfand-Naimark Theorem

$$\boxed{\underset{lclC}{HTop} \simeq \underset{abl}{C^*Alg}^{\bowtie}}$$

Trivial examples of C*-algebras

abelian $\mathbb{C}$ $Mat(n,\mathbb{C})$ *non-abelian*

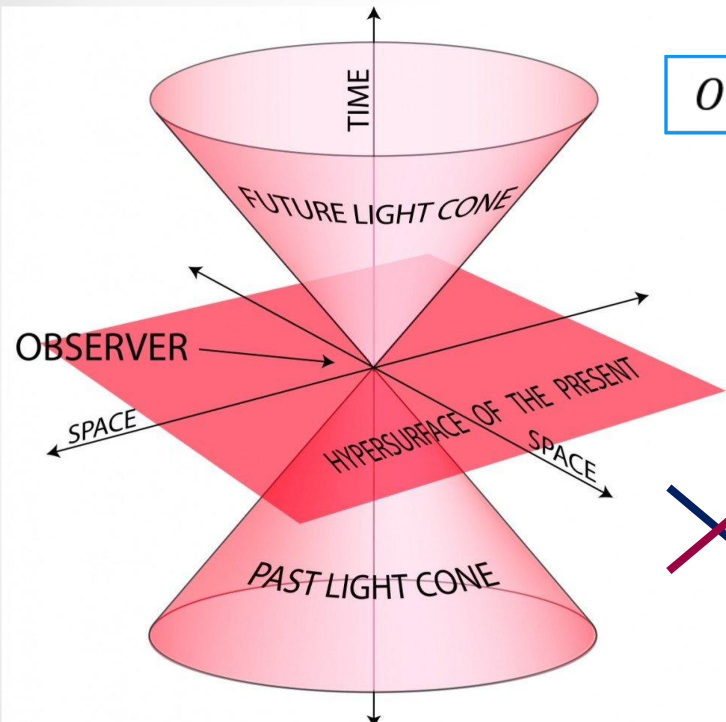
Algebraic Quantum Field Theory (AQFT)

Net of observable algebras $\{\mathfrak{A}(O) : O \in K\}$ [1,2]

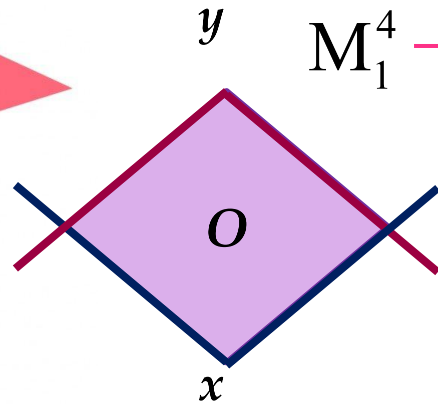
$$\boxed{\overline{V}_+ \equiv \left\{ x \in M_1^4 \mid x \cdot x \in \mathbb{R}_{\geq 0}, x^0 \in \mathbb{R}_{>0} \right\}}$$

Haag-Kastler axioms

- ***Isotony***
- ***Causality***
- ***Covariance***
- ***Time slice axiom***
- ***Spectrum condition***



$$\boxed{O = I_+(x) \cap I_-(y)}$$



Finite contractible region

$$M_1^4 \longrightarrow \mathfrak{A} \xrightarrow{\text{GNS-theorem}} \pi(\mathfrak{A})$$

$\mathfrak{A}(O)$ — *Abstract C*-algebra*

$\pi(\mathfrak{A}(O))$ — *Operator algebra on* $\mathcal{H}$

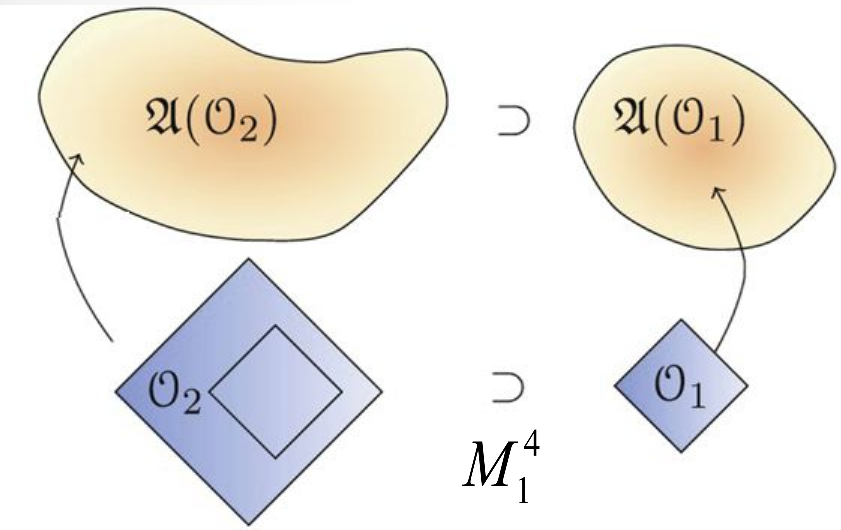
1 *Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. 5(7), 848–861 (1964)*

2 *Araki, H.: Mathematical Theory of Quantum Fields, vol. 101. Oxford UniversityPress, Oxford (1999)*

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

Isotony



$$\boxed{O_1 \subseteq O_2 \;\Rightarrow \exists \alpha_{12}\colon \mathfrak{A}(O_1) \to \mathfrak{A}(O_2)}$$

- For $''U \subseteq {}'U$ $\exists$ an **embedding**

$$in_{''U}^{'U}: \quad \mathfrak{A}(''U) \to \mathfrak{A}('U)$$

- For $'''U \subseteq {}''U \subseteq {}'U$, the embedding satisfy **the consistency conditions**

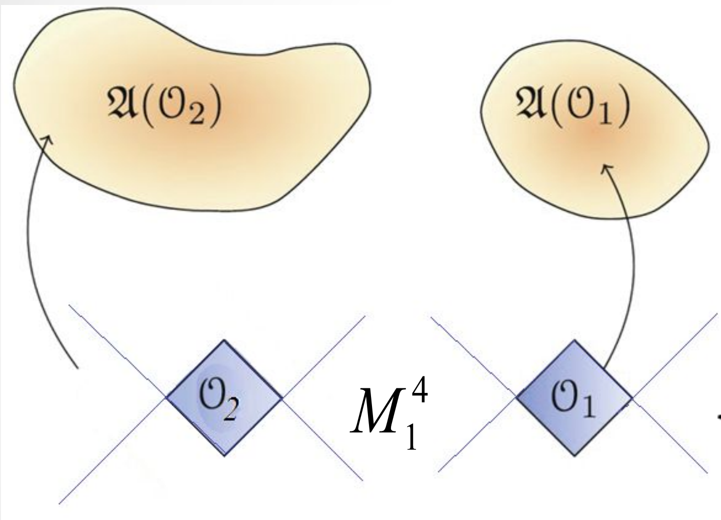
$$in_{''U}^{'U} \circ in_{'''U}^{''U} = in_{'''U}^{'U}$$

$$\begin{tikzcd} \mathfrak{A}('''U) \arrow[r, "in_{'''U}^{''U}"] \arrow[rr, bend left] & \mathfrak{A}(''U) \arrow[r, "in_{''U}^{'U}"] & \mathfrak{A}('U) \end{tikzcd}$$

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

Microcausality (locality)



$$[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = 0$$
O_1, O_2 – spatially separated diamonds

Connected component of the Poincare group identity $\Pi^{\uparrow}_{+}$,
is represented by authomorphisms $\rho_{(\Lambda,a)}$:

$$\mathfrak{A} \to \mathfrak{A}$$

$$A \mapsto \rho_{(\Lambda,a)}(A)$$

such that

$$\begin{cases} \rho_{(\Lambda,a)} \circ in^{'U}_{''U} = in^{(\Lambda,a)'U}_{(\Lambda,a)''U} \circ \rho_{(\Lambda,a)} \\ \rho_{(\Lambda',a')} \circ \rho_{(\Lambda,a)} = \rho_{(\Lambda',a')\cdot(\Lambda,a)} \end{cases}$$

$$\forall A \in \mathfrak{A}(''U)$$
$$''U \subseteq {}'U$$

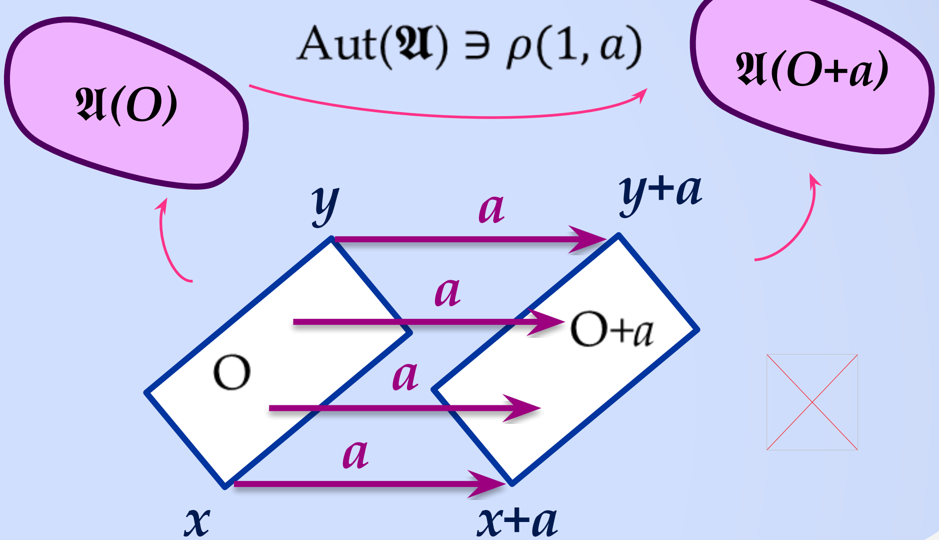
Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

- ***Covariance***
- $\rho_{(\Lambda,a)}(\mathfrak{A}(O))=\mathfrak{A}(\Lambda O+a)$ $\quad \forall(\Lambda,a)\in\Pi_{+}^{\uparrow}$
- $\rho_{(\Lambda,a)}\in Aut(\mathfrak{A})$

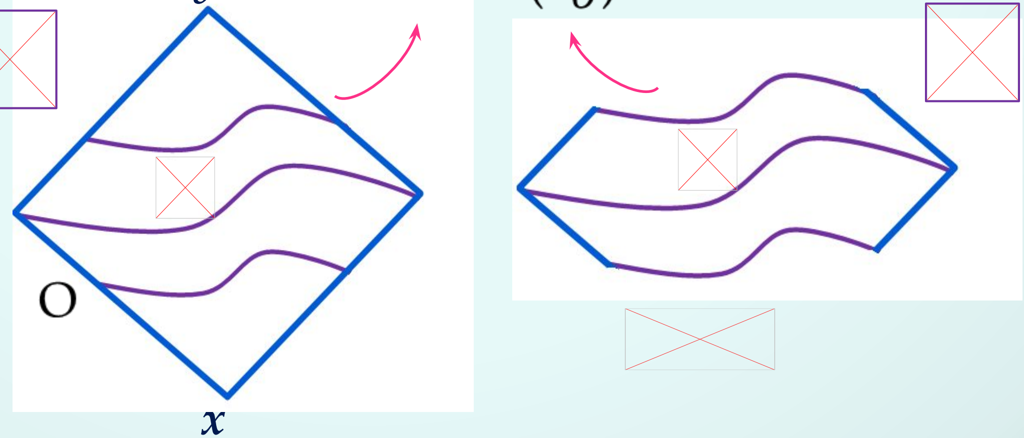
- ***Translational covariance***

$$\rho_{(1,a)}(\mathfrak{A}(O))=\mathfrak{A}(O+a)$$



- ***Time slice axiom***

$$\mathfrak{A}(O)=\mathfrak{A}(S_O)$$



Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

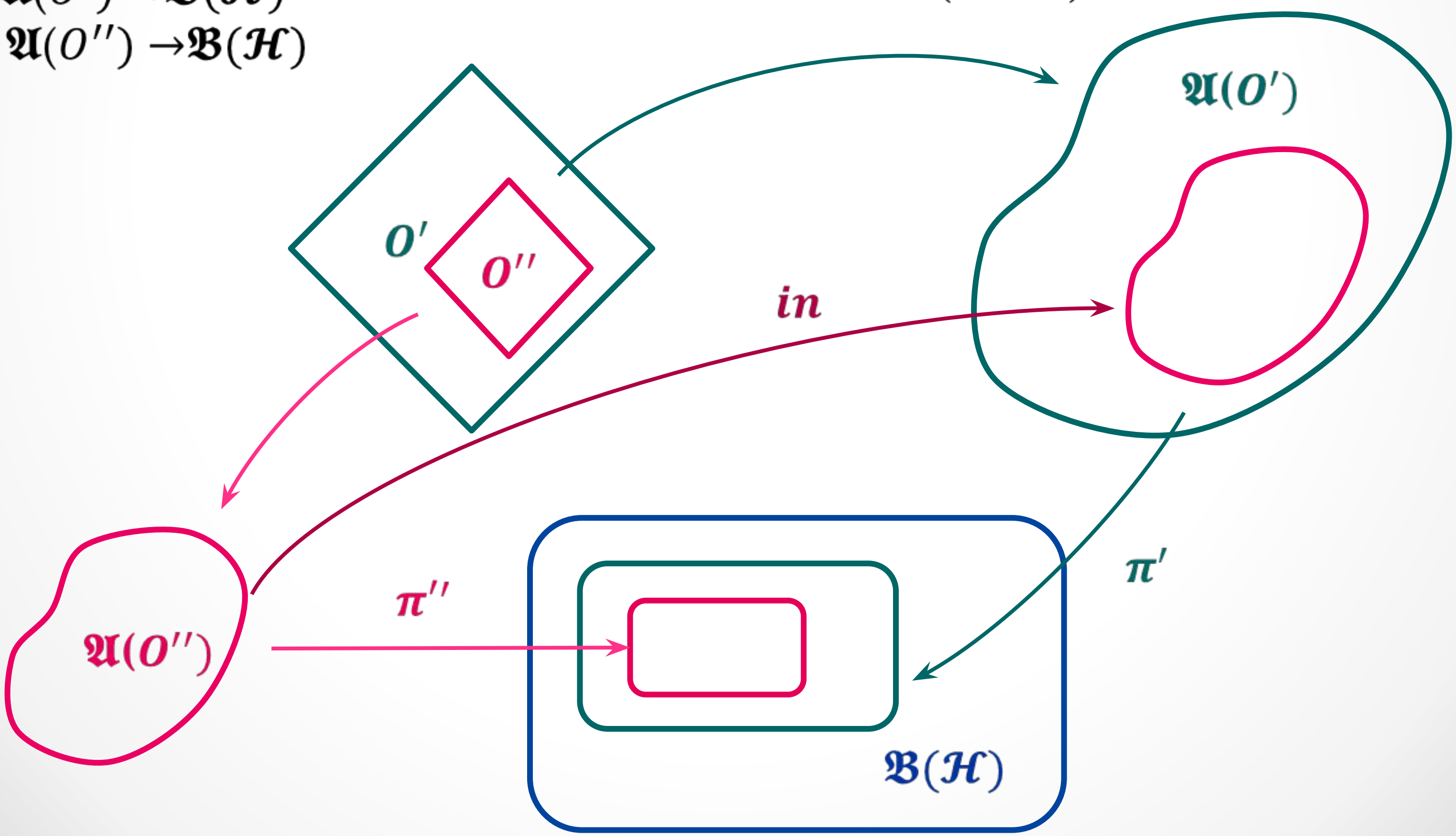
- *Spectrum condition*

- $spec(P_\beta) \subset \overline{V_+}$
- P_β – *generators of translations*

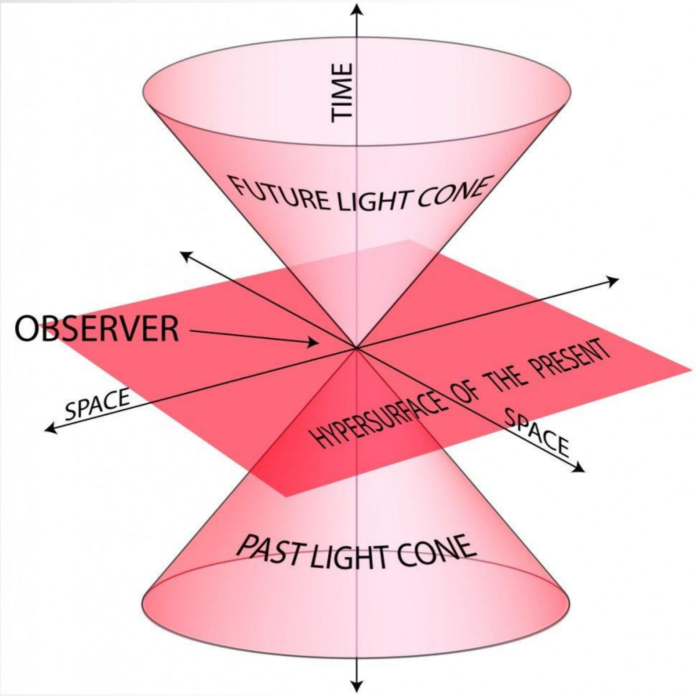
Consistent family of representations

$O' \supseteq O''$
$\pi': \mathfrak{A}(O') \to \mathfrak{B}(\mathcal{H})$
$\pi'': \mathfrak{A}(O'') \to \mathfrak{B}(\mathcal{H})$

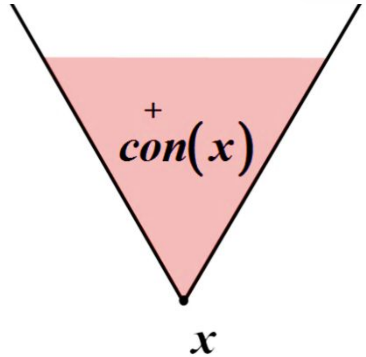
$\pi'|_{in(\mathfrak{A}(O''))} = \pi''$



Minkowski space-time

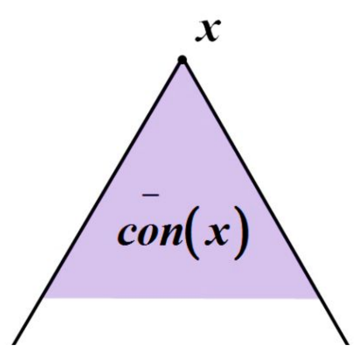


$$x \cdot x \equiv \eta_{ab} x^a x^b$$
$$a, b = \overline{0, D-1}$$
$$(\eta_{ab}) \equiv diag(+,-,-,\dots,-)$$



the family of *upper* **cones**

$$\overset{+}{Con}(M_1^D) \equiv \left\{ \emptyset, \overset{+}{con}_x \,\middle|\, x \in M_1^D, x \cdot x \geq 0, x^0 \geq 0 \right\}$$



the family of *lower* **cones**

$$\overset{-}{Con}(M_1^D) \equiv \left\{ \emptyset, \overset{-}{con}_x \,\middle|\, x \in M_1^D, x \cdot x \geq 0, x^0 \leq 0 \right\}$$

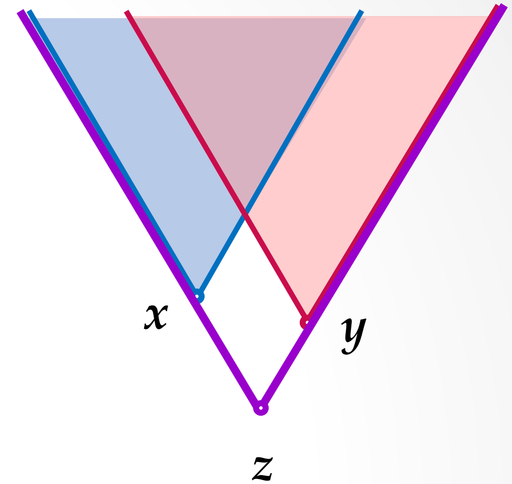
Operations on upper cones

addition

$$\overset{+}{\vee}: \overset{+}{Con}\left(M_1^D\right)\times \overset{+}{Con}\left(M_1^D\right)\to \overset{+}{Con}\left(M_1^D\right)$$

$$\left(\overset{+}{con}_x, \overset{+}{con}_y\right)\mapsto \overset{+}{con}_x \,\overset{+}{\vee}\, \overset{+}{con}_y \equiv$$

$\equiv$ *smallest upper cone, containing* $\overset{+}{con}_x$ and $\overset{+}{con}_y$ $= \overset{+}{con}_z$

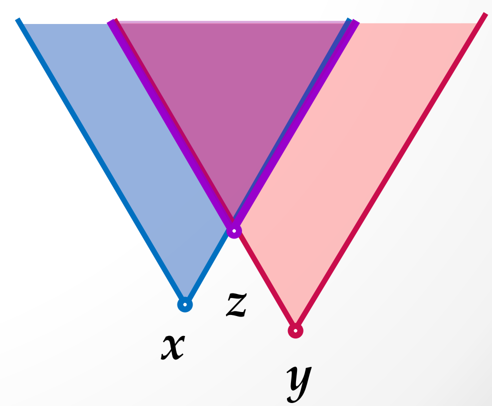


multiplication

$$\overset{+}{\wedge}: \overset{+}{Con}\left(M_1^D\right)\times \overset{+}{Con}\left(M_1^D\right)\to \overset{+}{Con}\left(M_1^D\right)$$

$$\left(\overset{+}{con}_x, \overset{+}{con}_y\right)\mapsto \overset{+}{con}_x \,\overset{+}{\wedge}\, \overset{+}{con}_y \equiv$$

$$\equiv \overset{+}{con}_x \cap \overset{+}{con}_y = \overset{+}{con}_z$$



Addition and multiplication on upper cones. Properties

Notation $\{\overset{+}{Ú}, \overset{+}{Ù}\} = \overset{+}{C}$

- *idempotency*

$$\overset{+}{con}_x \;\overset{+}{C}\; \overset{+}{con}_x \;°\; \overset{+}{con}_x$$

- *commutativity*

$$\overset{+}{con}_x \;\overset{+}{C}\; \overset{+}{con}_y \;°\; \overset{+}{con}_y \;\overset{+}{C}\; \overset{+}{con}_x$$

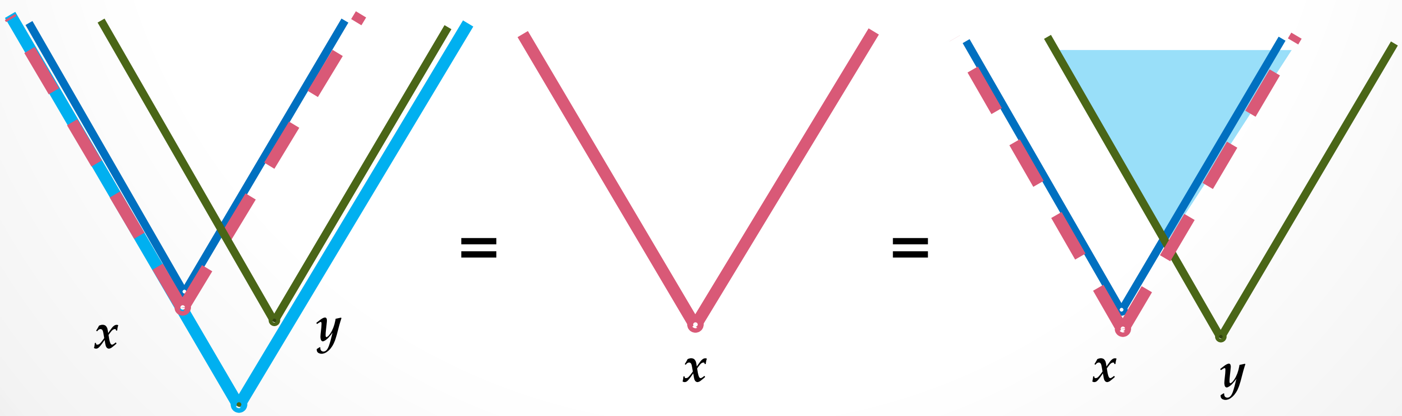
" x, y Î M_1^D

- *associativity*

$$\left(\overset{+}{con}_x \;\overset{+}{C}\; \overset{+}{con}_y\right) \overset{+}{C}\; \overset{+}{con}_z \;°\; \overset{+}{con}_x \;\overset{+}{C} \left(\overset{+}{con}_y \;\overset{+}{C}\; \overset{+}{con}_z\right)$$

- *absorption identity*

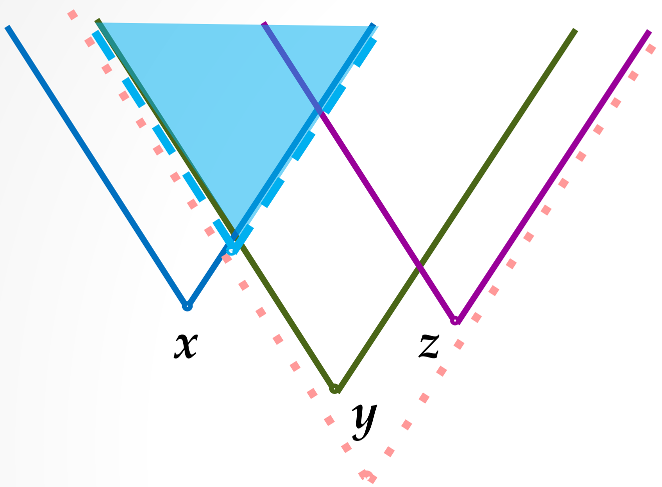
$$\overset{+}{con}_x \;\overset{+}{Ú} \left(\overset{+}{con}_x \;\overset{+}{Ù}\; \overset{+}{con}_y\right) °\; \overset{+}{con}_x \;°\; \overset{+}{con}_x \;\overset{+}{Ù} \left(\overset{+}{con}_x \;\overset{+}{Ú}\; \overset{+}{con}_y\right)$$



Distributivity of the obtained lattice

$\left(Con^{+}(M_1^D), \wedge, \vee\right)$

$$con_x^{+} \wedge \left(con_y^{+} \vee con_z^{+}\right) = \left(con_y^{+} \wedge con_z^{+}\right) \vee \left(con_y^{+} \wedge con_z^{+}\right)$$



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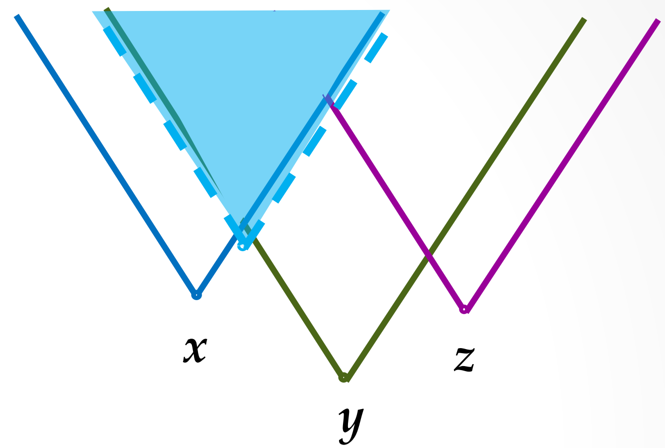
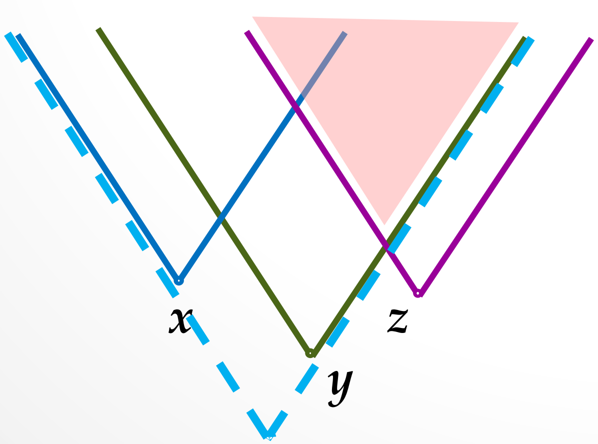


$$con_x^{+} \vee \left(con_y^{+} \wedge con_z^{+}\right) = \left(con_y^{+} \vee con_z^{+}\right) \wedge \left(con_y^{+} \vee con_z^{+}\right)$$



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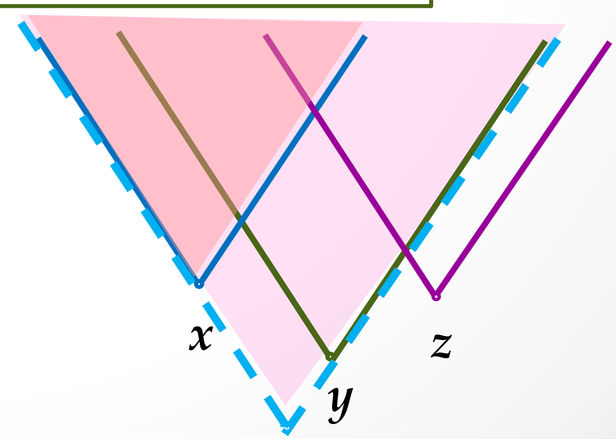


Bijection of distributive lattices $\left(Con\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right)$ and $\left(Con\left(M_1^D\right), \overset{-}{\wedge}, \overset{-}{\vee}\right)$

Bijection T
transforms one cone into another without changing the vertex

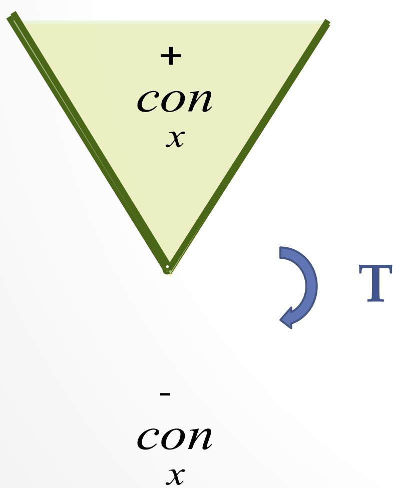
$$T : \left(Con\left(M_1^D\right), \overset{+}{\wedge}, \overset{+}{\vee}\right) \to \left(Con\left(M_1^D\right), \overset{-}{\vee}, \overset{-}{\wedge}\right)$$

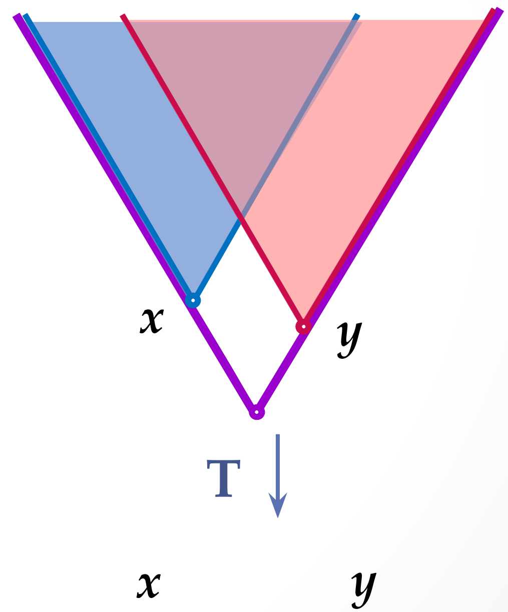
$$\overset{+}{con}_x \mapsto T\left(\overset{+}{con}_x\right) \equiv \overset{-}{con}_x$$

$$T\left(\overset{+}{con}_x \overset{+}{\vee} \overset{+}{con}_y\right) = T\left(\overset{+}{con}_x\right) \overset{-}{\wedge} T\left(\overset{+}{con}_y\right)$$

$$T\left(\overset{+}{con}_x \overset{+}{\wedge} \overset{+}{con}_y\right) = T\left(\overset{+}{con}_x\right) \overset{-}{\vee} T\left(\overset{+}{con}_y\right)$$

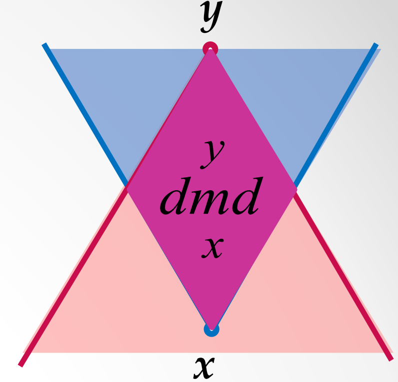
$$\overset{+}{con}_x \subseteq \overset{+}{con}_y \Rightarrow T\left(\overset{+}{con}_x\right) \supseteq T\left(\overset{+}{con}_y\right)$$





Operations on diamonds

$$Dmd\left(M_1^D\right) \equiv \left\{ \begin{array}{l} \overset{y}{\underset{x}{dmd}} = \overset{+}{\underset{x}{con}} \cap \overset{-}{\underset{y}{con}} = \overset{+}{\underset{x}{con}} \cap T\left(\overset{+}{\underset{y}{con}}\right) = \\ = T^{-1}\left(\overset{-}{\underset{x}{con}}\right) \cap \overset{-}{\underset{y}{con}} \end{array} \middle| \; x, y \in M_1^D \right\}$$

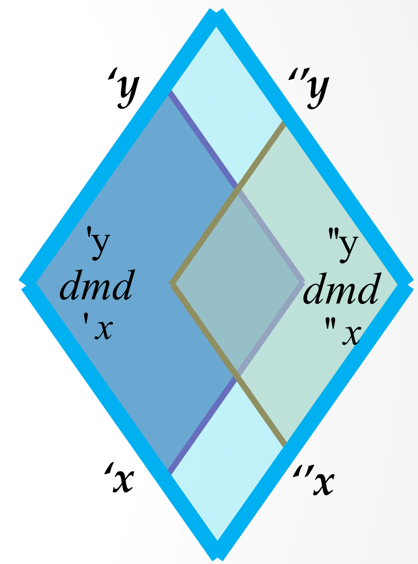


addition

$$\vee : Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$

$$\left(\overset{'y}{\underset{'x}{dmd}}, \overset{''y}{\underset{''x}{dmd}}\right) \mapsto \overset{'y}{\underset{'x}{dmd}} \vee \overset{''y}{\underset{''x}{dmd}} \equiv$$

$$\equiv \left(\overset{+}{\underset{'x}{con}} \vee \overset{+}{\underset{''x}{con}}\right) \cap \left(\overset{-}{\underset{'y}{con}} \vee \overset{-}{\underset{''y}{con}}\right)$$

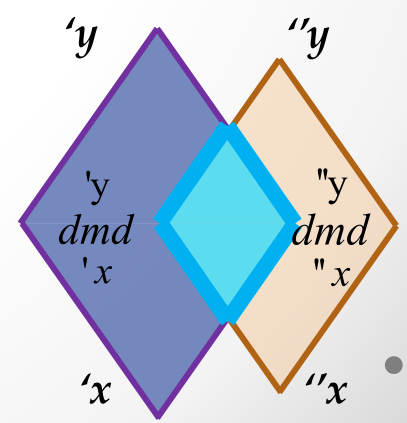


multiplication

$$\wedge : Dmd\left(M_1^D\right) \times Dmd\left(M_1^D\right) \to Dmd\left(M_1^D\right)$$

$$\left(\overset{'y}{\underset{'x}{dmd}}, \overset{''y}{\underset{''x}{dmd}}\right) \mapsto \overset{'y}{\underset{'x}{dmd}} \wedge \overset{''y}{\underset{''x}{dmd}} \equiv$$

$$\equiv \left(\overset{+}{\underset{'x}{con}} \wedge \overset{+}{\underset{''x}{con}}\right) \cap \left(\overset{-}{\underset{'y}{con}} \wedge \overset{-}{\underset{''y}{con}}\right)$$



Addition and multiplication on diamonds. Properties

Notation $\{\vee, \wedge\} = C$

- *idempotency*

$$\underset{x}{\overset{y}{dmd}} \; C \; \underset{x}{\overset{y}{dmd}} \equiv \underset{x}{\overset{y}{dmd}}$$

- *commutativity*

$$\underset{'x}{\overset{'y}{dmd}} \; C \; \underset{''x}{\overset{''y}{dmd}} \equiv \underset{''x}{\overset{''y}{dmd}} \; C \; \underset{'x}{\overset{'y}{dmd}}$$

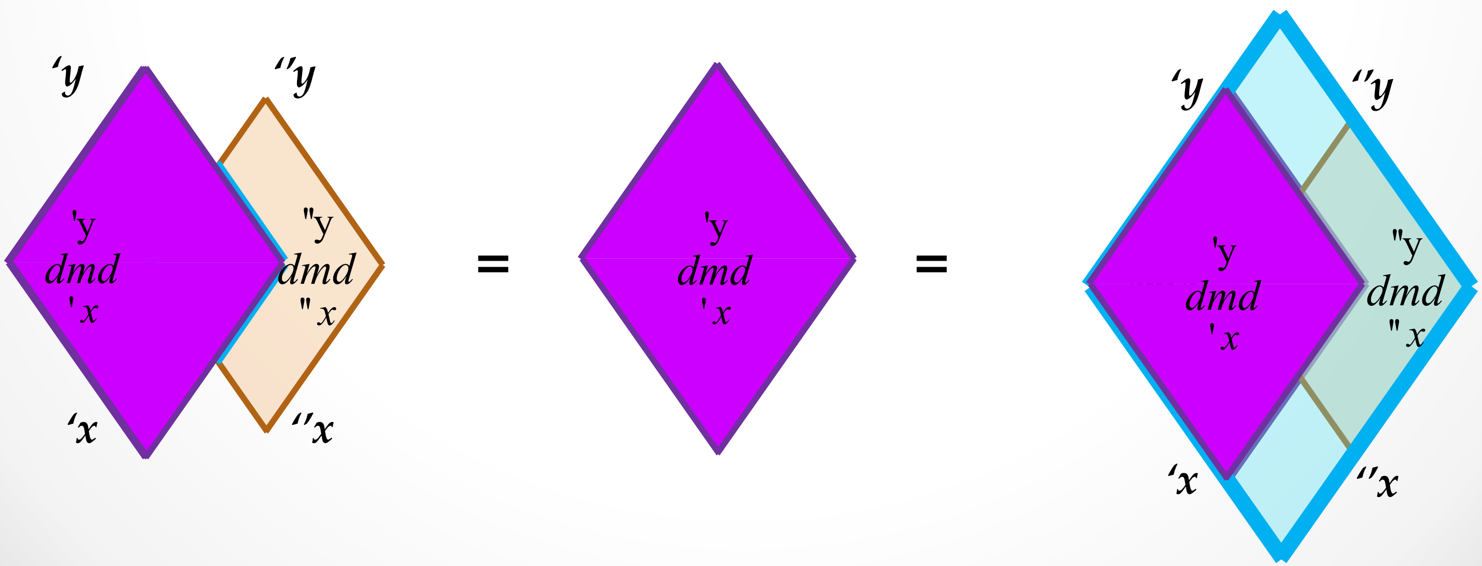
$$\forall \; x, y \in M_1^D$$

- *associativity*

$$\left(\underset{'x}{\overset{'y}{dmd}} \; C \; \underset{''x}{\overset{''y}{dmd}} \right) C \; \underset{'''x}{\overset{'''y}{dmd}} \equiv \underset{'x}{\overset{'y}{dmd}} \; C \left(\underset{''x}{\overset{''y}{dmd}} \; C \; \underset{'''x}{\overset{'''y}{dmd}} \right)$$

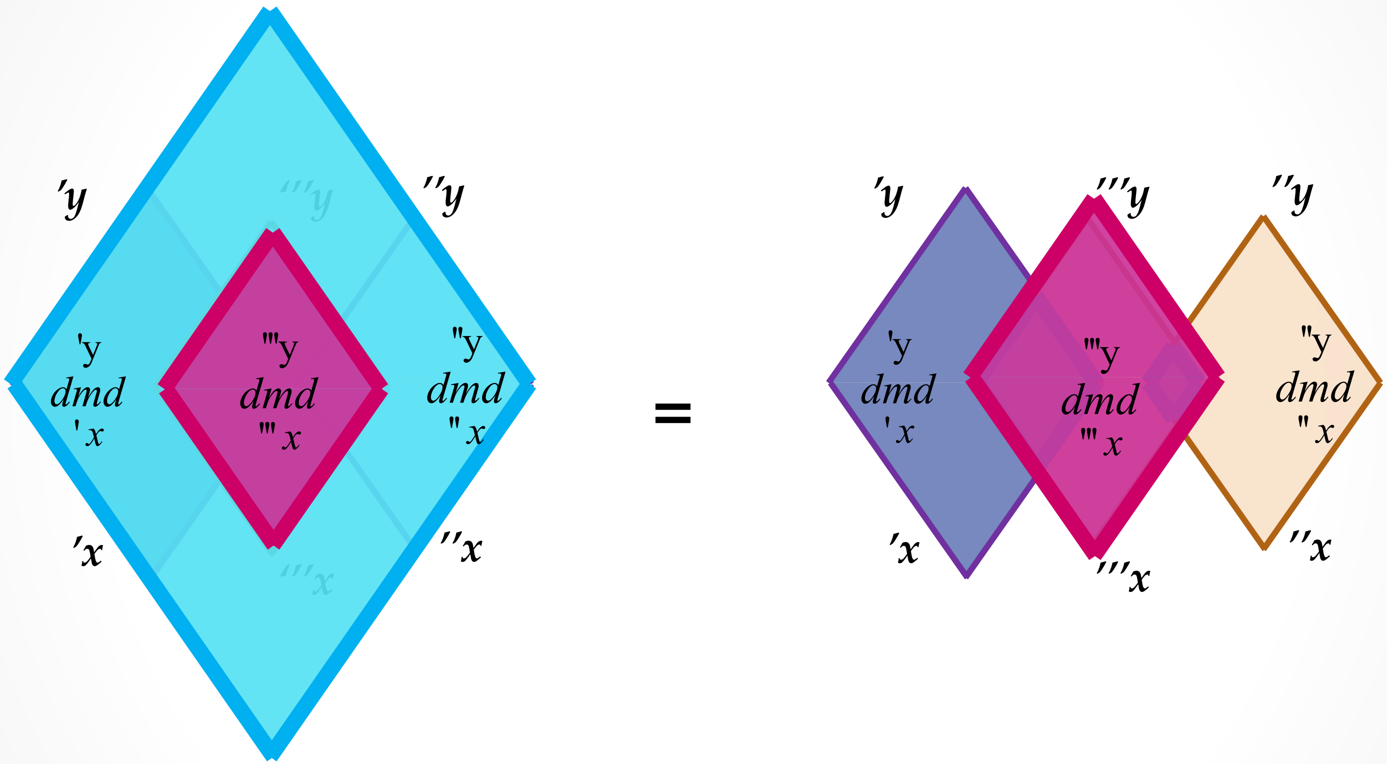
- *absorption identity*

$$\underset{'x}{\overset{'y}{dmd}} \vee \left(\underset{'x}{\overset{'y}{dmd}} \wedge \underset{''x}{\overset{''y}{dmd}} \right) \equiv \underset{'x}{\overset{'y}{dmd}} \equiv \underset{'x}{\overset{'y}{dmd}} \wedge \left(\underset{'x}{\overset{'y}{dmd}} \vee \underset{''x}{\overset{''y}{dmd}} \right)$$



Distributivity of the obtained lattice $\left(Dmd\left(M_1^D\right), \wedge, \vee\right)$

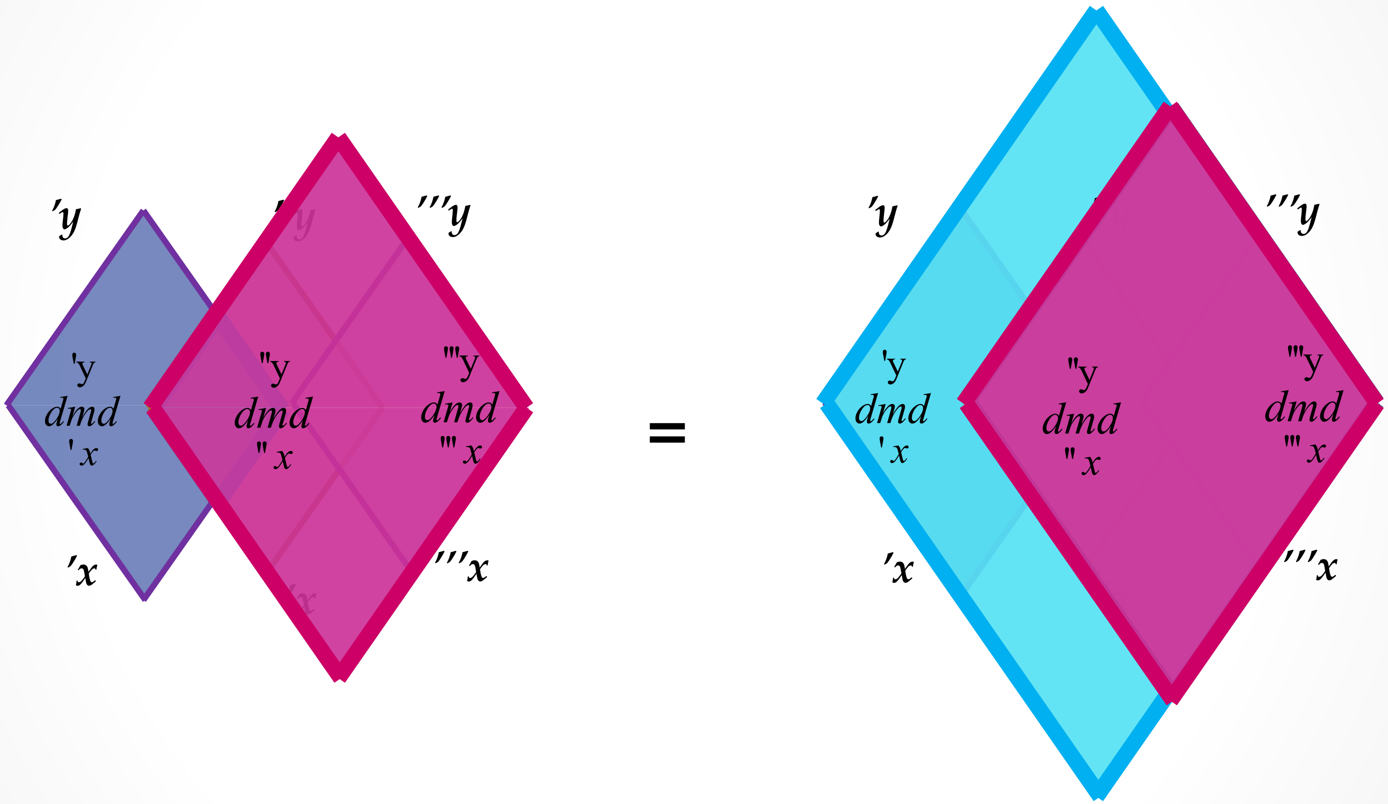
$$\left(dmd_{x'}^{y'} \vee dmd_{x''}^{y''}\right) \wedge dmd_{x'''}^{y'''} = \left(dmd_{x'}^{y'} \wedge dmd_{x'''}^{y'''}\right) \vee \left(dmd_{x''}^{y''} \wedge dmd_{x'''}^{y'''}\right)$$



The ***distributive identities*** *for the introduced operations:*
addition *= "**union**" and* ***multiplication****=" **intersection**"*
hold only when the involved ***intersections of diamonds are nonempty***

Distributivity of the obtained lattice $\left(Dmd\left(M_1^D\right), \wedge, \vee\right)$

$$\left(dmd_{'x}^{'y} \wedge dmd_{''x}^{''y}\right) \vee dmd_{'''x}^{'''y} = \left(dmd_{'x}^{'y} \vee dmd_{'''x}^{'''y}\right) \wedge \left(dmd_{''x}^{''y} \vee dmd_{'''x}^{'''y}\right)$$



*The **distributive identities** for the introduced operations:*
***addition** = "**union**" and **multiplication**="**intersection**"*
*hold only when the involved **intersections of diamonds are nonempty***

Thank you
for your attention!