

Algebraic constructions generated by causal structure of space-times

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Algebraic Quantum Field Theory (AQFT). Tools



C*-algebras

C*-algebra is an involutive Banach algebra \mathfrak{A} satisfying $\|A^*A\| = \|A\|^2 \forall A \in \mathfrak{A}$.

***-homomorphism**

$$\rho: \mathbb{C}^* \rightarrow \mathbb{C}^*$$

$$\rho(A \otimes B) = \rho(A) \otimes \rho(B),$$

$$\bullet \rho(\lambda(A_1 + A_2)) = \lambda \rho(A_1) + \rho(A_2)$$

$$\bullet \rho(A_1 A_2) = \rho(A_2) \cdot \rho(A_1)$$

$$\bullet \rho(A^*) = (\rho(A))^*$$

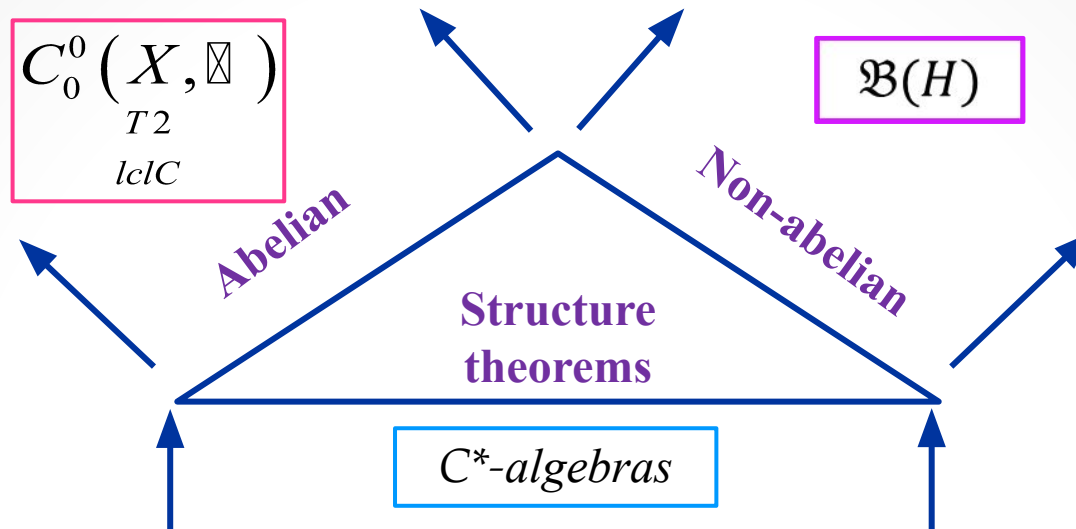
• *is \mathbb{C} -linear*

• *respects multiplication*

• *respects involution*

Representation of a C*-algebra $\pi: \mathbb{C}^* \rightarrow \mathfrak{B}(\mathcal{H})$ linear bounded operators on a \mathbb{C} -Hilbert space \mathcal{H}
**-homomorphism*

Structure theorems for C^* - algebras



Gelfand-Naimark Theorem

$$HTop \simeq C^*Alg \begin{matrix} \mathbb{K} \\ lclC \quad abl \end{matrix}$$

Trivial examples of C^* -algebras

abelian \mathbb{K} $Mat(n, \mathbb{K})$ non-abelian

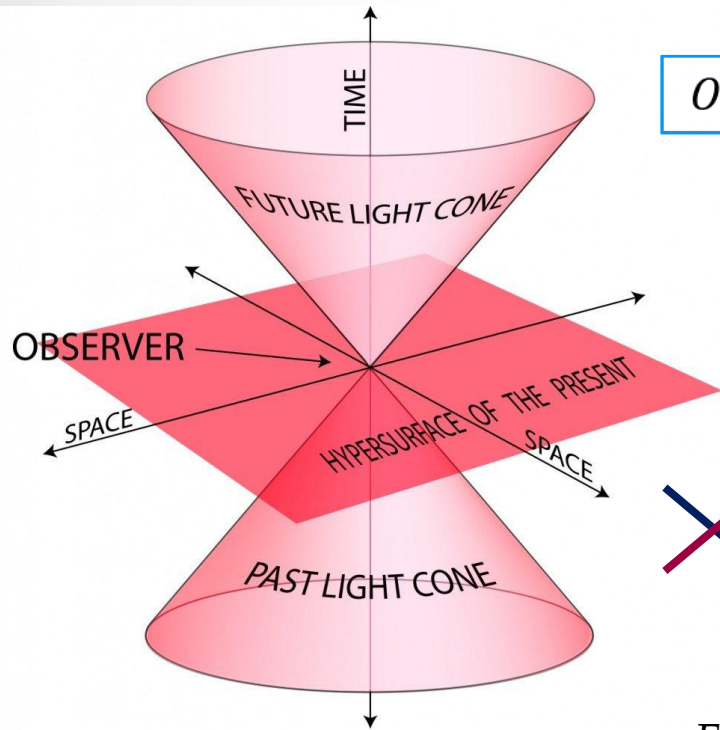
Algebraic Quantum Field Theory (AQFT)

Net of observable algebras $\{\mathfrak{A}(O): O \in K\}$ ^{1,2}

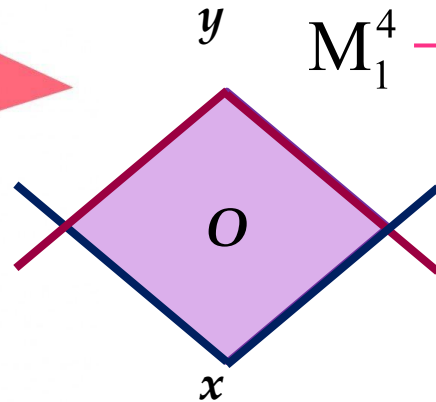
$$\overline{V}_+ \equiv \left\{ x \in M_1^4 \mid x \cdot x \in \mathbb{R}_{\geq 0}, x^0 \in \mathbb{R}_{> 0} \right\}$$

Haag-Kastler axioms

- *Isotony*
- *Causality*
- *Covariance*
- *Time slice axiom*
- *Spectrum condition*



$$O = I_+(x) \cap I_-(y)$$



Finite contractible region

$$M_1^4 \xrightarrow{\text{GNS-theorem}} \mathfrak{A} \xrightarrow{\text{GNS-theorem}} \pi(\mathfrak{A})$$

$\mathfrak{A}(O)$

$\pi(\mathfrak{A}(O))$

Abstract C*-algebra

Operator algebra on \mathcal{H}

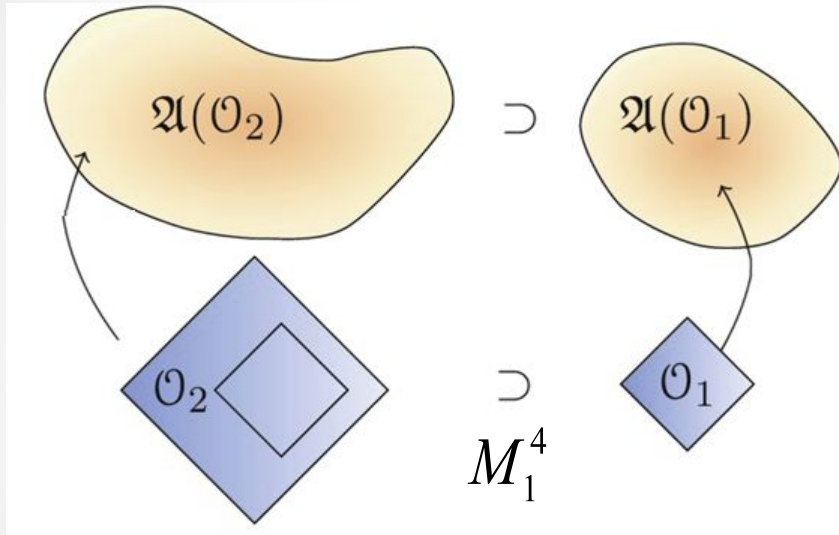
1 Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. 5(7), 848–861 (1964)

2 Araki, H.: Mathematical Theory of Quantum Fields, vol. 101. Oxford University Press, Oxford (1999)

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

Isotony



- For $''U \subseteq 'U \exists$ an **embedding**

$$in_{\mathcal{U}}^{\mathcal{U}'} : \mathfrak{A}(''U) \rightarrow \mathfrak{A}('U)$$

- For $'''U \subseteq ''U \subseteq 'U$, the embeddings satisfy **the consistency conditions**

$$in_{\mathcal{U}}^{\mathcal{U}''} \boxtimes in_{\mathcal{U}''}^{\mathcal{U}'} = in_{\mathcal{U}}^{\mathcal{U}'}$$

$$\mathfrak{A}('''U) \xrightarrow{in_{\mathcal{U}}^{\mathcal{U}''}} \mathfrak{A}(''U) \xrightarrow{in_{\mathcal{U}''}^{\mathcal{U}'}} \mathfrak{A}('U)$$

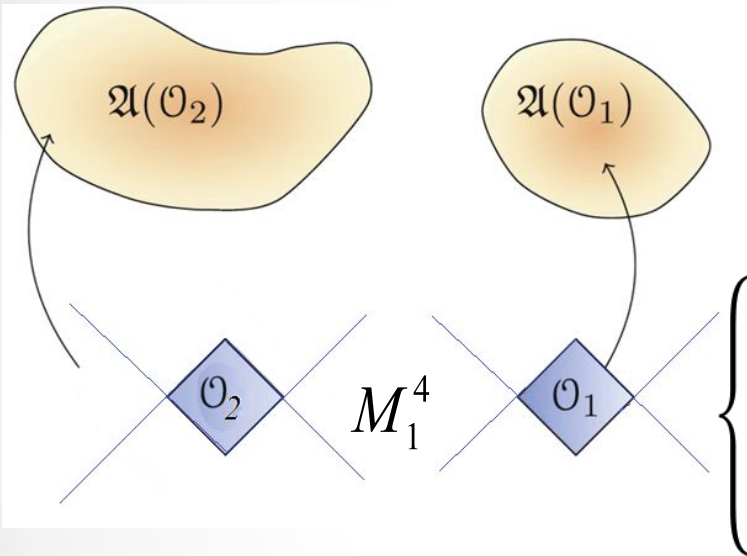
The diagram shows a sequence of embeddings: $\mathfrak{A}('''U) \xrightarrow{in_{\mathcal{U}}^{\mathcal{U}''}} \mathfrak{A}(''U) \xrightarrow{in_{\mathcal{U}''}^{\mathcal{U}'}} \mathfrak{A}('U)$. A blue curved arrow above the sequence indicates that the composition of the two embeddings is equal to the direct embedding from $\mathfrak{A}('''U)$ to $\mathfrak{A}('U)$.

$$O_1 \subseteq O_2 \Rightarrow \exists \alpha_{12}: \mathfrak{A}(O_1) \rightarrow \mathfrak{A}(O_2)$$

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

*Microcausality
(locality)*



$$[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = 0$$

O_1, O_2 – spatially separated diamonds

Connected component of the Poincare group identity Π_+^\uparrow , is represented by automorphisms $\rho_{(\Lambda, a)}$:

$$\mathfrak{A} \rightarrow \mathfrak{A}$$

$$A \mapsto \rho_{(\Lambda, a)}(A)$$

such that

$$\rho_{(\Lambda, a)} \boxtimes in_{\mathcal{U}}^{\mathcal{U}} = in_{(\Lambda, a)\mathcal{U}}^{(\Lambda, a)\mathcal{U}} \boxtimes \rho_{(\Lambda, a)}$$

$$\rho_{(\Lambda, a)} \boxtimes \rho_{(\Lambda, a)} = \rho_{(\Lambda, a) \boxtimes (\Lambda, a)}$$

$$\forall A \in \mathfrak{A}(\mathcal{U})$$

$$\mathcal{U}' \subseteq \mathcal{U}$$

Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

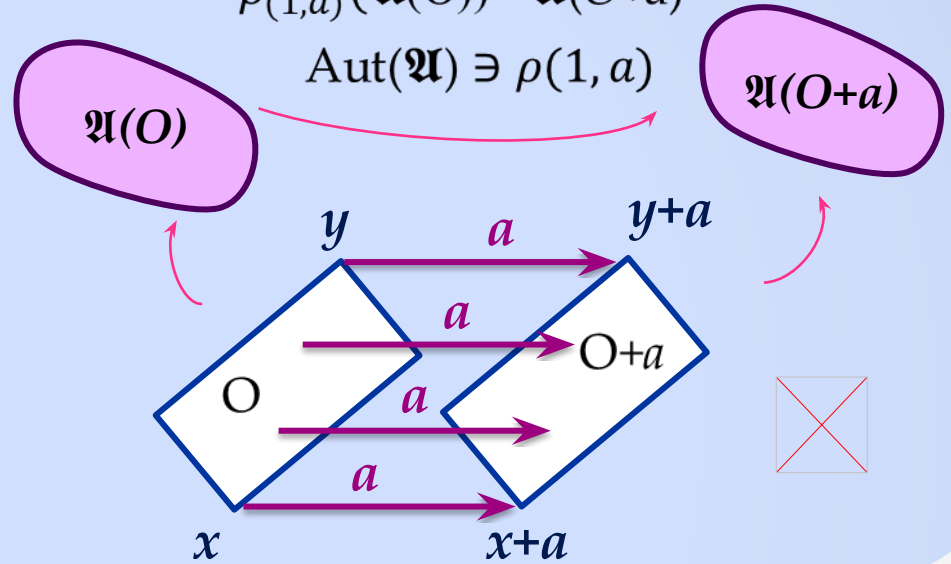
• Covariance

- $\rho_{(\Lambda, a)}(\mathfrak{A}(O)) = \mathfrak{A}(\Lambda O + a)$
 $\forall (\Lambda, a) \in \Pi_+^\uparrow$
- $\rho_{(\Lambda, a)} \in \text{Aut}(\mathfrak{A})$

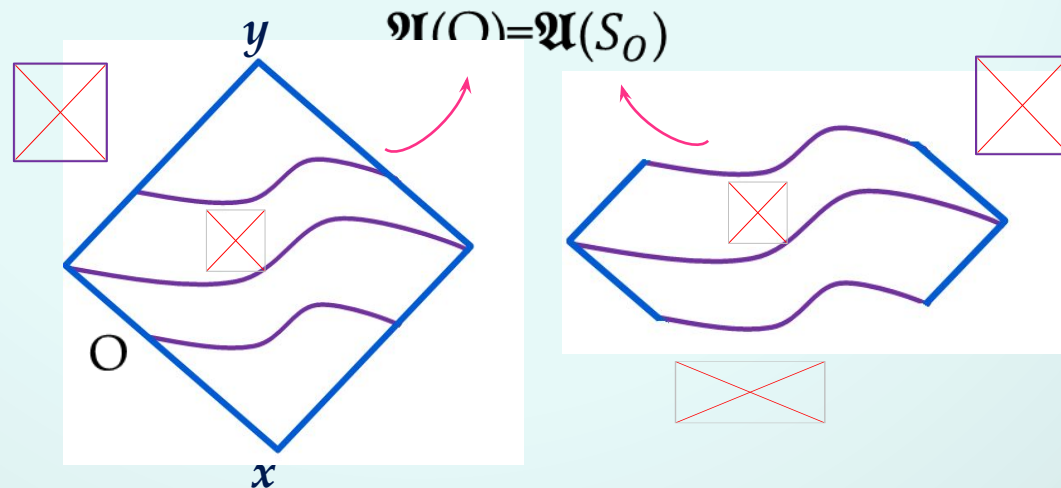
• Translational covariance

$$\rho_{(1, a)}(\mathfrak{A}(O)) = \mathfrak{A}(O + a)$$

$$\text{Aut}(\mathfrak{A}) \ni \rho(1, a)$$



• Time slice axiom



Algebraic Quantum Field Theory (AQFT)

Haag-Kastler axioms

- *Spectrum condition*

- $\text{spec}(P_\beta) \subset \overline{V}_+$
- P_β – generators of translations

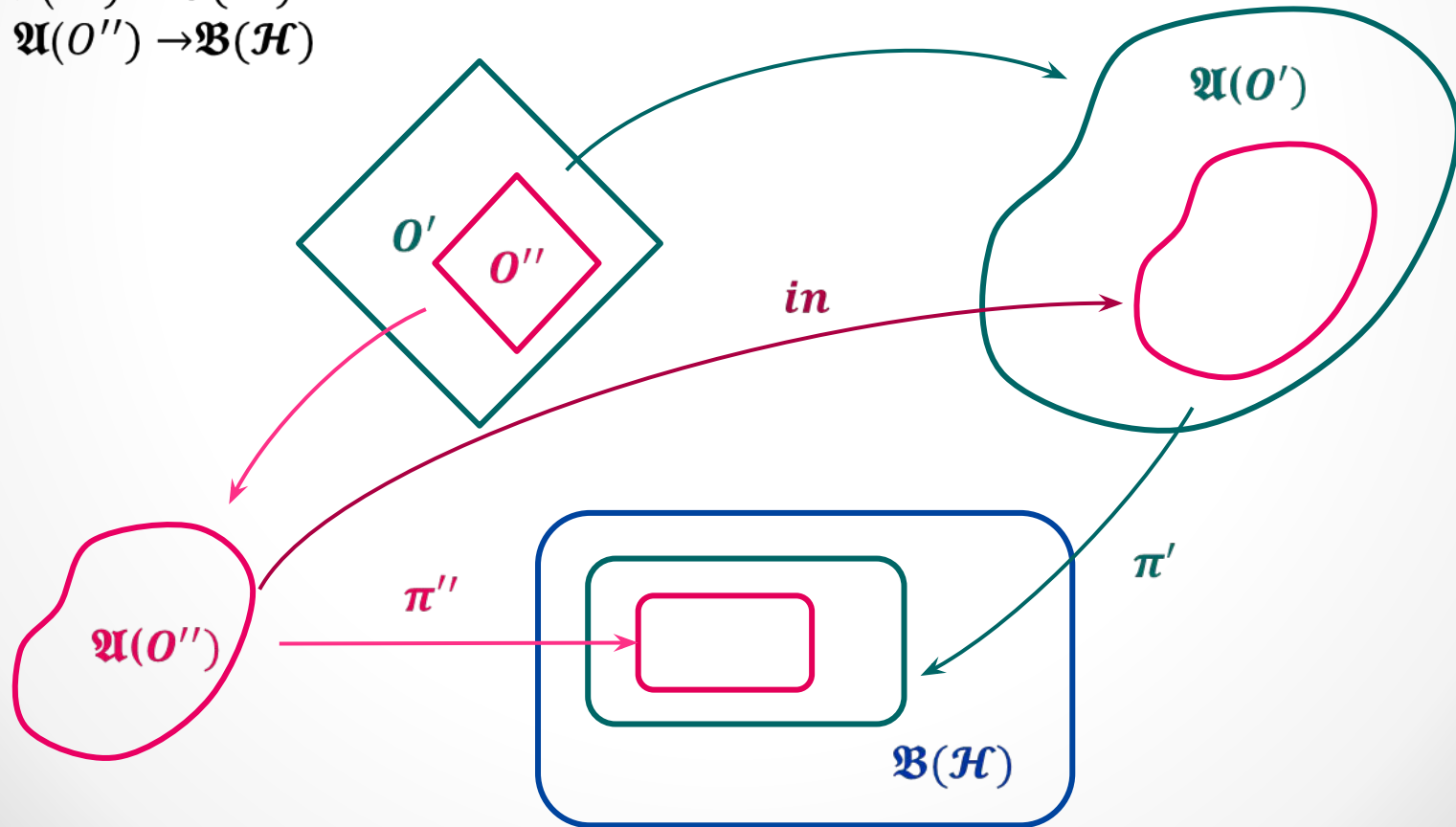
Consistent family of representations

$$O' \supseteq O''$$

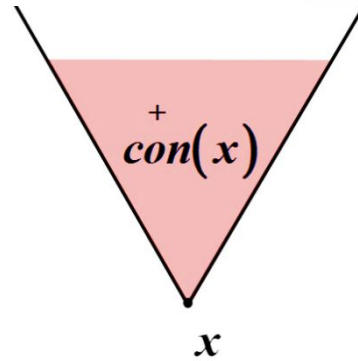
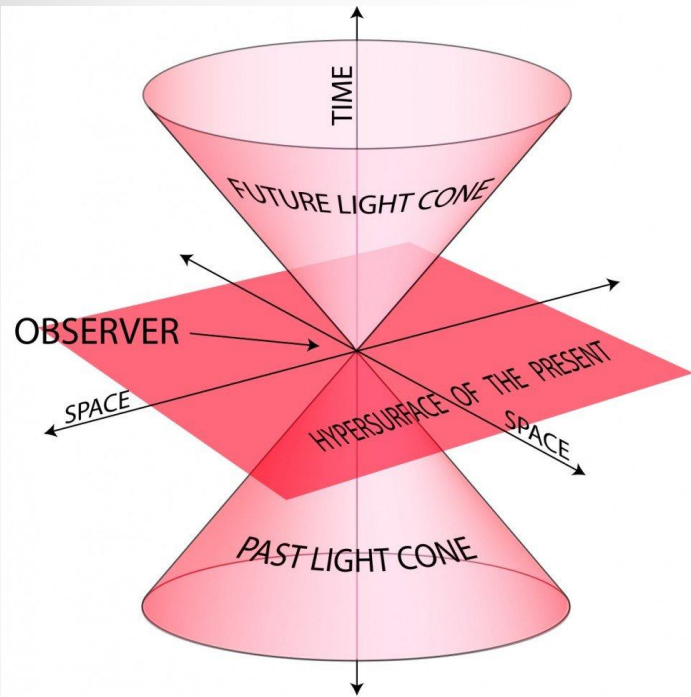
$$\pi': \mathfrak{A}(O') \rightarrow \mathfrak{B}(\mathcal{H})$$

$$\pi'': \mathfrak{A}(O'') \rightarrow \mathfrak{B}(\mathcal{H})$$

$$\pi' \circ \text{in} = \pi''$$

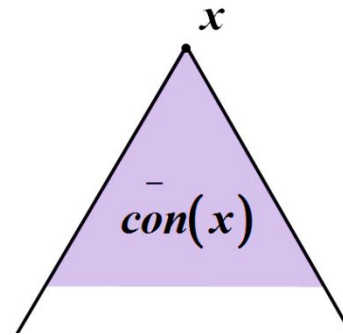


Minkowski space-time



*the family of
upper cones*

$${}^+ \text{Con}(M_1^D) \circ \left[\begin{array}{c} \mathcal{E} \\ \vdots \\ \vdots \end{array} \right] \Big|_x \hat{=} M_1^D, x \times x^3 \ 0, x^0 \ 3 \ 0 \Big|_b$$



*the family of
lower cones*

$${}^- \text{Con}(M_1^D) \circ \left[\begin{array}{c} \mathcal{E} \\ \vdots \\ \vdots \end{array} \right] \Big|_x \hat{=} M_1^D, x \times x^3 \ 0, x^0 \ \mathcal{E} \ 0 \Big|_b$$

$$x \times x^0 \ h_{ab} x^a x^b$$

$$a, b = \overline{0, D-1}$$

$$(h_{ab}) \circ \text{diag}(+, -, -, \dots, -)$$

Operations on upper cones

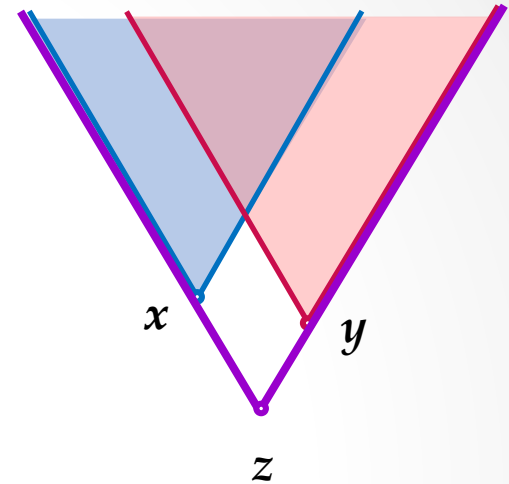
addition

$$\dot{U}: \text{Con}(M_1^D) \oplus \text{Con}(M_1^D) \oplus \text{Con}(M_1^D)$$

$$\text{con}_x^+, \text{con}_y^+ \oplus \text{con}_x^+ \dot{U} \text{con}_y^+$$

smallest upper cone,

$$\text{con}_x^+ \text{ and } \text{con}_y^+ = \text{con}_z^+$$

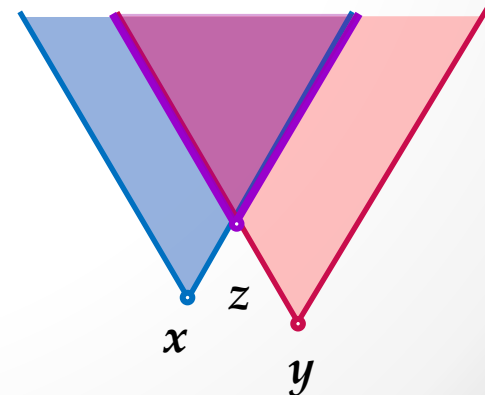


multiplication

$$\dot{U}: \text{Con}(M_1^D) \oplus \text{Con}(M_1^D) \oplus \text{Con}(M_1^D)$$

$$\text{con}_x^+, \text{con}_y^+ \dot{U} \text{con}_x^+ \text{con}_y^+$$

$$\text{con}_x^+ \mid \text{con}_y^+ = \text{con}_z^+$$



Addition and multiplication on upper cones. Properties

- *idempotency*

$$\begin{array}{cccc} + & + & + & + \\ \text{con } C & \text{con } C & \text{con } \circ & \text{con } C \\ x & & x & x \end{array}$$

Notation $\left\{ \overset{+}{U}, \overset{+}{U} \right\} = \overset{+}{C}$

- *commutativity*

$$\begin{array}{cccccc} + & + & + & + & + & + \\ \text{con } C & \text{con } C & \text{con } \circ & \text{con } C & \text{con } C & \text{con } C \\ x & & y & y & & x \end{array}$$

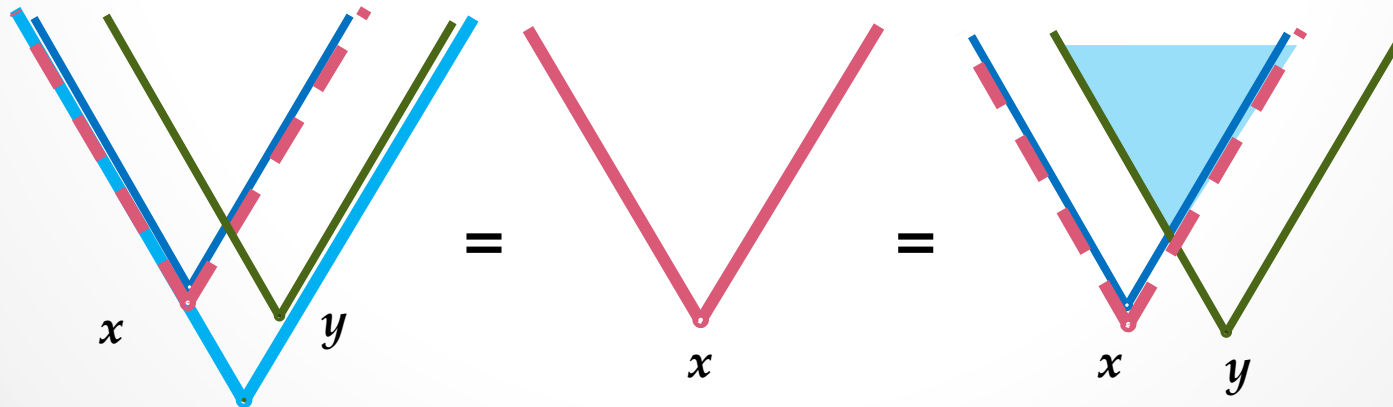
" $x, y \hat{=} M_1^D$

- *associativity*

$$\begin{array}{ccccccccc} \text{con } C & + & + & \text{con } \circ & + & + & \text{con } \circ & + & + & \text{con } C & + & + & \text{con } \circ \\ x & & y & & z & & x & & y & & z & & \end{array}$$

- *absorption identity*

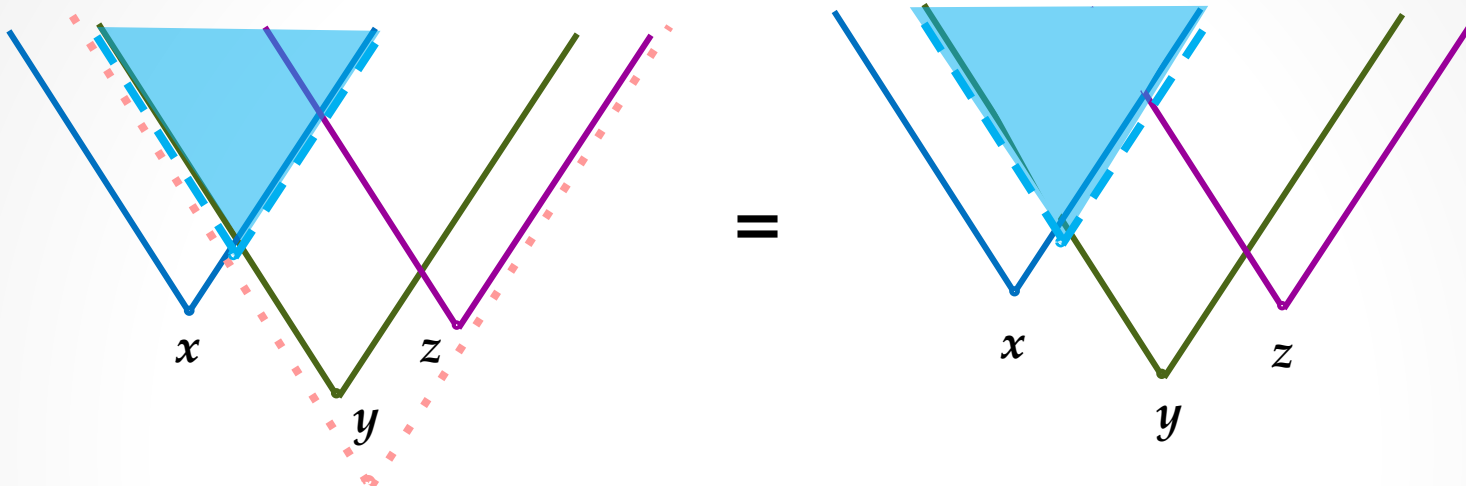
$$\begin{array}{ccccccccccc} + & + & \text{con } C & + & + & \text{con } \circ & + & + & \text{con } C & + & + & \text{con } \circ \\ \text{con } U & \text{con } U & \text{con } U & \text{con } \circ & \text{con } \circ & \text{con } \circ & \text{con } U & \text{con } U & \text{con } U & \text{con } \circ & \text{con } \circ & \text{con } \circ \\ x & & x & & y & & x & & x & & y & & \end{array}$$



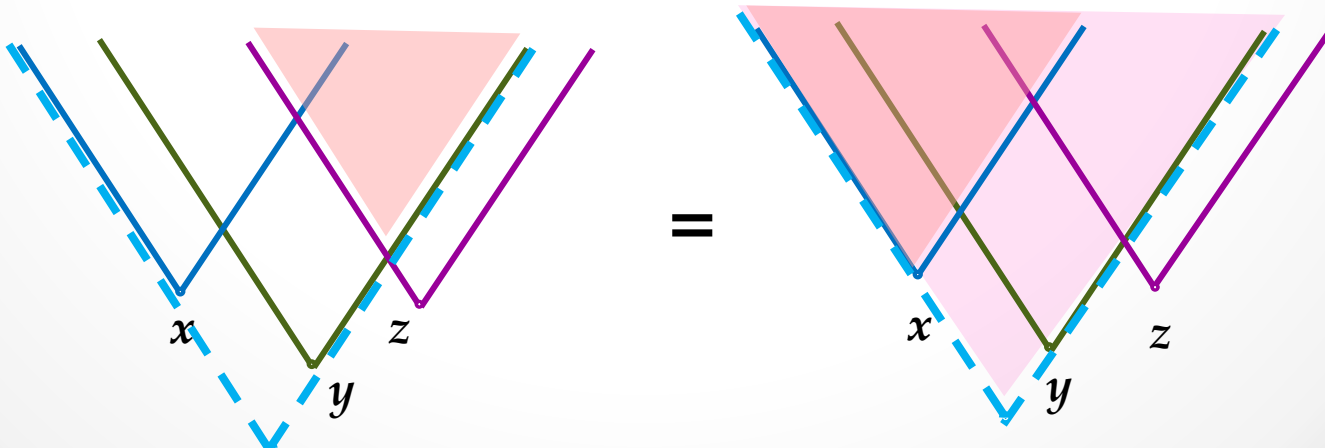
Distributivity of the obtained lattice

$\text{Con}^+(M_1^D), \dot{U}, \dot{U}^{\circ}$

$$\begin{matrix} + & + & \text{Con}^+ & + & + & \dot{U} & = & \text{Con}^+ & + & + & \dot{U} & \text{Con}^+ & + & + & \dot{U} \\ \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} & & \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} \\ x & & y & & z & & & y & & z & & y & & z & \end{matrix}$$



$$\begin{matrix} + & + & \text{Con}^+ & + & + & \dot{U} & = & \text{Con}^+ & + & + & \dot{U} & \text{Con}^+ & + & + & \dot{U} \\ \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} & & \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} & \text{con} & \dot{U} \\ x & & y & & z & & & y & & z & & y & & z & \end{matrix}$$



Bijection of distributive lattices

$$\text{Con}^+(M_1^D), \dot{U}, \dot{U}^+$$

and

$$\text{Con}^-(M_1^D), \dot{U}, \dot{U}^-$$

Bijection T

transforms one cone into another without changing the vertex

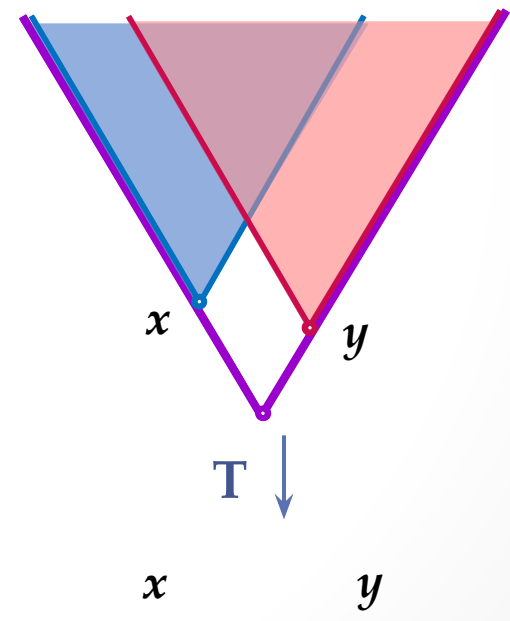
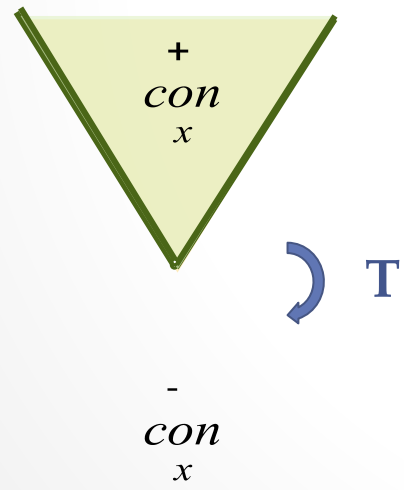
$$T : \text{Con}^+(M_1^D), \dot{U}, \dot{U}^+ \rightarrow \text{Con}^-(M_1^D), \dot{U}, \dot{U}^-$$

$$\text{con}_x^+ \xrightarrow{T} \text{con}_x^-$$

$$T \text{con}_x^+ \dot{U} \text{con}_y^+ = T \text{con}_x^+ \dot{U} T \text{con}_y^+$$

$$T \text{con}_x^+ \dot{U} \text{con}_y^+ = T \text{con}_x^+ \dot{U} T \text{con}_y^+$$

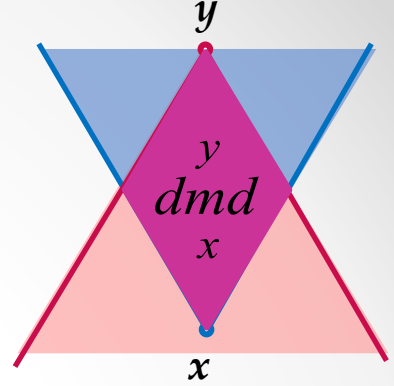
$$\text{con}_x^+ \dot{U} \text{con}_y^+ = T \text{con}_x^+ \dot{U} T \text{con}_y^+$$



Operations on diamonds

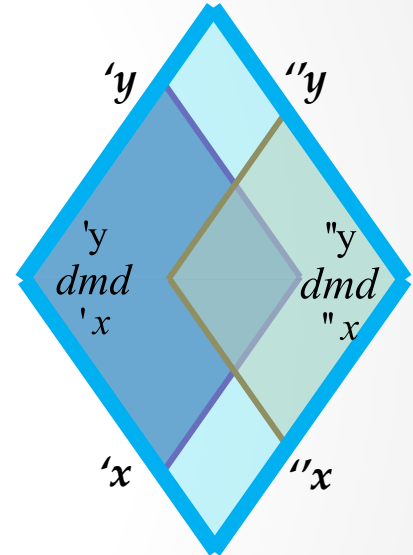
$$Dmd(M_1^D) \circledast = T^{-1} \left(\begin{array}{c|c} \begin{array}{c} y \\ \text{con} \\ x \end{array} & \begin{array}{c} + \\ \text{con} \\ x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} - \\ \text{con} \\ y \end{array} & \begin{array}{c} + \\ \text{con} \\ x \end{array} \end{array} \right) = \left(\begin{array}{c|c} \begin{array}{c} y \\ \text{con} \\ x \end{array} & \begin{array}{c} + \\ \text{con} \\ y \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} - \\ \text{con} \\ y \end{array} & \begin{array}{c} + \\ \text{con} \\ x \end{array} \end{array} \right)$$

addition



$$\dot{\cup} : Dmd(M_1^D) \circledast Dmd(M_1^D) \circledast Dmd(M_1^D)$$

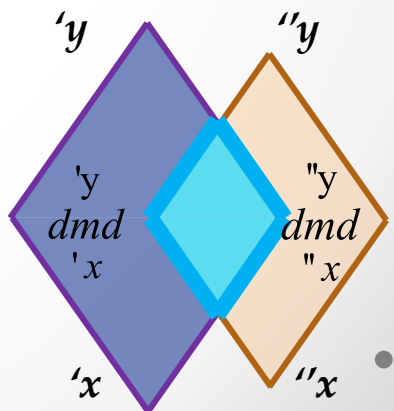
$$\left(\begin{array}{c|c} \begin{array}{c} 'y \\ \text{con} \\ 'x \end{array} & \begin{array}{c} + \\ \text{con} \\ 'x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} ''y \\ \text{con} \\ ''x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} 'y \\ \text{con} \\ 'x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} ''y \\ \text{con} \\ ''x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right)$$



multiplication

$$\dot{\cup} : Dmd(M_1^D) \circledast Dmd(M_1^D) \circledast Dmd(M_1^D)$$

$$\left(\begin{array}{c|c} \begin{array}{c} 'y \\ \text{con} \\ 'x \end{array} & \begin{array}{c} + \\ \text{con} \\ 'x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} ''y \\ \text{con} \\ ''x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} 'y \\ \text{con} \\ 'x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right) \circledast \left(\begin{array}{c|c} \begin{array}{c} ''y \\ \text{con} \\ ''x \end{array} & \begin{array}{c} + \\ \text{con} \\ ''x \end{array} \end{array} \right)$$



Addition and multiplication on diamonds. Properties

- *idempotency*

$$\begin{matrix} y \\ dmd \\ x \end{matrix} \mathbb{C} \begin{matrix} y \\ dmd \\ x \end{matrix} \circ \begin{matrix} y \\ dmd \\ x \end{matrix}$$

Notation $\left\{ \acute{U}, \grave{U} \right\} = \mathbb{C}$

- *commutativity*

$$\begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \mathbb{C} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \circ \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \mathbb{C} \begin{matrix} 'y \\ dmd \\ 'x \end{matrix}$$

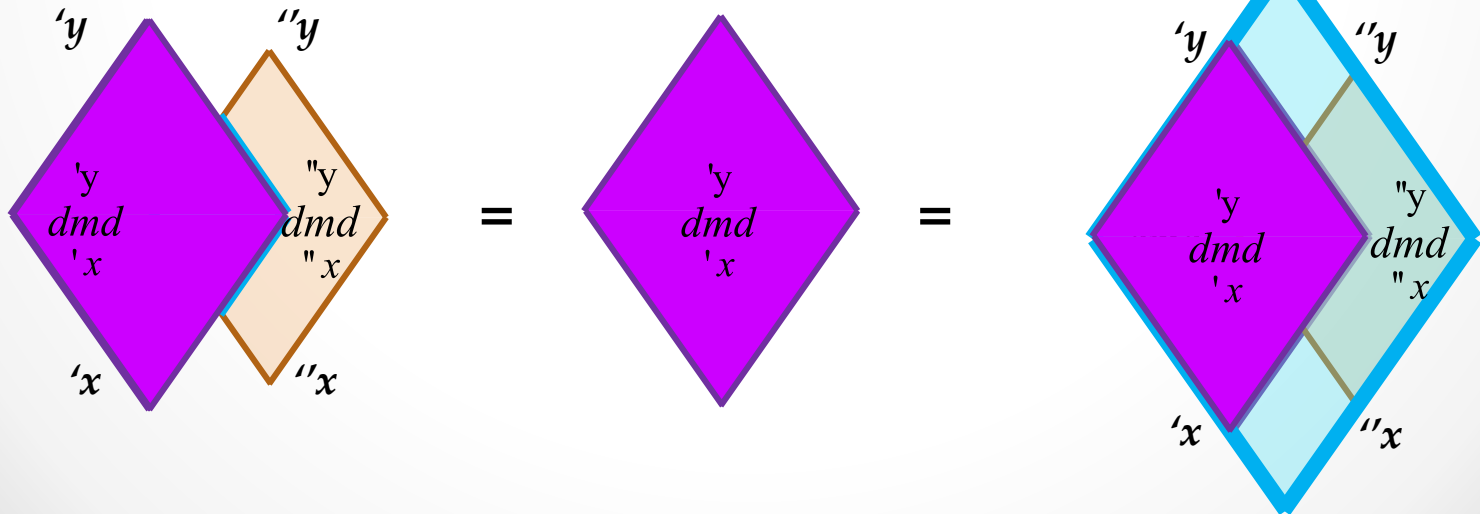
$$''x, y \hat{=} M_1^D$$

- *associativity*

$$\begin{matrix} \text{æ} \\ dmd \\ \text{æ} \end{matrix} \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \mathbb{C} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \circ \begin{matrix} \text{ö} \\ dmd \\ \text{ö} \end{matrix} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \mathbb{C} \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \mathbb{C} \begin{matrix} \text{æ} \\ dmd \\ \text{æ} \end{matrix} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \mathbb{C} \begin{matrix} \text{ö} \\ dmd \\ \text{ö} \end{matrix} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix}$$

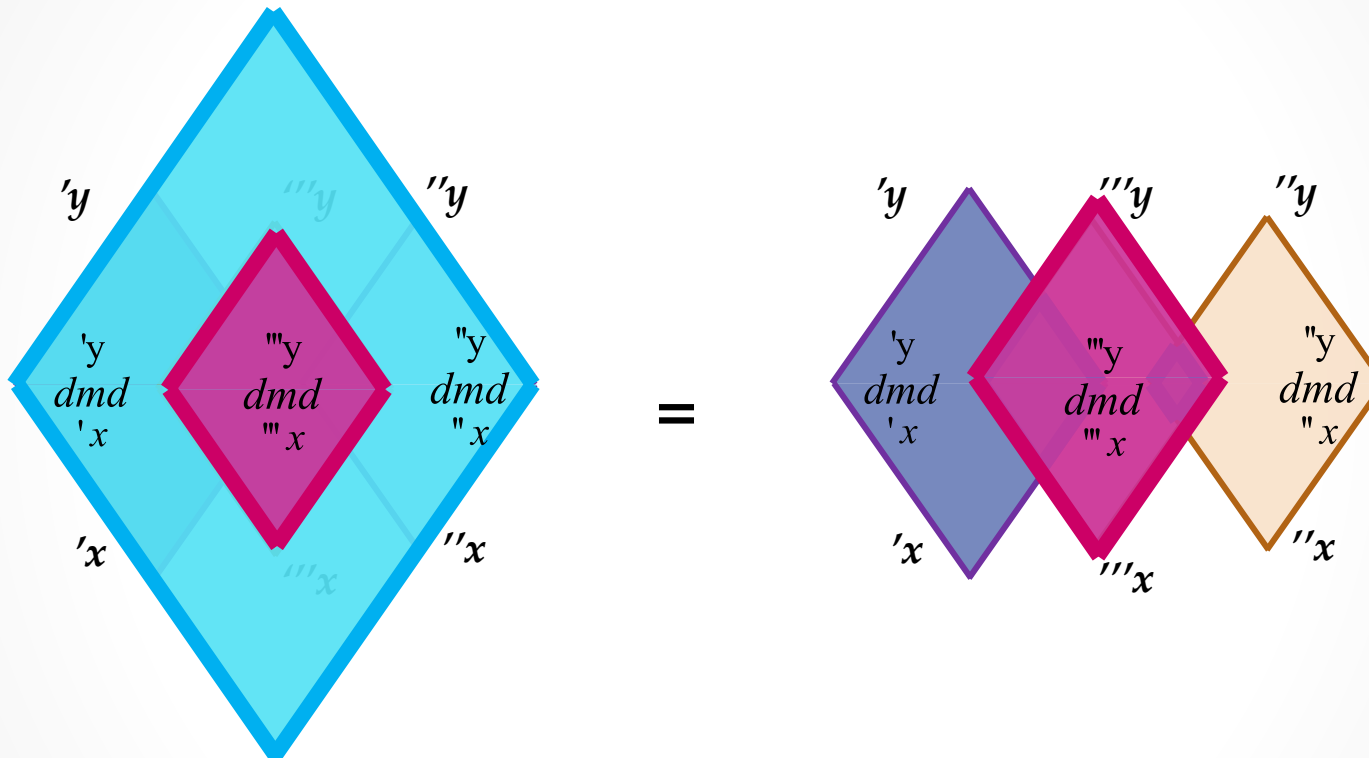
- *absorption identity*

$$\begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \acute{U} \begin{matrix} \text{æ} \\ dmd \\ \text{æ} \end{matrix} \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \grave{U} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \circ \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \circ \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \grave{U} \begin{matrix} \text{æ} \\ dmd \\ \text{æ} \end{matrix} \begin{matrix} 'y \\ dmd \\ 'x \end{matrix} \acute{U} \begin{matrix} ''y \\ dmd \\ ''x \end{matrix} \circ \begin{matrix} 'y \\ dmd \\ 'x \end{matrix}$$



Distributivity of the obtained lattice $(Dmd(M_1^D), \dot{\cup}, \dot{\cup})$

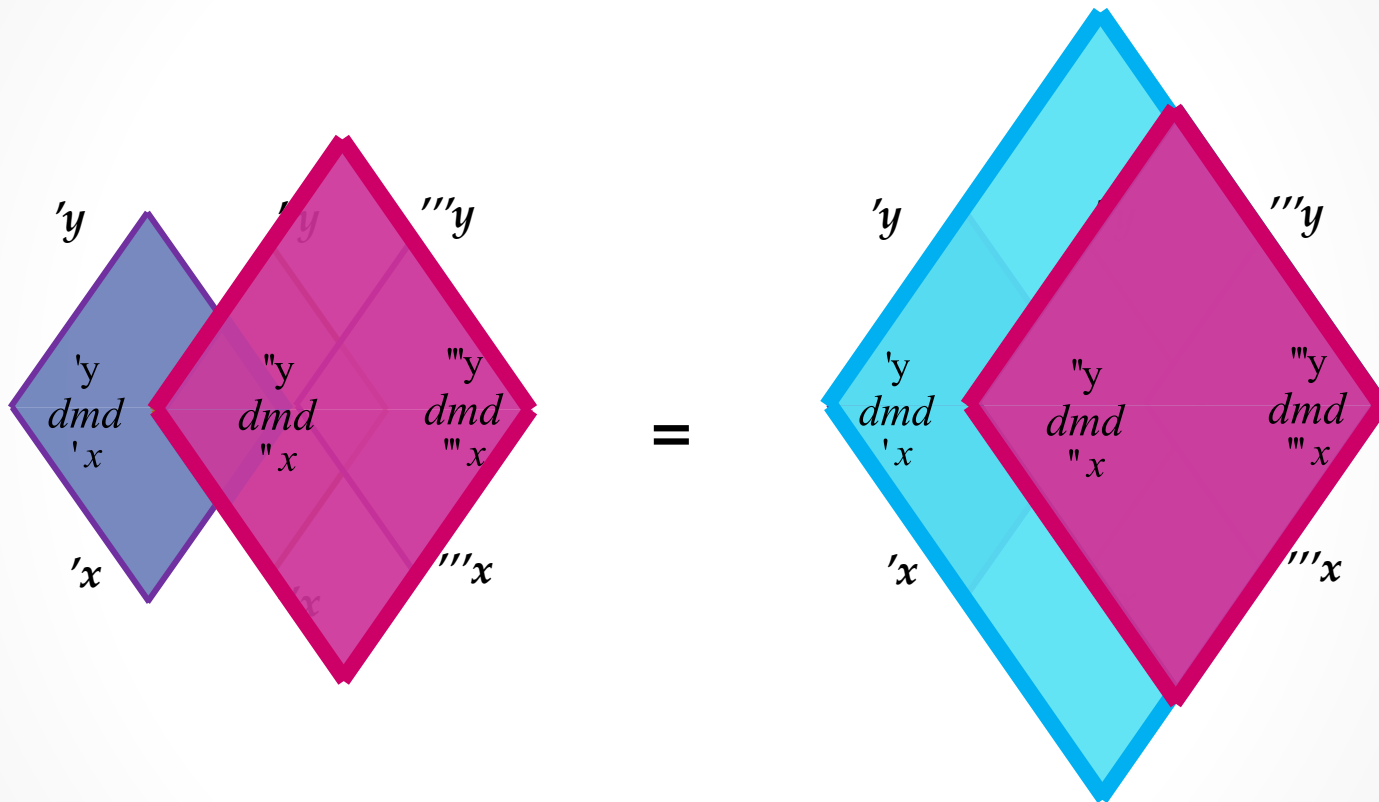
$$\begin{array}{c}
 \text{æ} \text{'y} \quad \text{"y} \quad \ddot{\text{ö}} \quad \text{'y} \\
 \text{æ} \text{'x} \quad \text{"x} \quad \ddot{\text{o}} \quad \text{'x}
 \end{array}
 \dot{\cup}
 \begin{array}{c}
 \text{æ} \text{'y} \quad \text{"y} \quad \ddot{\text{ö}} \\
 \text{æ} \text{'x} \quad \text{"x} \quad \ddot{\text{o}}
 \end{array}
 \dot{\cup}
 \begin{array}{c}
 \text{æ} \text{'y} \quad \text{"y} \quad \ddot{\text{ö}} \\
 \text{æ} \text{'x} \quad \text{"x} \quad \ddot{\text{o}}
 \end{array}
 =
 \begin{array}{c}
 \text{æ} \text{'y} \quad \text{"y} \quad \ddot{\text{ö}} \quad \text{æ} \text{'y} \quad \text{"y} \quad \text{'y} \\
 \text{æ} \text{'x} \quad \text{"x} \quad \ddot{\text{o}} \quad \text{æ} \text{'x} \quad \text{"x} \quad \text{'x}
 \end{array}$$



The distributive identities for the introduced operations:
addition = “*union*” and *multiplication* = “*intersection*”
 hold only when the involved *intersections of diamonds are nonempty*

Distributivity of the obtained lattice $(Dmd(M_1^D), \dot{\cup}, \dot{\cup})$

$$\begin{array}{c}
 \text{æ} \text{ 'y} \text{ "y} \text{ ö} \text{ "'y} \\
 \text{æ} \text{ dmd} \dot{\cup} \text{ dmd} \dot{\cup} \text{ dmd} = \text{æ} \text{ 'y} \text{ "y} \text{ ö} \text{ "'y} \\
 \text{æ} \text{ 'x} \text{ "x} \text{ ø} \text{ "'x} \text{ æ} \text{ 'y} \text{ "y} \text{ ö} \text{ "'y} \\
 \text{æ} \text{ 'x} \text{ "x} \text{ ø} \text{ "'x} \text{ æ} \text{ 'x} \text{ "x} \text{ ø} \text{ "'x}
 \end{array}$$



The distributive identities for the introduced operations:
addition = “**union**” and *multiplication* = “**intersection**”
 hold only when the involved **intersections of diamonds are nonempty**

Thank you
for your attention!