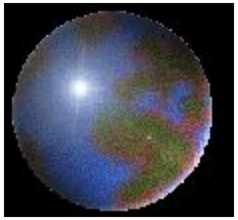
Linear Algebra



Chapter 2

Matrix Algebra

2.1 Addition, Scalar Multiplication, and Multiplication of Matrices

- a_{ij}: the element of matrix A in row i and column j.
- For a square $n\times n$ matrix A, the **main diagonal** is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

Two matrices are **equal** if they are of the same size and if their corresponding elements are equal.

Thus $A = B$ if $a_{ij} = b_{ij}$ $\forall$ i, j.

($\forall$ for every, for all)

Addition of Matrices

Definition

Let A and B be matrices of the same size.

Their **sum** $A + B$ is the matrix obtained by adding together the corresponding elements of A and B.

The matrix $A + B$ will be of the same size as A and B.

If A and B are not of the same size, they cannot be added, and we say that **the sum does not exist**.

Thus if $C = A + B$, then $c_{ij} = a_{ij} + b_{ij} \ \forall i, j$.

Example 1

Let $A=\begin{bmatrix}1 & 4 & 7\\ 0 & -2 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 5 & -6\\ -3 & 1 & 8\end{bmatrix}$, and $C=\begin{bmatrix}-5 & 4\\ 2 & 7\end{bmatrix}$.

Determine $A + B$ and $A + C$, if the sum exist.

Solution

$$(1)\ A+B=\begin{bmatrix}1 & 4 & 7\\ 0 & -2 & 3\end{bmatrix}+\begin{bmatrix}2 & 5 & -6\\ -3 & 1 & 8\end{bmatrix}$$

$$=\begin{bmatrix}1+2 & 4+5 & 7-6\\ 0-3 & -2+1 & 3+8\end{bmatrix}$$

$$=\begin{bmatrix}3 & 9 & 1\\ -3 & -1 & 11\end{bmatrix}.$$

(2) Because A is 2×3 matrix and C is a 2×2 matrix, they are not of the same size, $A + C$ does not exist.

Scalar Multiplication of matrices

Definition

Let A be a matrix and c be a scalar. The **scalar multiple** of A by c, denoted cA, is the matrix obtained by multiplying every element of A by c. The matrix cA will be the same size as A.

Thus if $B = cA$, then $b_{ij} = ca_{ij}\ \forall i, j$.

Example 2

Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 7 & -3 & 0 \end{bmatrix}$.

$$3A = \begin{bmatrix} 3\times 1 & 3\times(-2) & 3\times 4 \\ 3\times 7 & 3\times(-3) & 3\times 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 21 & -9 & 0 \end{bmatrix}.$$

Observe that A and $3A$ are both 2×3 matrices.

Negation and Subtraction

Definition

We now define subtraction of matrices in such a way that makes it compatible with addition, scalar multiplication, and negative. Let

$$A - B = A + (-1)B$$

Example 3

Suppose $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -1 \\ 0 & 4 & 6 \end{bmatrix}$.

$$A - B = \begin{bmatrix} 5-2 & 0-8 & -2-(-1) \\ 3-0 & 6-4 & -5-6 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1 \\ 3 & 2 & -11 \end{bmatrix}.$$

Multiplication of Matrices

Definition

Let the number of columns in a matrix A be the same as the number of rows in a matrix B. The product AB then exists.

Let A: $m \times n$ matrix, B: $n \times k$ matrix,
The **product matrix** $\boldsymbol{C=AB}$ has elements

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

C is a $m \times k$ matrix.

If the number of columns in A does not equal the number of row B, we say that **the product does not exist**.

Example 4

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -2 & 5 \end{bmatrix}$.

Determine AB, BA, and AC, if the products exist.

Solution.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1\times5)+(3\times3) & (1\times0)+(3\times(-2)) & (1\times1)+(3\times6) \\ (2\times5)+(0\times3) & (2\times0)+(0\times(-2)) & (2\times1)+(0\times6) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}.$$

BA and AC do not exist. **Note.** In general, $AB \neq BA$.

Example 5

Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$. Determine AB.

$$AB = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 0+5 \\ -7+0 & 0+0 \\ 3-6 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -7 & 0 \\ -3 & -10 \end{bmatrix}$$

Example 6

Let $C = AB$, $A = \begin{bmatrix} 2 & 1 \\ \boxed{-3 \quad 4} \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 3 & \boxed{2} \\ 5 & 0 & \boxed{1} \end{bmatrix}$ Determine c_{23}.

$$c_{23} = \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$

Size of a Product Matrix

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

$$\begin{array}{ccc} A & B & = AB \\ m \times r & r \times n & m \times n \end{array}$$

Example 7

If A is a 5×6 matrix and B is an 6×7 matrix.
Because A has six columns and B has six rows. Thus AB exits.
And AB will be a 5×7 matrix.

Special Matrices

Definition

A **zero matrix** is a matrix in which all the elements are zeros.

A **diagonal matrix** is a square matrix in which all the elements not on the main diagonal are zeros.

An **identity matrix** is a diagonal matrix in which every diagonal element is 1.

$$0_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

zero matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

diaginal matrix A

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

identity matrix

Theorem 2.1

Let A be $m \times n$ matrix and O_{mn} be the zero $m \times n$ matrix. Let B be an $n \times n$ square matrix. O_n and I_n be the zero and identity $n \times n$ matrices. Then

$$A + O_{mn} = O_{mn} + A = A$$
$$BO_n = O_n B = O_n$$
$$BI_n = I_n B = B$$

Example 8

Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$.

$$A + O_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} = A$$

$$BO_2 = \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$

Homework

- Exercises will be given by the teachers of the practical classes.

Exercise

Let A be a matrix whose third row is all zeros. Let B be any matrix such that the product AB exists.
Prove that the third row of AB is all zeros.

Solution

$$(AB)_{3i} = [a_{31}\ a_{32}\ \cdots\ a_{3n}]\begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = [0\ 0\ \cdots\ 0]\begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = 0, \forall i.$$

2.2 Algebraic Properties of Matrix Operations

Theorem 2.2 -1

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Addition and scalar Multiplication

1. $A + B = B + A$ *Commutative property of addition*
2. $A + (B + C) = (A + B) + C$ *Associative property of addition*
3. $A + O = O + A = A$ (*where O is the appropriate zero matrix*)
4. $c(A + B) = cA + cB$ *Distributive property of addition*
5. $(a + b)C = aC + bC$ *Distributive property of addition*
6. $(ab)C = a(bC)$

Theorem 2.2 -2

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Multiplication

1. $A(BC) = (AB)C$ *Associative property of multiplication*
2. $A(B + C) = AB + AC$ *Distributive property of multiplication*
3. $(A + B)C = AC + BC$ *Distributive property of multiplication*
4. $AI_n = I_nA = A$ (*where* I_n *is the appropriate identity matrix*)
5. $c(AB) = (cA)B = A(cB)$

Note: $AB \neq BA$ in general. *Multiplication of matrices is not commutative.*

Proof of Theorem 2.2 (A+B=B+A)

Consider the (i,j)th elements of matrices $A+B$ and $B+A$:

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

$$\therefore$$

$$A+B=B+A$$

Example 9

Let $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$.

$$A+B+C = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}.$$

Arithmetic Operations

If A is an $m \times r$ matrix and B is $r \times n$ matrix, the number of scalar multiplications involved in computing the product AB is mrn.

Consider three matrices A, B and C such that the product ABC exists.
Compare the number of multiplications involved in the two ways $(AB)C$ and $A(BC)$ of computing the product ABC

Example 10

Let $A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$, $B=\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & -2\end{bmatrix}$, and $C=\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}$. Compute ABC.

Solution.

Which method is better?

Count the number of multiplications.

(1) $(AB)C$

$$AB=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & -2\end{bmatrix}=\begin{bmatrix}-2 & 1 & -1\\ 1 & 3 & 11\end{bmatrix}.$$

$$(AB)C=\begin{bmatrix}-2 & 1 & -1\\ 1 & 3 & 11\end{bmatrix}\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}-9\\ 1\end{bmatrix}.$$

$2\times6+3\times2$
$=12+6=18$

(2) $A(BC)$

$$BC=\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & 2\end{bmatrix}\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ -4\end{bmatrix}$$

$$A(BC)=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}-1\\ -4\end{bmatrix}=\begin{bmatrix}-9\\ 1\end{bmatrix}.$$

$3\times2+2\times2$
$=6+4=10$

$\therefore$ $A(BC)$ is better.

Caution

In algebra we know that the following cancellation laws apply.

- If $ab = ac$ and $a \neq 0$ then $b = c$.
- If $pq = 0$ then $p = 0$ or $q = 0$.

However the corresponding results are not true for matrices.

- $AB = AC$ does not imply that $B = C$.
- $PQ = O$ does not imply that $P = O$ or $Q = O$.

Example 11

(1) Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$.

Observe that $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$.

(2) Consider the matrices $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$.

Observe that $PQ = O$, but $P \neq O$ and $Q \neq O$.

Powers of Matrices

Definition

If A is a square matrix, then

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Theorem 2.3

If A is an $n \times n$ square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.
3. $A^0 = I_n$ (by definition)

Example 12

$$\text{If } A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}, \text{ compute } A^4.$$

Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB$$

Solution

$$\begin{aligned} &A(A+2B)+3B(2A-B)-A^2+7B^2-5AB \\ &= A^2+2AB+6BA-3B^2-A^2+7B^2-5AB \\ &= \underline{-3AB+6BA}+4B^2 \end{aligned}$$

We can’t add the two matrices

Systems of Linear Equations

A system of m linear equations in n variables as follows

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$\boldsymbol{AX = B}$$

Idempotent and Nilpotent Matrices

Definition

(1) A square matrix A is said to be **idempotent** if $A^2=A$.

(2) A square matrix A is said to **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

$$(1)\ A=\begin{bmatrix}3 & -6\\ 1 & -2\end{bmatrix},\ A^2=\begin{bmatrix}3 & -6\\ 1 & -2\end{bmatrix}=A.$$

$$(2)\ B=\begin{bmatrix}3 & -9\\ 1 & -3\end{bmatrix},\ B^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.\ \text{The degree of nilpotency}:2$$

Homework

- **Exercises** will be given by the teachers of the practical classes.

2.3 Symmetric Matrices

Definition

The **transpose** of a matrix A, denoted A^t, is the matrix whose columns are the rows of the given matrix A.

$$\text{i.e., } A: m\times n \;\Rightarrow\; A^t: n\times m,\; (A^t)_{ij} = A_{ji} \quad \forall i, j.$$

Example 15

$$A = \begin{bmatrix} 2 & 7 \\ -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -7 \\ 4 & 5 & 6 \end{bmatrix}, \text{and } C = \begin{bmatrix} -1 & 3 & 4 \end{bmatrix}.$$

$$A^t = \begin{bmatrix} 2 & -8 \\ 7 & 0 \end{bmatrix} \qquad B^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -7 & 6 \end{bmatrix} \qquad C^t = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

Theorem 2.4: Properties of Transpose

Let A and B be matrices and c be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.

1. $(A+B)^t = A^t + B^t$ *Transpose of a sum*
2. $(cA)^t = cA^t$ *Transpose of a scalar multiple*
3. $(AB)^t = B^tA^t$ *Transpose of a product*
4. $(A^t)^t = A$

Symmetric Matrix

Definition

A **symmetric matrix** is a matrix that is equal to its transpose.

$$A = A^t, \quad \text{i.e.,} \quad a_{ij} = a_{ji} \; \forall i, j$$

Example 16

$$\begin{bmatrix} 2 & 5 \\ 5 & -4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -4 \\ 1 & 7 & 8 \\ -4 & 8 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 7 & 3 & 9 \\ -2 & 3 & 2 & -3 \\ 4 & 9 & -3 & 6 \end{bmatrix}$$

match

match

Remark: *If and only if*

- Let p and q be statements.
  Suppose that p implies q (if p then q), written $p \Rightarrow q$, and that also $q \Rightarrow p$, we say that

  "p if and only if q" (*in short* iff)

Example 17

Let A and B be symmetric matrices of the same size. Prove that the product AB is symmetric if and only if $AB = BA$.

Proof

*We have to show (a) AB is symmetric $\Rightarrow$ $AB = BA$, and the converse, (b) AB is symmetric $\Leftarrow$ $AB = BA$.

($\Rightarrow$) Let AB be symmetric, then

$AB = (AB)^t$ by definition of symmetric matrix

$= B^t A^t$ by Thm 2.4 (3)

$= BA$ since A and B are symmetric

($\Leftarrow$) Let $AB = BA$, then

$(AB)^t = (BA)^t$

$= A^t B^t$ by Thm 2.4 (3)

$= AB$ since A and B are symmetric

Example 18

Let A be a symmetric matrix. Prove that A^2 is symmetric.

Proof

$$(A^2)^t = (AA)^t = (A^t A^t) = AA = A^2$$

Homework

- **Exercises** will be given by the teachers of the practical classes.

2.4 The Inverse of a Matrix

Definition

Let A be an $n \times n$ matrix. If a matrix B can be found such that $AB = BA = I_n$, then A is said to be **invertible** and B is called the **inverse** of A. If such a matrix B does not exist, then A has no inverse. (denote $B = A^{-1}$, and $A^{-k}=(A^{-1})^k$)

Example 19

Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has inverse $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Proof

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus $AB = BA = I_2$, proving that the matrix A has inverse B.

Theorem 2.5

The inverse of an invertible matrix is unique.

Proof

Let B and C be inverses of A.

Thus $AB = BA = I_n$, and $AC = CA = I_n$.

Multiply both sides of the equation $AB = I_n$ by C.

$$C(AB) = CI_n$$

$$(CA)B = C \longleftarrow \boxed{\text{Thm2.2}}$$

$$I_n B = C$$

$$B = C$$

Thus an invertible matrix has only one inverse.

Gauss-Jordan Elimination for finding the Inverse of a Matrix

Let A be an $n \times n$ matrix.

1. Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
2. Compute the reduced echelon form of $[A : I_n]$.

   If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.

   If the reduced echelon form is not of the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n, then A has no inverse.

An $n \times n$ matrix A is invertible if and only if its reduced echelon form is I_n.

Example 20

Determine the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Solution

$$[A:I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{(-1)R2}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx} \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+R3 \\ R2+(-1)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Thus, $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$.

Example 21

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

Solution

$$[A : I_3] = \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix} \underset{\substack{R2+(-1)R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+(-1)R2 \\ R3+3R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & -5 & 3 & 1 \end{bmatrix}$$

There is no need to proceed further.
The reduced echelon form cannot have a one in the (3, 3) location.
The reduced echelon form cannot be of the form $[I_n : B]$.
Thus A^{-1} does not exist.

Properties of Matrix Inverse

Let A and B be invertible matrices and c a nonzero scalar, Then

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \dfrac{1}{c}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^n)^{-1} = (A^{-1})^n$
5. $(A^t)^{-1} = (A^{-1})^t$

Proof

1. By definition, $AA^{-1} = A^{-1}A = I$.

2. $(cA)(\frac{1}{c}A^{-1}) = I = (\frac{1}{c}A^{-1})(cA)$

3. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I = (B^{-1}A^{-1})(AB)$

4. $A^n(A^{-1})^n = \underbrace{A\cdots A}_{n\text{ times}} \cdot \underbrace{A^{-1}\cdots A^{-1}}_{n\text{ times}} = I = (A^{-1})^n A^n$

5. $AA^{-1} = I,\ (AA^{-1})^t = \underline{(A^{-1})^t A^t = I},$

   $A^{-1}A = I,\ (A^{-1}A)^t = \underline{A^t(A^{-1})^t = I},$

Example 22

If $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, then it can be shown that $A^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Use this information to compute $(A^t)^{-1}$.

Solution

$$(A^t)^{-1} = (A^{-1})^t = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$

Theorem 2.6

Let $AX = B$ be a system of n linear equations in n variables. If A^{-1} exists, the solution is unique and is given by $X = A^{-1}B$.

Proof

($X = A^{-1}B$ is a solution.)
Substitute $X = A^{-1}B$ into the matrix equation.

$$AX = A(A^{-1}B) = (AA^{-1})B = I_n B = B.$$

(The solution is unique.)
Let Y be any solution, thus $AY = B$. Multiplying both sides of this equation by A^{-1} gives

$$A^{-1}A\,Y = A^{-1}B$$
$$I_n Y = A^{-1}B$$
$$Y = A^{-1}B. \quad Then \quad Y = X.$$

Example 22

Solve the system of equations
$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 - 5x_3 &= 3 \\ -x_1 + 3x_2 + 5x_3 &= -2 \end{aligned}$$

Solution

This system can be written in the following matrix form:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

If the matrix of coefficients is invertible, the unique solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

This inverse has already been found in Example 20. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The unique solution is $x_1 = 1, x_2 = -2, x_3 = 1.$

Elementary Matrices

Definition

An **elementary matrix** is one that can be obtained from the identity matrix I_n through a single elementary row operation.

Example 23

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{5R2} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2+2R1} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

- Elementary row operation
- Elementary matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot A = E_1 A$$

$$\xrightarrow{5R2} \begin{bmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_2 A$$

$$\xrightarrow{R2+2R1} \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_3 A$$

Notes for elementary matrices

- Each elementary matrix is invertible.

Example 24

$$I \underset{R1+2R2}{\approx} E_1 \Rightarrow E_1 \underset{R1-2R2}{\approx} I, \quad \text{i.e., } E_2E_1 = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boxed{E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

- If A and B are row equivalent matrices and A is invertible, then B is invertible.

Proof

If $A \approx \ldots \approx B$, then

$B = E_n \ldots E_2 E_1 A$ for some elementary matrices $E_n, \ldots, E_2$ and E_1.

So $B^{-1} = (E_n \ldots E_2 E_1 A)^{-1} = A^{-1} E_1^{-1} E_2^{-1} \ldots E_n^{-1}$.

Homework

- **Exercises** will be given by the teachers of the practical classes.

Exercise

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that $A^{-1}=\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.