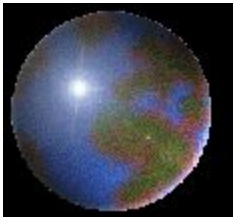


# Linear Algebra



## ***Chapter 2*** ***Matrix Algebra***



## 2.1 Addition, Scalar Multiplication, and Multiplication of Matrices

- $a_{ij}$ : the element of matrix  $A$  in row  $i$  and column  $j$ .
- For a square  $n \times n$  matrix  $A$ , the **main diagonal** is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \boxtimes & a_{1n} \\ a_{21} & a_{22} & \boxtimes & a_{2n} \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ a_{n1} & a_{n2} & \boxtimes & a_{nn} \end{bmatrix}$$

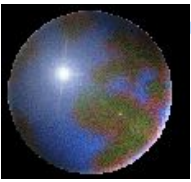
A red oval highlights the main diagonal elements  $a_{11}, a_{22}, \dots, a_{nn}$  in the matrix above.

### Definition

Two matrices are **equal** if they are of the same size and if their corresponding elements are equal.

Thus  $A = B$  if  $a_{ij} = b_{ij} \quad \forall i, j.$

( $\forall$  for every, for all)



# Addition of Matrices

## Definition

Let  $A$  and  $B$  be matrices of the same size.

Their **sum**  $A + B$  is the matrix obtained by adding together the corresponding elements of  $A$  and  $B$ .

The matrix  $A + B$  will be of the same size as  $A$  and  $B$ .

If  $A$  and  $B$  are not of the same size, they cannot be added, and we say that **the sum does not exist**.

Thus if  $C = A + B$ , then  $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$ .



## Example 1

$$\text{Let } A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}.$$

Determine  $A + B$  and  $A + C$ , if the sum exist.

### Solution

$$\begin{aligned} (1) A + B &= \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 4+5 & 7-6 \\ 0-3 & -2+1 & 3+8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 9 & 1 \\ -3 & -1 & 11 \end{bmatrix}. \end{aligned}$$

(2) Because  $A$  is  $2 \times 3$  matrix and  $C$  is a  $2 \times 2$  matrix, they are not of the same size,  $A + C$  does not exist.



# Scalar Multiplication of matrices

## Definition

Let  $A$  be a matrix and  $c$  be a scalar. The **scalar multiple** of  $A$  by  $c$ , denoted  $cA$ , is the matrix obtained by multiplying every element of  $A$  by  $c$ . The matrix  $cA$  will be the same size as  $A$ .

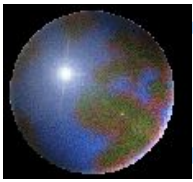
Thus if  $B = cA$ , then  $b_{ij} = ca_{ij} \forall i, j$ .

## Example 2

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 4 \\ 7 & -3 & 0 \end{bmatrix}.$$

$$3A = \begin{bmatrix} 3 \times 1 & 3 \times (-2) & 3 \times 4 \\ 3 \times 7 & 3 \times (-3) & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 21 & -9 & 0 \end{bmatrix}.$$

Observe that  $A$  and  $3A$  are both  $2 \times 3$  matrices.



# Negation and Subtraction

## Definition

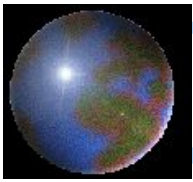
We now define subtraction of matrices in such a way that makes it compatible with addition, scalar multiplication, and negative. Let

$$A - B = A + (-1)B$$

## Example 3

Suppose  $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 6 & -5 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 8 & -1 \\ 0 & 4 & 6 \end{bmatrix}$ .

$$A - B = \begin{bmatrix} 5-2 & 0-8 & -2-(-1) \\ 3-0 & 6-4 & -5-6 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1 \\ 3 & 2 & -11 \end{bmatrix}.$$



# Multiplication of Matrices

## Definition

Let the number of columns in a matrix  $A$  be the same as the number of rows in a matrix  $B$ . The product  $AB$  then exists.

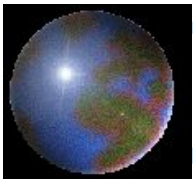
Let  $A: m \times n$  matrix,  $B: n \times k$  matrix,

The **product matrix**  $C=AB$  has elements

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \boxtimes & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \boxtimes \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \boxtimes + a_{in}b_{nj}$$

$C$  is a  $m \times k$  matrix.

If the number of columns in  $A$  does not equal the number of row  $B$ , we say that **the product does not exist**.



## Example 4

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}, \text{ and } C = [6 \quad -2 \quad 5].$$

Determine  $AB$ ,  $BA$ , and  $AC$ , if the products exist.

### Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} [1 \quad 3] \begin{bmatrix} 5 \\ 3 \end{bmatrix} & [1 \quad 3] \begin{bmatrix} 0 \\ -2 \end{bmatrix} & [1 \quad 3] \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ [2 \quad 0] \begin{bmatrix} 5 \\ 3 \end{bmatrix} & [2 \quad 0] \begin{bmatrix} 0 \\ -2 \end{bmatrix} & [2 \quad 0] \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (1 \times 5) + (3 \times 3) & (1 \times 0) + (3 \times (-2)) & (1 \times 1) + (3 \times 6) \\ (2 \times 5) + (0 \times 3) & (2 \times 0) + (0 \times (-2)) & (2 \times 1) + (0 \times 6) \end{bmatrix} \\ &= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}. \end{aligned}$$

$BA$  and  $AC$  do not exist. **Note.** In general,  $AB \neq BA$ .





## Example 5

Let  $A = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$ . Determine  $AB$ .

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} [2 \ 1] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [2 \ 1] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [7 \ 0] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [7 \ 0] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [-3 \ -2] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [-3 \ -2] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} -2+3 & 0+5 \\ -7+0 & 0+0 \\ 3-6 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -7 & 0 \\ -3 & -10 \end{bmatrix} \end{aligned}$$

## Example 6

Let  $C = AB$ ,  $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} -7 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$ . Determine  $c_{23}$ .

$$c_{23} = [-3 \ 4] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$



# Size of a Product Matrix

If  $A$  is an  $m \times r$  matrix and  $B$  is an  $r \times n$  matrix, then  $AB$  will be an  $m \times n$  matrix.

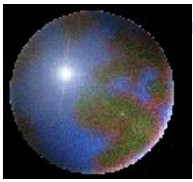
$$\begin{array}{ccc} A & B & = AB \\ m \times r & r \times n & m \times n \end{array}$$

## Example 7

If  $A$  is a  $5 \times 6$  matrix and  $B$  is an  $6 \times 7$  matrix.

Because  $A$  has six columns and  $B$  has six rows. Thus  $AB$  exists.

And  $AB$  will be a  $5 \times 7$  matrix.



# Special Matrices

## Definition

A **zero matrix** is a matrix in which all the elements are zeros.

A **diagonal matrix** is a square matrix in which all the elements not on the main diagonal are zeros.

An **identity matrix** is a diagonal matrix in which every diagonal element is 1.

$$O_{mn} = \begin{bmatrix} 0 & 0 & \boxtimes & 0 \\ 0 & 0 & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & 0 \end{bmatrix}$$

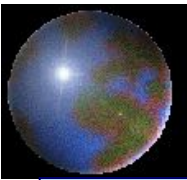
zero matrix

$$A = \begin{bmatrix} a_{11} & 0 & \boxtimes & 0 \\ 0 & a_{22} & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & a_{mn} \end{bmatrix}$$

diagonal matrix A

$$I_n = \begin{bmatrix} 1 & 0 & \boxtimes & 0 \\ 0 & 1 & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & 1 \end{bmatrix}$$

identity matrix



# Theorem 2.1

Let  $A$  be  $m \times n$  matrix and  $O_{mn}$  be the zero  $m \times n$  matrix. Let  $B$  be an  $n \times n$  square matrix.  $O_n$  and  $I_n$  be the zero and identity  $n \times n$  matrices. Then

$$\begin{aligned}A + O_{mn} &= O_{mn} + A = A \\BO_n &= O_n B = O_n \\BI_n &= I_n B = B\end{aligned}$$

## Example 8

Let  $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ .

$$A + O_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} = A$$

$$BO_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$



# Homework

- Exercises will be given by the teachers of the practical classes.

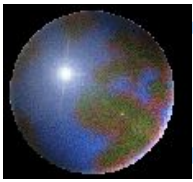
## Exercise

Let  $A$  be a matrix whose third row is all zeros. Let  $B$  be any matrix such that the product  $AB$  exists.

Prove that the third row of  $AB$  is all zeros.

## Solution

$$(AB)_{3i} = [a_{31} \ a_{32} \ \dots \ a_{3n}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = [0 \ 0 \ \dots \ 0] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = 0, \forall i.$$



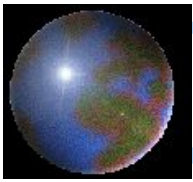
## 2.2 Algebraic Properties of Matrix Operations

### Theorem 2.2 -1

Let  $A$ ,  $B$ , and  $C$  be matrices and  $a$ ,  $b$ , and  $c$  be scalars. Assume that the size of the matrices are such that the operations can be performed.

#### **Properties of Matrix Addition and scalar Multiplication**

1.  $A + B = B + A$       *Commutative property of addition*
2.  $A + (B + C) = (A + B) + C$       *Associative property of addition*
3.  $A + O = O + A = A$       *(where  $O$  is the appropriate zero matrix)*
4.  $c(A + B) = cA + cB$       *Distributive property of addition*
5.  $(a + b)C = aC + bC$       *Distributive property of addition*
6.  $(ab)C = a(bC)$



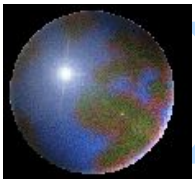
## Theorem 2.2 -2

Let  $A$ ,  $B$ , and  $C$  be matrices and  $a$ ,  $b$ , and  $c$  be scalars. Assume that the size of the matrices are such that the operations can be performed.

### Properties of Matrix Multiplication

1.  $A(BC) = (AB)C$       *Associative property of multiplication*
2.  $A(B + C) = AB + AC$       *Distributive property of multiplication*
3.  $(A + B)C = AC + BC$       *Distributive property of multiplication*
4.  $AI_n = I_n A = A$       (*where  $I_n$  is the appropriate identity matrix*)
5.  $c(AB) = (cA)B = A(cB)$

**Note:**  $AB \neq BA$  in general. *Multiplication of matrices is not commutative.*



## **Proof of Theorem 2.2 ( $A+B=B+A$ )**

Consider the  $(i,j)$ th elements of matrices  $A+B$  and  $B+A$ :

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

$\therefore$

$$A+B=B+A$$

### **Example 9**

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}.$$

$$\begin{aligned} A+B+C &= \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}. \end{aligned}$$





# Arithmetic Operations

If  $A$  is an  $m \times r$  matrix and  $B$  is  $r \times n$  matrix, the number of scalar multiplications involved in computing the product  $AB$  is  $mrn$ .

Consider three matrices  $A$ ,  $B$  and  $C$  such that the product  $ABC$  exists.

Compare the number of multiplications involved in the two ways  $(AB)C$  and  $A(BC)$  of computing the product  $ABC$



# Example 10

Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ , and  $C = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ . Compute  $ABC$ .

## Solution.

(1)  $(AB)C$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

(2)  $A(BC)$

$$BC = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

Which method is better?

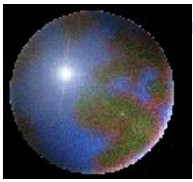
Count the number of

multiplications.

$$2 \times 6 + 3 \times 2 \\ = 12 + 6 = 18$$

$$3 \times 2 + 2 \times 2 \\ = 6 + 4 = 10$$

$\therefore A(BC)$  is better.



# Caution

In algebra we know that the following cancellation laws apply.

- If  $ab = ac$  and  $a \neq 0$  then  $b = c$ .
- If  $pq = 0$  then  $p = 0$  or  $q = 0$ .

However the corresponding results are not true for matrices.

- $AB = AC$  **does not imply** that  $B = C$ .
- $PQ = O$  **does not imply** that  $P = O$  or  $Q = O$ .

## Example 11

(1) Consider the matrices  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ , and  $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$ .

Observe that  $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ , but  $B \neq C$ .

(2) Consider the matrices  $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ , and  $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$ .

Observe that  $PQ = O$ , but  $P \neq O$  and  $Q \neq O$ .



# Powers of Matrices

## Definition

If  $A$  is a square matrix, then

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_{k \text{ times}}$$

## Theorem 2.3

If  $A$  is an  $n \times n$  square matrix and  $r$  and  $s$  are nonnegative integers, then

1.  $A^r A^s = A^{r+s}$ .
2.  $(A^r)^s = A^{rs}$ .
3.  $A^0 = I_n$  (by definition)



## Example 12

If  $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$ , compute  $A^4$ .

### Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

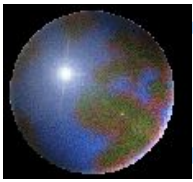
**Example 13** Simplify the following matrix expression.

$$A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB$$

### Solution

$$\begin{aligned} & A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB \\ &= A^2 + 2AB + 6BA - 3B^2 - A^2 + 7B^2 - 5AB \\ &= \underline{-3AB + 6BA} + 4B^2 \end{aligned}$$

We can't add the two matrices



# Systems of Linear Equations

A system of  $m$  linear equations in  $n$  variables as follows

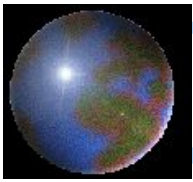
$$\begin{aligned} a_{11}x_1 + \boxed{\phantom{0}} + a_{1n}x_n &= b_1 \\ \boxed{\phantom{0}} \quad \boxed{\phantom{0}} \quad \boxed{\phantom{0}} \quad \boxed{\phantom{0}} \\ a_{m1}x_1 + \boxed{\phantom{0}} + a_{mn}x_n &= b_m \end{aligned}$$

Let

$$A = \begin{bmatrix} a_{11} & \boxed{\phantom{0}} & a_{1n} \\ \boxed{\phantom{0}} & \boxed{\phantom{0}} & \boxed{\phantom{0}} \\ a_{m1} & \boxed{\phantom{0}} & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \boxed{\phantom{0}} \\ x_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ \boxed{\phantom{0}} \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$\mathbf{AX} = \mathbf{B}$$



# Idempotent and Nilpotent Matrices

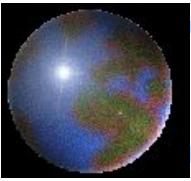
## Definition

- (1) A square matrix  $A$  is said to be **idempotent** if  $A^2=A$ .
- (2) A square matrix  $A$  is said to be **nilpotent** if there is a positive integer  $p$  such that  $A^p=0$ . The least integer  $p$  such that  $A^p=0$  is called the **degree of nilpotency** of the matrix.

## Example 14

$$(1) A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = A.$$

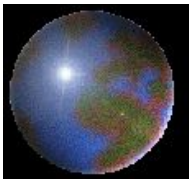
$$(2) B = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The degree of nilpotency : 2}$$



## *Homework*

- **Exercises** will be given by the teachers of the practical classes.





## 2.3 Symmetric Matrices

### Definition

The **transpose** of a matrix  $A$ , denoted  $A^t$ , is the matrix whose columns are the rows of the given matrix  $A$ .

i.e.,  $A : m \times n \Rightarrow A^t : n \times m, (A^t)_{ij} = A_{ji} \quad \forall i, j.$

### Example 15

$$A = \begin{bmatrix} 2 & 7 \\ -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -7 \\ 4 & 5 & 6 \end{bmatrix}, \text{ and } C = [-1 \quad 3 \quad 4].$$

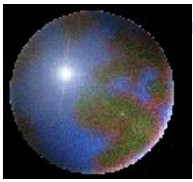
$$A^t = \begin{bmatrix} 2 & -8 \\ 7 & 0 \end{bmatrix} \quad B^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -7 & 6 \end{bmatrix} \quad C^t = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$



## Theorem 2.4: Properties of Transpose

Let  $A$  and  $B$  be matrices and  $c$  be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.

1.  $(A + B)^t = A^t + B^t$       *Transpose of a sum*
2.  $(cA)^t = cA^t$       *Transpose of a scalar multiple*
3.  $(AB)^t = B^tA^t$       *Transpose of a product*
4.  $(A^t)^t = A$



# Symmetric Matrix

## Definition

A **symmetric matrix** is a matrix that is equal to its transpose.

$$A = A^t, \quad \text{i.e., } a_{ij} = a_{ji} \quad \forall i, j$$

## Example 16

$$\begin{bmatrix} 2 & 5 \\ 5 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 \\ 1 & 7 & 8 \\ -4 & 8 & -3 \end{bmatrix}$$

match

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 7 & 3 & 9 \\ -2 & 3 & 2 & -3 \\ 4 & 9 & -3 & 6 \end{bmatrix}$$

match



## *Remark: If and only if*

- Let  $p$  and  $q$  be statements.  
Suppose that  $p$  implies  $q$  (if  $p$  then  $q$ ), written  $p \Rightarrow q$ ,  
and that also  $q \Rightarrow p$ , we say that

“ $p$  if and only if  $q$ ” (in short **iff** )



## Example 17

Let  $A$  and  $B$  be symmetric matrices of the same size. Prove that the product  $AB$  is symmetric if and only if  $AB = BA$ .

### Proof

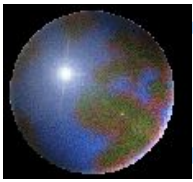
\*We have to show (a)  $AB$  is symmetric  $\Rightarrow AB = BA$ ,  
and the converse, (b)  $AB$  is symmetric  $\Leftarrow AB = BA$ .

( $\Rightarrow$ ) Let  $AB$  be symmetric, then

$$\begin{aligned} AB &= (AB)^t && \text{by definition of symmetric matrix} \\ &= B^t A^t && \text{by Thm 2.4 (3)} \\ &= BA && \text{since } A \text{ and } B \text{ are symmetric} \end{aligned}$$

( $\Leftarrow$ ) Let  $AB = BA$ , then

$$\begin{aligned} (AB)^t &= (BA)^t \\ &= A^t B^t && \text{by Thm 2.4 (3)} \\ &= AB && \text{since } A \text{ and } B \text{ are symmetric} \end{aligned}$$

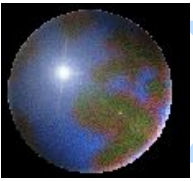


## Example 18

Let  $A$  be a symmetric matrix. Prove that  $A^2$  is symmetric.

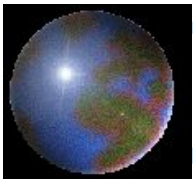
### Proof

$$(A^2)^t = (AA)^t = (A^t A^t) = AA = A^2$$



# *Homework*

- **Exercises** will be given by the teachers of the practical classes.



## 2.4 The Inverse of a Matrix

### Definition

Let  $A$  be an  $n \times n$  matrix. If a matrix  $B$  can be found such that  $AB = BA = I_n$ , then  $A$  is said to be **invertible** and  $B$  is called the **inverse** of  $A$ . If such a matrix  $B$  does not exist, then  $A$  has no inverse. (denote  $B = A^{-1}$ , and  $A^{-k} = (A^{-1})^k$ )

### Example 19

Prove that the matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  has inverse  $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ .

### Proof

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus  $AB = BA = I_2$ , proving that the matrix  $A$  has inverse  $B$ .





## Theorem 2.5

The inverse of an invertible matrix is unique.

### Proof

Let  $B$  and  $C$  be inverses of  $A$ .

Thus  $AB = BA = I_n$ , and  $AC = CA = I_n$ .

Multiply both sides of the equation  $AB = I_n$  by  $C$ .

$$C(AB) = CI_n$$

$$(CA)B = C \quad \leftarrow \text{Thm2.2}$$

$$I_n B = C$$

$$B = C$$

Thus an invertible matrix has only one inverse.



# Gauss-Jordan Elimination for finding the Inverse of a Matrix

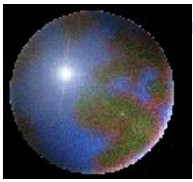
Let  $A$  be an  $n \times n$  matrix.

1. Adjoin the identity  $n \times n$  matrix  $I_n$  to  $A$  to form the matrix  $[A : I_n]$ .
2. Compute the reduced echelon form of  $[A : I_n]$ .

If the reduced echelon form is of the type  $[I_n : B]$ , then  $B$  is the inverse of  $A$ .

If the reduced echelon form is not of the type  $[I_n : B]$ , in that the first  $n \times n$  submatrix is not  $I_n$ , then  $A$  has no inverse.

An  $n \times n$  matrix  $A$  is invertible if and only if its reduced echelon form is  $I_n$ .



# Example 20

Determine the inverse of the matrix  $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

## Solution

$$[A : I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \text{R2} + (-2)\text{R1} \\ \text{R3} + \text{R1} \end{array} \approx \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} \\ (-1)\text{R2} \end{array} \approx \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \text{R1} + \text{R2} \\ \text{R3} + (-2)\text{R2} \end{array} \approx \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ \text{R1} + \text{R3} \\ \text{R2} + (-1)\text{R3} \end{array} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Thus,  $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$ .



## Example 21

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

### Solution

$$\begin{aligned} [A : I_3] &= \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \approx \\ R2 + (-1)R1 \\ R3 + (-2)R1 \end{array} \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix} \\ & \begin{array}{l} \approx \\ R1 + (-1)R2 \\ R3 + 3R2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{bmatrix} \end{aligned}$$

There is no need to proceed further.

The reduced echelon form cannot have a one in the (3, 3) location.

The reduced echelon form cannot be of the form  $[I_n : B]$ .

Thus  $A^{-1}$  does not exist.



# Properties of Matrix Inverse

Let  $A$  and  $B$  be invertible matrices and  $c$  a nonzero scalar, Then

$$1. (A^{-1})^{-1} = A$$

$$4. (A^n)^{-1} = (A^{-1})^n$$

$$2. (cA)^{-1} = \frac{1}{c} A^{-1}$$

$$5. (A^t)^{-1} = (A^{-1})^t$$

$$3. (AB)^{-1} = B^{-1} A^{-1}$$

## Proof

1. By definition,

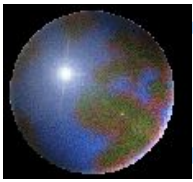
$$AA^{-1} = A^{-1}A = I$$

$$3. (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I = (B^{-1}A^{-1})(AB)$$

$$4. \underbrace{A \cdot A \cdot \dots \cdot A}_{n \text{ times}} \cdot \underbrace{A^{-1} \cdot A^{-1} \cdot \dots \cdot A^{-1}}_{n \text{ times}} = I = (A^{-1})^n A^n$$

$$5. AA^{-1} = I, (AA^{-1})^t = \underline{(A^{-1})^t A^t = I},$$

$$A^{-1}A = I, (A^{-1}A)^t = \underline{A^t (A^{-1})^t = I},$$

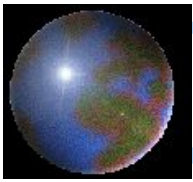


## Example 22

If  $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ , then it can be shown that  $A^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ . Use this information to compute  $(A^t)^{-1}$ .

## Solution

$$(A^t)^{-1} = (A^{-1})^t = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$



## Theorem 2.6

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables. If  $A^{-1}$  exists, the solution is unique and is given by  $X = A^{-1}B$ .

### Proof

( $X = A^{-1}B$  is a solution.)

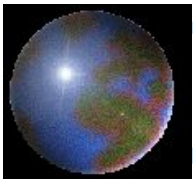
Substitute  $X = A^{-1}B$  into the matrix equation.

$$AX = A(A^{-1}B) = (AA^{-1})B = I_n B = B.$$

(The solution is unique.)

Let  $Y$  be any solution, thus  $AY = B$ . Multiplying both sides of this equation by  $A^{-1}$  gives

$$\begin{aligned} A^{-1}A Y &= A^{-1}B \\ I_n Y &= A^{-1}B \\ Y &= A^{-1}B. \quad \text{Then } Y=X. \end{aligned}$$



## Example 22

Solve the system of equations

$$\begin{aligned}x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 - 5x_3 &= 3 \\ -x_1 + 3x_2 + 5x_3 &= -2\end{aligned}$$

### Solution

This system can be written in the following matrix form:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

If the matrix of coefficients is invertible, the unique solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

This inverse has already been found in Example 20. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The unique solution is  $x_1 = 1$ ,  $x_2 = -2$ ,  $x_3 = 1$ .





# Elementary Matrices

## Definition

An **elementary matrix** is one that can be obtained from the identity matrix  $I_n$  through a single elementary row operation.

## Example 23

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

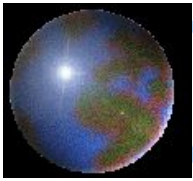
$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$5R_2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + 2R_1$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Elementary Matrices

- Elementary row operation
- Elementary matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$R2 \leftrightarrow R3$

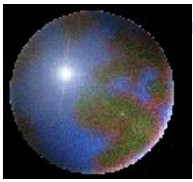
$$\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot A = E_1 A$$

$5R2$

$$\begin{bmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_2 A$$

$R2 + 2R1$

$$\begin{bmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_3 A$$



# Notes for elementary matrices

- Each elementary matrix is invertible.

## Example 24

$$I \underset{R1+2R2}{\approx} E_1 \Rightarrow E_1 \underset{R1-2R2}{\approx} I, \text{ i.e., } E_2 E_1 = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

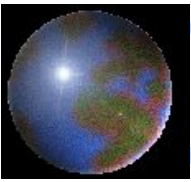
- If  $A$  and  $B$  are row equivalent matrices and  $A$  is invertible, then  $B$  is invertible.

## Proof

If  $A \approx \dots \approx B$ , then

$B = E_n \dots E_2 E_1 A$  for some elementary matrices  $E_n, \dots, E_2$  and  $E_1$ .

So  $B^{-1} = (E_n \dots E_2 E_1 A)^{-1} = A^{-1} E_1^{-1} E_2^{-1} \dots E_n^{-1}$ .



# Homework

- Exercises will be given by the teachers of the practical classes.

## Exercise

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , show that  $A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .