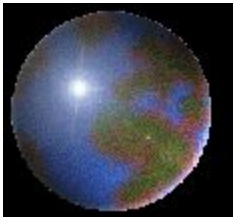


Linear Algebra



Chapter 2 ***Matrix Algebra***



2.1 Addition, Scalar Multiplication, and Multiplication of Matrices

- a_{ij} : the element of matrix A in row i and column j .
- For a square $n \times n$ matrix A , the **main diagonal** is:

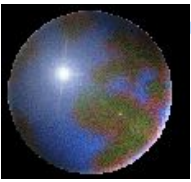
$$A = \begin{bmatrix} a_{11} & a_{12} & \boxtimes & a_{1n} \\ a_{21} & a_{22} & \boxtimes & a_{2n} \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ a_{n1} & a_{n2} & \boxtimes & a_{nn} \end{bmatrix}$$

Definition

Two matrices are **equal** if they are of the same size and if their corresponding elements are equal.

Thus $A = B$ if $a_{ij} = b_{ij} \quad \forall i, j$.

(\forall for every, for all)



Addition of Matrices

Definition

Let A and B be matrices of the same size.

Their **sum** $A + B$ is the matrix obtained by adding together the corresponding elements of A and B .

The matrix $A + B$ will be of the same size as A and B .

If A and B are not of the same size, they cannot be added, and we say that **the sum does not exist**.

Thus if $C = A + B$, then $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$.



Example 1

$$\text{Let } A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}.$$

Determine $A + B$ and $A + C$, if the sum exist.

Solution

$$\begin{aligned} (1) A + B &= \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 4+5 & 7-6 \\ 0-3 & -2+1 & 3+8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 9 & 1 \\ -3 & -1 & 11 \end{bmatrix}. \end{aligned}$$

(2) Because A is 2×3 matrix and C is a 2×2 matrix, they are not of the same size, $A + C$ does not exist.



Scalar Multiplication of matrices

Definition

Let A be a matrix and c be a scalar. The **scalar multiple** of A by c , denoted cA , is the matrix obtained by multiplying every element of A by c . The matrix cA will be the same size as A .

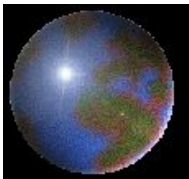
Thus if $B = cA$, then $b_{ij} = ca_{ij} \forall i, j$.

Example 2

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 4 \\ 7 & -3 & 0 \end{bmatrix}.$$

$$3A = \begin{bmatrix} 3 \times 1 & 3 \times (-2) & 3 \times 4 \\ 3 \times 7 & 3 \times (-3) & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 21 & -9 & 0 \end{bmatrix}.$$

Observe that A and $3A$ are both 2×3 matrices.



Negation and Subtraction

Definition

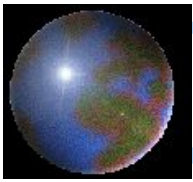
We now define subtraction of matrices in such a way that makes it compatible with addition, scalar multiplication, and negative. Let

$$A - B = A + (-1)B$$

Example 3

Suppose $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -1 \\ 0 & 4 & 6 \end{bmatrix}$.

$$A - B = \begin{bmatrix} 5-2 & 0-8 & -2-(-1) \\ 3-0 & 6-4 & -5-6 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1 \\ 3 & 2 & -11 \end{bmatrix}.$$



Multiplication of Matrices

Definition

Let the number of columns in a matrix A be the same as the number of rows in a matrix B . The product AB then exists.

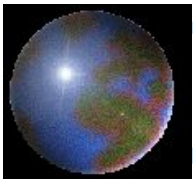
Let $A: m \times n$ matrix, $B: n \times k$ matrix,

The **product matrix** $C=AB$ has elements

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \boxtimes & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \boxtimes \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \boxtimes + a_{in}b_{nj}$$

C is a $m \times k$ matrix.

If the number of columns in A does not equal the number of row B , we say that **the product does not exist**.



Example 4

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}, \text{ and } C = [6 \quad -2 \quad 5].$$

Determine AB , BA , and AC , if the products exist.

Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} [1 \quad 3] \begin{bmatrix} 5 \\ 3 \end{bmatrix} & [1 \quad 3] \begin{bmatrix} 0 \\ -2 \end{bmatrix} & [1 \quad 3] \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ [2 \quad 0] \begin{bmatrix} 5 \\ 3 \end{bmatrix} & [2 \quad 0] \begin{bmatrix} 0 \\ -2 \end{bmatrix} & [2 \quad 0] \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (1 \times 5) + (3 \times 3) & (1 \times 0) + (3 \times (-2)) & (1 \times 1) + (3 \times 6) \\ (2 \times 5) + (0 \times 3) & (2 \times 0) + (0 \times (-2)) & (2 \times 1) + (0 \times 6) \end{bmatrix} \\ &= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}. \end{aligned}$$

BA and AC do not exist. **Note.** In general, $AB \neq BA$.



Example 5

Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$. Determine AB .

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} [2 \ 1] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [2 \ 1] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [7 \ 0] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [7 \ 0] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [-3 \ -2] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [-3 \ -2] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} -2+3 & 0+5 \\ -7+0 & 0+0 \\ 3-6 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -7 & 0 \\ -3 & -10 \end{bmatrix} \end{aligned}$$

Example 6

Let $C = AB$, $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$. Determine c_{23} .

$$c_{23} = [-3 \ 4] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$



Size of a Product Matrix

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

$$\begin{array}{ccc} A & B & = AB \\ m \times r & r \times n & m \times n \end{array}$$

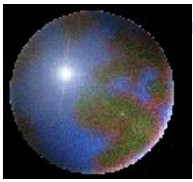
The diagram illustrates the multiplication of two matrices. Matrix A has dimensions $m \times r$, matrix B has dimensions $r \times n$, and their product AB has dimensions $m \times n$. The r in A and the r in B are circled in red, and a bracket connects them, indicating that the number of columns in A must equal the number of rows in B for the product to exist. Another bracket connects the m in A to the m in AB , and a third bracket connects the n in B to the n in AB .

Example 7

If A is a 5×6 matrix and B is an 6×7 matrix.

Because A has six columns and B has six rows. Thus AB exists.

And AB will be a 5×7 matrix.



Special Matrices

Definition

A **zero matrix** is a matrix in which all the elements are zeros.

A **diagonal matrix** is a square matrix in which all the elements not on the main diagonal are zeros.

An **identity matrix** is a diagonal matrix in which every diagonal element is 1.

$$O_{mn} = \begin{bmatrix} 0 & 0 & \boxtimes & 0 \\ 0 & 0 & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & 0 \end{bmatrix}$$

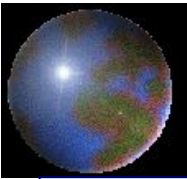
zero matrix

$$A = \begin{bmatrix} a_{11} & 0 & \boxtimes & 0 \\ 0 & a_{22} & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & a_{mn} \end{bmatrix}$$

diagonal matrix A

$$I_n = \begin{bmatrix} 1 & 0 & \boxtimes & 0 \\ 0 & 1 & \boxtimes & 0 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & \boxtimes & 1 \end{bmatrix}$$

identity matrix



Theorem 2.1

Let A be $m \times n$ matrix and O_{mn} be the zero $m \times n$ matrix. Let B be an $n \times n$ square matrix. O_n and I_n be the zero and identity $n \times n$ matrices. Then

$$\begin{aligned}A + O_{mn} &= O_{mn} + A = A \\BO_n &= O_n B = O_n \\BI_n &= I_n B = B\end{aligned}$$

Example 8

Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$.

$$A + O_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} = A$$

$$BO_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$



Homework

- Exercises will be given by the teachers of the practical classes.

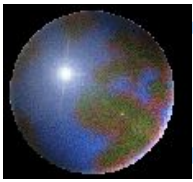
Exercise

Let A be a matrix whose third row is all zeros. Let B be any matrix such that the product AB exists.

Prove that the third row of AB is all zeros.

Solution

$$(AB)_{3i} = [a_{31} \ a_{32} \ \dots \ a_{3n}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = [0 \ 0 \ \dots \ 0] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = 0, \forall i.$$



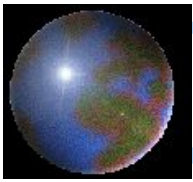
2.2 Algebraic Properties of Matrix Operations

Theorem 2.2 -1

Let A , B , and C be matrices and a , b , and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Addition and scalar Multiplication

1. $A + B = B + A$ *Commutative property of addition*
2. $A + (B + C) = (A + B) + C$ *Associative property of addition*
3. $A + O = O + A = A$ *(where O is the appropriate zero matrix)*
4. $c(A + B) = cA + cB$ *Distributive property of addition*
5. $(a + b)C = aC + bC$ *Distributive property of addition*
6. $(ab)C = a(bC)$



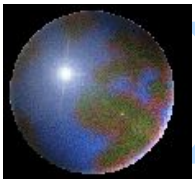
Theorem 2.2 -2

Let A , B , and C be matrices and a , b , and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Multiplication

1. $A(BC) = (AB)C$ *Associative property of multiplication*
2. $A(B + C) = AB + AC$ *Distributive property of multiplication*
3. $(A + B)C = AC + BC$ *Distributive property of multiplication*
4. $AI_n = I_n A = A$ (where I_n is the appropriate identity matrix)
5. $c(AB) = (cA)B = A(cB)$

Note: $AB \neq BA$ in general. *Multiplication of matrices is not commutative.*



Proof of Theorem 2.2 ($A+B=B+A$)

Consider the (i,j) th elements of matrices $A+B$ and $B+A$:

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

\therefore

$$A+B=B+A$$

Example 9

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}.$$

$$\begin{aligned} A+B+C &= \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}. \end{aligned}$$



Arithmetic Operations

If A is an $m \times r$ matrix and B is $r \times n$ matrix, the number of scalar multiplications involved in computing the product AB is mrn .

Consider three matrices A , B and C such that the product ABC exists.

Compare the number of multiplications involved in the two ways $(AB)C$ and $A(BC)$ of computing the product ABC



Example 10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. Compute ABC .

Solution.

(1) $(AB)C$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

(2) $A(BC)$

$$BC = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

Which method is better?

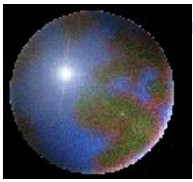
Count the number of

multiplications.

$$2 \times 6 + 3 \times 2 \\ = 12 + 6 = 18$$

$$3 \times 2 + 2 \times 2 \\ = 6 + 4 = 10$$

$\therefore A(BC)$ is better.



Caution

In algebra we know that the following cancellation laws apply.

- If $ab = ac$ and $a \neq 0$ then $b = c$.
- If $pq = 0$ then $p = 0$ or $q = 0$.

However the corresponding results are not true for matrices.

- $AB = AC$ **does not imply** that $B = C$.
- $PQ = O$ **does not imply** that $P = O$ or $Q = O$.

Example 11

(1) Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$.

Observe that $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$.

(2) Consider the matrices $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$.

Observe that $PQ = O$, but $P \neq O$ and $Q \neq O$.



Powers of Matrices

Definition

If A is a square matrix, then

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_{k \text{ times}}$$

Theorem 2.3

If A is an $n \times n$ square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.
3. $A^0 = I_n$ (by definition)



Example 12

If $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4 .

Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

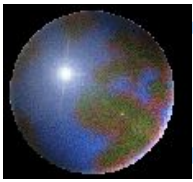
Example 13 Simplify the following matrix expression.

$$A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB$$

Solution

$$\begin{aligned} & A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB \\ &= A^2 + 2AB + 6BA - 3B^2 - A^2 + 7B^2 - 5AB \\ &= \underline{-3AB + 6BA} + 4B^2 \end{aligned}$$

We can't add the two matrices



Systems of Linear Equations

A system of m linear equations in n variables as follows

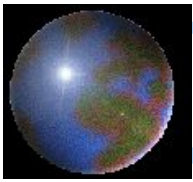
$$\begin{aligned} a_{11}x_1 + \boxed{} + a_{1n}x_n &= b_1 \\ \boxed{} \quad \boxed{} \quad \boxed{} \quad \boxed{} \\ a_{m1}x_1 + \boxed{} + a_{mn}x_n &= b_m \end{aligned}$$

Let

$$A = \begin{bmatrix} a_{11} & \boxed{} & a_{1n} \\ \boxed{} & \boxed{} & \boxed{} \\ a_{m1} & \boxed{} & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \boxed{} \\ x_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ \boxed{} \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$\mathbf{AX} = \mathbf{B}$$



Idempotent and Nilpotent Matrices

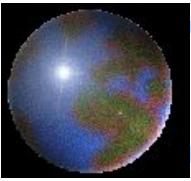
Definition

- (1) A square matrix A is said to be **idempotent** if $A^2=A$.
- (2) A square matrix A is said to be **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

$$(1) A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = A.$$

$$(2) B = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The degree of nilpotency : 2}$$



Homework

- **Exercises** will be given by the teachers of the practical classes.



2.3 Symmetric Matrices

Definition

The **transpose** of a matrix A , denoted A^t , is the matrix whose columns are the rows of the given matrix A .

i.e., $A : m \times n \Rightarrow A^t : n \times m, (A^t)_{ij} = A_{ji} \quad \forall i, j.$

Example 15

$$A = \begin{bmatrix} 2 & 7 \\ -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -7 \\ 4 & 5 & 6 \end{bmatrix}, \text{ and } C = [-1 \quad 3 \quad 4].$$

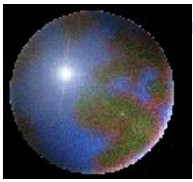
$$A^t = \begin{bmatrix} 2 & -8 \\ 7 & 0 \end{bmatrix} \quad B^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -7 & 6 \end{bmatrix} \quad C^t = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$



Theorem 2.4: Properties of Transpose

Let A and B be matrices and c be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.

1. $(A + B)^t = A^t + B^t$ *Transpose of a sum*
2. $(cA)^t = cA^t$ *Transpose of a scalar multiple*
3. $(AB)^t = B^tA^t$ *Transpose of a product*
4. $(A^t)^t = A$



Symmetric Matrix

Definition

A **symmetric matrix** is a matrix that is equal to its transpose.

$$A = A^t, \quad \text{i.e., } a_{ij} = a_{ji} \quad \forall i, j$$

Example 16

$$\begin{bmatrix} 2 & 5 \\ 5 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 \\ 1 & 7 & 8 \\ -4 & 8 & -3 \end{bmatrix}$$

match

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 7 & 3 & 9 \\ -2 & 3 & 2 & -3 \\ 4 & 9 & -3 & 6 \end{bmatrix}$$

match



Remark: If and only if

- Let p and q be statements.
Suppose that p implies q (if p then q), written $p \Rightarrow q$,
and that also $q \Rightarrow p$, we say that

“ p if and only if q ” (in short **iff**)



Example 17

Let A and B be symmetric matrices of the same size. Prove that the product AB is symmetric if and only if $AB = BA$.

Proof

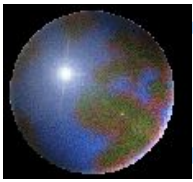
*We have to show (a) AB is symmetric $\Rightarrow AB = BA$,
and the converse, (b) AB is symmetric $\Leftarrow AB = BA$.

(\Rightarrow) Let AB be symmetric, then

$$\begin{aligned} AB &= (AB)^t && \text{by definition of symmetric matrix} \\ &= B^t A^t && \text{by Thm 2.4 (3)} \\ &= BA && \text{since } A \text{ and } B \text{ are symmetric} \end{aligned}$$

(\Leftarrow) Let $AB = BA$, then

$$\begin{aligned} (AB)^t &= (BA)^t \\ &= A^t B^t && \text{by Thm 2.4 (3)} \\ &= AB && \text{since } A \text{ and } B \text{ are symmetric} \end{aligned}$$

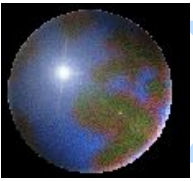


Example 18

Let A be a symmetric matrix. Prove that A^2 is symmetric.

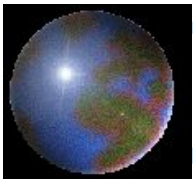
Proof

$$(A^2)^t = (AA)^t = (A^t A^t) = AA = A^2$$



Homework

- **Exercises** will be given by the teachers of the practical classes.



2.4 The Inverse of a Matrix

Definition

Let A be an $n \times n$ matrix. If a matrix B can be found such that $AB = BA = I_n$, then A is said to be **invertible** and B is called the **inverse** of A . If such a matrix B does not exist, then A has no inverse. (denote $B = A^{-1}$, and $A^{-k} = (A^{-1})^k$)

Example 19

Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has inverse $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Proof

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus $AB = BA = I_2$, proving that the matrix A has inverse B .



Theorem 2.5

The inverse of an invertible matrix is unique.

Proof

Let B and C be inverses of A .

Thus $AB = BA = I_n$, and $AC = CA = I_n$.

Multiply both sides of the equation $AB = I_n$ by C .

$$C(AB) = CI_n$$

$$(CA)B = C \quad \leftarrow \text{Thm2.2}$$

$$I_n B = C$$

$$B = C$$

Thus an invertible matrix has only one inverse.



Gauss-Jordan Elimination for finding the Inverse of a Matrix

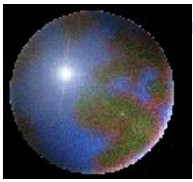
Let A be an $n \times n$ matrix.

1. Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
2. Compute the reduced echelon form of $[A : I_n]$.

If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A .

If the reduced echelon form is not of the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n , then A has no inverse.

An $n \times n$ matrix A is invertible if and only if its reduced echelon form is I_n .



Example 20

Determine the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Solution

$$[A : I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \text{R2} + (-2)\text{R1} \\ \text{R3} + \text{R1} \end{array} \approx \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} \\ (-1)\text{R2} \\ \\ \end{array} \approx \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \text{R1} + \text{R2} \\ \text{R3} + (-2)\text{R2} \end{array} \approx \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ \text{R1} + \text{R3} \\ \text{R2} + (-1)\text{R3} \end{array} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\text{Thus, } A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}.$$



Example 21

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

Solution

$$\begin{aligned} [A : I_3] &= \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \approx \\ R2 + (-1)R1 \\ R3 + (-2)R1 \end{array} \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix} \\ & \begin{array}{l} \approx \\ R1 + (-1)R2 \\ R3 + 3R2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{bmatrix} \end{aligned}$$

There is no need to proceed further.

The reduced echelon form cannot have a one in the (3, 3) location.

The reduced echelon form cannot be of the form $[I_n : B]$.

Thus A^{-1} does not exist.



Properties of Matrix Inverse

Let A and B be invertible matrices and c a nonzero scalar, Then

$$1. (A^{-1})^{-1} = A$$

$$4. (A^n)^{-1} = (A^{-1})^n$$

$$2. (cA)^{-1} = \frac{1}{c} A^{-1}$$

$$5. (A^t)^{-1} = (A^{-1})^t$$

$$3. (AB)^{-1} = B^{-1} A^{-1}$$

Proof

1. By definition,

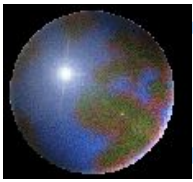
$$AA^{-1} = A^{-1}A = I$$

$$3. (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I = (B^{-1}A^{-1})(AB)$$

$$4. \underbrace{A \cdots A}_{n \text{ times}} \cdot \underbrace{A^{-1} \cdots A^{-1}}_{n \text{ times}} = I = (A^{-1})^n A^n$$

$$5. AA^{-1} = I, (AA^{-1})^t = \underline{(A^{-1})^t A^t} = I,$$

$$A^{-1}A = I, (A^{-1}A)^t = \underline{A^t (A^{-1})^t} = I,$$

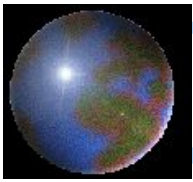


Example 22

If $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, then it can be shown that $A^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Use this information to compute $(A^t)^{-1}$.

Solution

$$(A^t)^{-1} = (A^{-1})^t = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$



Theorem 2.6

Let $AX = B$ be a system of n linear equations in n variables. If A^{-1} exists, the solution is unique and is given by $X = A^{-1}B$.

Proof

($X = A^{-1}B$ is a solution.)

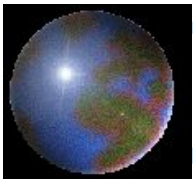
Substitute $X = A^{-1}B$ into the matrix equation.

$$AX = A(A^{-1}B) = (AA^{-1})B = I_n B = B.$$

(The solution is unique.)

Let Y be any solution, thus $AY = B$. Multiplying both sides of this equation by A^{-1} gives

$$\begin{aligned} A^{-1}A Y &= A^{-1}B \\ I_n Y &= A^{-1}B \\ Y &= A^{-1}B. \quad \text{Then } Y=X. \end{aligned}$$



Example 22

Solve the system of equations

$$\begin{aligned}x_1 - x_2 - 2x_3 &= 1 \\2x_1 - 3x_2 - 5x_3 &= 3 \\-x_1 + 3x_2 + 5x_3 &= -2\end{aligned}$$

Solution

This system can be written in the following matrix form:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

If the matrix of coefficients is invertible, the unique solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

This inverse has already been found in Example 20. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The unique solution is $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.



Elementary Matrices

Definition

An **elementary matrix** is one that can be obtained from the identity matrix I_n through a single elementary row operation.

Example 23

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

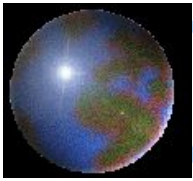
$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$5R_2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + 2R_1$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Elementary Matrices

- Elementary row operation
- Elementary matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$R2 \leftrightarrow R3$

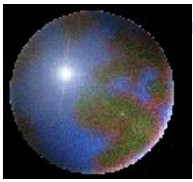
$$\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot A = E_1 A$$

$5R2$

$$\begin{bmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_2 A$$

$R2 + 2R1$

$$\begin{bmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_3 A$$



Notes for elementary matrices

- Each elementary matrix is invertible.

Example 24

$$I \underset{R1+2R2}{\approx} E_1 \Rightarrow E_1 \underset{R1-2R2}{\approx} I, \text{ i.e., } E_2 E_1 = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

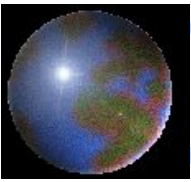
- If A and B are row equivalent matrices and A is invertible, then B is invertible.

Proof

If $A \approx \dots \approx B$, then

$B = E_n \dots E_2 E_1 A$ for some elementary matrices E_n, \dots, E_2 and E_1 .

So $B^{-1} = (E_n \dots E_2 E_1 A)^{-1} = A^{-1} E_1^{-1} E_2^{-1} \dots E_n^{-1}$.



Homework

- Exercises will be given by the teachers of the practical classes.

Exercise

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that $A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.