Дифференциальные операции первого порядка

$f(\vec{r}) = f(x, y, z)$ – скалярное поле $\quad \vec{u}(\vec{r}) = \{P(\vec{r}), Q(\vec{r}), R(\vec{r})\} = \{P(x,y,z), Q(x,y,z), R(x,y,z)\}$

векторное поле

Пусть $f, \vec{u} \in C_1$ в $V \subset E_3$. Тогда

$$\operatorname{grad} f = \left\{\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right\} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$ – градиент

$$\operatorname{div} \vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ – дивергенция

$$\operatorname{rot} \vec{u} = \left\{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}; \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}$$ – ротор (вихрь)

Введём $\nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ – оператор Гамильтона (набла)

Тогда $\operatorname{grad} f = \nabla f$, $\operatorname{div} \vec{u} = (\nabla, \vec{u})$, $\operatorname{rot} \vec{u} = [\nabla, \vec{u}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Системы координат – ортогональные и правые.

$\vec{i}, \vec{j}, \vec{k}$ – старый базис $\quad \vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} =$

$\vec{i}_1, \vec{i}_2, \vec{i}_3$ – новый базис $\quad = a_1 \vec{i}_1 + a_2 \vec{i}_2 + a_3 \vec{i}_3 \quad = x_1 \vec{i}_1 + x_2 \vec{i}_2 + x_3 \vec{i}_3$

$$T = (t_{ij}) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} -$$

матрица перехода от $\vec{i}, \vec{j}, \vec{k}$ к $\vec{i}_1, \vec{i}_2, \vec{i}_3$

это ортогон. матрица ($T^{-1} = T^T$), причём $\det T = 1$

$\vec{i}_1 = t_{11}\vec{i} + t_{21}\vec{j} + t_{31}\vec{k} \quad a_x = t_{11}a_1 + t_{12}a_2 + t_{13}a_3 \quad x = t_{11}x_1 + t_{12}x_2 + t_{13}x_3$

$\vec{i}_2 = t_{12}\vec{i} + t_{22}\vec{j} + t_{32}\vec{k} \quad a_y = t_{21}a_1 + t_{22}a_2 + t_{23}a_3 \quad y = t_{21}x_1 + t_{22}x_2 + t_{23}x_3$

$\vec{i}_3 = t_{13}\vec{i} + t_{23}\vec{j} + t_{33}\vec{k} \quad a_z = t_{31}a_1 + t_{32}a_2 + t_{33}a_3 \quad z = t_{31}x_1 + t_{32}x_2 + t_{33}x_3$

$\vec{i} = t_{11}\vec{i}_1 + t_{12}\vec{i}_2 + t_{13}\vec{i}_3 \quad a_1 = t_{11}a_x + t_{21}a_y + t_{31}a_z \quad x_1 = t_{11}x + t_{21}y + t_{31}z$

$\vec{j} = t_{21}\vec{i}_1 + t_{22}\vec{i}_2 + t_{23}\vec{i}_3 \quad a_2 = t_{12}a_x + t_{22}a_y + t_{32}a_z \quad x_2 = t_{12}x + t_{22}y + t_{32}z$

$\vec{k} = t_{31}\vec{i}_1 + t_{32}\vec{i}_2 + t_{33}\vec{i}_3 \quad a_3 = t_{13}a_x + t_{23}a_y + t_{33}a_z \quad x_3 = t_{13}x + t_{23}y + t_{33}z$

Системы координат – ортогональные и правые.

$\vec{i}, \vec{j}, \vec{k}$ – старый базис $\quad \vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} =$

$\vec{i}_1, \vec{i}_2, \vec{i}_3$ – новый базис $\quad = a_1 \vec{i}_1 + a_2 \vec{i}_2 + a_3 \vec{i}_3 \quad = x_1 \vec{i}_1 + x_2 \vec{i}_2 + x_3 \vec{i}_3$

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матрица перехода от $\vec{i}, \vec{j}, \vec{k}$ к $\vec{i}_1, \vec{i}_2, \vec{i}_3$ – это ортогон. матрица ($T^{-1} = T^T$), причём $\det T = 1$

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$\vec{i}_3 = t_{13}\vec{i} + t_{23}\vec{j} + t_{33}\vec{k} \quad a_z = t_{31}a_1 + t_{32}a_2 + t_{33}a_3 \quad z = t_{31}x_1 + t_{32}x_2 + t_{33}x_3$

$\vec{i} = t_{11}\vec{i}_1 + t_{12}\vec{i}_2 + t_{13}\vec{i}_3 \quad a_1 = t_{11}a_x + t_{21}a_y + t_{31}a_z \quad x_1 = t_{11}x + t_{21}y + t_{31}z$

$\vec{j} = t_{21}\vec{i}_1 + t_{22}\vec{i}_2 + t_{23}\vec{i}_3 \quad a_2 = t_{12}a_x + t_{22}a_y + t_{32}a_z \quad x_2 = t_{12}x + t_{22}y + t_{32}z$

$\vec{k} = t_{31}\vec{i}_1 + t_{32}\vec{i}_2 + t_{33}\vec{i}_3 \quad a_3 = t_{13}a_x + t_{23}a_y + t_{33}a_z \quad x_3 = t_{13}x + t_{23}y + t_{33}z$

$\nabla f = \operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$. Рассмотрим $\frac{\partial f}{\partial x_1}\vec{i}_1 + \frac{\partial f}{\partial x_2}\vec{i}_2 + \frac{\partial f}{\partial x_3}\vec{i}_3$

$$\frac{\partial f}{\partial x_1} = t_{11}\frac{\partial f}{\partial x} + t_{21}\frac{\partial f}{\partial y} + t_{31}\frac{\partial f}{\partial z}$$
$$\frac{\partial f}{\partial x_2} = t_{12}\frac{\partial f}{\partial x} + t_{22}\frac{\partial f}{\partial y} + t_{32}\frac{\partial f}{\partial z}$$
$$\frac{\partial f}{\partial x_3} = t_{13}\frac{\partial f}{\partial x} + t_{23}\frac{\partial f}{\partial y} + t_{33}\frac{\partial f}{\partial z}$$

координаты grad f меняются как к-ты любого вектора, т.е. grad f – инвар.

$$\left(\begin{aligned} \frac{\partial}{\partial x_1} &= t_{11}\frac{\partial}{\partial x} + t_{21}\frac{\partial}{\partial y} + t_{31}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_2} &= t_{12}\frac{\partial}{\partial x} + t_{22}\frac{\partial}{\partial y} + t_{32}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_3} &= t_{13}\frac{\partial}{\partial x} + t_{23}\frac{\partial}{\partial y} + t_{33}\frac{\partial}{\partial z} \end{aligned}\right)$$

$$\frac{\partial f}{\partial x_1} = t_{11}\frac{\partial f}{\partial x} + t_{21}\frac{\partial f}{\partial y} + t_{31}\frac{\partial f}{\partial z}$$
$$\frac{\partial f}{\partial x_2} = t_{12}\frac{\partial f}{\partial x} + t_{22}\frac{\partial f}{\partial y} + t_{32}\frac{\partial f}{\partial z}$$
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$$\vec{u} = P\vec{i} + Q\vec{j} + R\vec{k} = P_1\vec{i}_1 + P_2\vec{i}_2 + P_3\vec{i}_3$$

$$P_1 = t_{11}P + t_{21}Q + t_{31}R$$
$$P_2 = t_{12}P + t_{22}Q + t_{32}R$$
$$P_3 = t_{13}P + t_{23}Q + t_{33}R$$

$$\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \frac{\partial P_3}{\partial x_3} =$$

$$= \left[t_{11}\left(t_{11}\frac{\partial P}{\partial x} + t_{21}\frac{\partial P}{\partial y} + t_{31}\frac{\partial P}{\partial z}\right) + t_{21}\left(t_{11}\frac{\partial Q}{\partial x} + t_{21}\frac{\partial Q}{\partial y} + t_{31}\frac{\partial Q}{\partial z}\right) + t_{31}\left(t_{11}\frac{\partial R}{\partial x} + t_{21}\frac{\partial R}{\partial y} + t_{31}\frac{\partial R}{\partial z}\right)\right] +$$
$$+ \left[t_{12}\left(t_{12}\frac{\partial P}{\partial x} + t_{22}\frac{\partial P}{\partial y} + t_{32}\frac{\partial P}{\partial z}\right) + t_{22}\left(t_{12}\frac{\partial Q}{\partial x} + t_{22}\frac{\partial Q}{\partial y} + t_{32}\frac{\partial Q}{\partial z}\right) + t_{32}\left(t_{12}\frac{\partial R}{\partial x} + t_{22}\frac{\partial R}{\partial y} + t_{32}\frac{\partial R}{\partial z}\right)\right] +$$
$$+ \left[t_{13}\left(t_{13}\frac{\partial P}{\partial x} + t_{23}\frac{\partial P}{\partial y} + t_{33}\frac{\partial P}{\partial z}\right) + t_{23}\left(t_{13}\frac{\partial Q}{\partial x} + t_{23}\frac{\partial Q}{\partial y} + t_{33}\frac{\partial Q}{\partial z}\right) + t_{33}\left(t_{13}\frac{\partial R}{\partial x} + t_{23}\frac{\partial R}{\partial y} + t_{33}\frac{\partial R}{\partial z}\right)\right] =$$
$$= (t_{11}^2 + t_{12}^2 + t_{13}^2)\frac{\partial P}{\partial x} + (t_{11}t_{21} + t_{12}t_{22} + t_{13}t_{23})\frac{\partial P}{\partial y} + (t_{11}t_{31} + t_{12}t_{32} + t_{13}t_{33})\frac{\partial P}{\partial z} +$$
$$+ (t_{21}t_{11} + t_{22}t_{12} + t_{23}t_{13})\frac{\partial Q}{\partial x} + (t_{21}^2 + t_{22}^2 + t_{23}^2)\frac{\partial Q}{\partial y} + (t_{21}t_{31} + t_{22}t_{32} + t_{23}t_{33})\frac{\partial Q}{\partial z} +$$
$$+ (t_{31}t_{11} + t_{32}t_{12} + t_{33}t_{13})\frac{\partial R}{\partial x} + (t_{31}t_{21} + t_{32}t_{22} + t_{33}t_{23})\frac{\partial R}{\partial y} + (t_{31}^2 + t_{32}^2 + t_{33}^2)\frac{\partial R}{\partial z} =$$
$$= (T - \text{ортог. матрица}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}\vec{u}$$

Рассмотрим вектор $\begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ P_1 & P_2 & P_3 \end{vmatrix} =$

$$\begin{vmatrix} t_{11}\vec{i} + t_{21}\vec{j} + t_{31}\vec{k} & t_{12}\vec{i} + t_{22}\vec{j} + t_{32}\vec{k} & t_{13}\vec{i} + t_{23}\vec{j} + t_{33}\vec{k} \\ t_{11}\dfrac{\partial}{\partial x} + t_{21}\dfrac{\partial}{\partial y} + t_{31}\dfrac{\partial}{\partial z} & t_{12}\dfrac{\partial}{\partial x} + t_{22}\dfrac{\partial}{\partial y} + t_{32}\dfrac{\partial}{\partial z} & t_{13}\dfrac{\partial}{\partial x} + t_{23}\dfrac{\partial}{\partial y} + t_{33}\dfrac{\partial}{\partial z} \\ t_{11}P + t_{21}Q + t_{31}R & t_{12}P + t_{22}Q + t_{32}R & t_{13}P + t_{23}Q + t_{33}R \end{vmatrix} =$$

$$= \det\left\{ \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \right\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \cdot \det T = \operatorname{rot} \vec{u}$$

Итак, grad, div, rot не зависят от ортог. преоб. с-мы к-т