

Дифференциальные операции первого порядка

$f(\vec{r}) = f(x, y, z)$ - скалярное поле $\vec{u}(\vec{r}) = \{P(\vec{r}), Q(\vec{r}), R(\vec{r})\} = \{P(x, y, z), Q(x, y, z), R(x, y, z)\}$ - векторное поле

Пусть $f, \vec{u} \in C, \forall V \subset \mathbb{E}_3$. Тогда

$$\text{grad } f = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\} = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \quad - \text{градиент}$$

$$\text{div } \vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad - \text{дивергенция}$$

$$\text{rot } \vec{u} = \left\{ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} \quad - \text{ротор (вихрь)}$$

Введём $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона (набла)

$$\text{Тогда } \text{grad } f = \nabla f, \quad \text{div } \vec{u} = (\nabla, \vec{u}), \quad \text{rot } \vec{u} = [\nabla, \vec{u}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Μετασχηματισμοί κεραινοειδή - ορθογώνια και γνήσια.

$\vec{i}, \vec{j}, \vec{k}$ - παλαιά βάση
 $\vec{t}_1, \vec{t}_2, \vec{t}_3$ - νέα βάση

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = a_1 \vec{t}_1 + a_2 \vec{t}_2 + a_3 \vec{t}_3$$

$$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k} = x_1 \vec{t}_1 + x_2 \vec{t}_2 + x_3 \vec{t}_3$$

$$T = (t_{ij}) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

Ματρίτσα μεταφοράς από $\vec{i}, \vec{j}, \vec{k}$ κ $\vec{t}_1, \vec{t}_2, \vec{t}_3$

στο ορθόγων. μετασχηματισμό ($T^{-1} = T^T$), όπου $\det T = 1$

$$\begin{aligned} \vec{t}_1 &= t_{11} \vec{i} + t_{21} \vec{j} + t_{31} \vec{k} \\ \vec{t}_2 &= t_{12} \vec{i} + t_{22} \vec{j} + t_{32} \vec{k} \\ \vec{t}_3 &= t_{13} \vec{i} + t_{23} \vec{j} + t_{33} \vec{k} \end{aligned}$$

$$a_x = t_{11} a_1 + t_{12} a_2 + t_{13} a_3$$

$$a_y = t_{21} a_1 + t_{22} a_2 + t_{23} a_3$$

$$a_z = t_{31} a_1 + t_{32} a_2 + t_{33} a_3$$

$$x = t_{11} x_1 + t_{12} x_2 + t_{13} x_3$$

$$y = t_{21} x_1 + t_{22} x_2 + t_{23} x_3$$

$$z = t_{31} x_1 + t_{32} x_2 + t_{33} x_3$$

$$\vec{i} = t_{11} \vec{t}_1 + t_{12} \vec{t}_2 + t_{13} \vec{t}_3$$

$$\vec{j} = t_{21} \vec{t}_1 + t_{22} \vec{t}_2 + t_{23} \vec{t}_3$$

$$\vec{k} = t_{31} \vec{t}_1 + t_{32} \vec{t}_2 + t_{33} \vec{t}_3$$

$$a_1 = t_{11} a_x + t_{21} a_y + t_{31} a_z$$

$$a_2 = t_{12} a_x + t_{22} a_y + t_{32} a_z$$

$$a_3 = t_{13} a_x + t_{23} a_y + t_{33} a_z$$

$$x_1 = t_{11} x + t_{21} y + t_{31} z$$

$$x_2 = t_{12} x + t_{22} y + t_{32} z$$

$$x_3 = t_{13} x + t_{23} y + t_{33} z$$

Сферическая координата - сферическая и правая.

$\vec{i}, \vec{j}, \vec{k}$ - старый базис
 $\vec{t}_1, \vec{t}_2, \vec{t}_3$ - новый базис

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = a_1 \vec{t}_1 + a_2 \vec{t}_2 + a_3 \vec{t}_3$$

$$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k} = x_1 \vec{t}_1 + x_2 \vec{t}_2 + x_3 \vec{t}_3$$

$$T = (t_{ij}) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

матрица перехода от $\vec{i}, \vec{j}, \vec{k}$ к $\vec{t}_1, \vec{t}_2, \vec{t}_3$

это ортонорм. матрица ($T^{-1} = T^T$), причем $\det T = 1$

$$\begin{aligned} \vec{t}_1 &= t_{11} \vec{i} + t_{21} \vec{j} + t_{31} \vec{k} \\ \vec{t}_2 &= t_{12} \vec{i} + t_{22} \vec{j} + t_{32} \vec{k} \\ \vec{t}_3 &= t_{13} \vec{i} + t_{23} \vec{j} + t_{33} \vec{k} \end{aligned}$$

$$a_x = t_{11} a_1 + t_{12} a_2 + t_{13} a_3$$

$$x = t_{11} x_1 + t_{12} x_2 + t_{13} x_3$$

$$a_y = t_{21} a_1 + t_{22} a_2 + t_{23} a_3$$

$$y = t_{21} x_1 + t_{22} x_2 + t_{23} x_3$$

$$a_z = t_{31} a_1 + t_{32} a_2 + t_{33} a_3$$

$$z = t_{31} x_1 + t_{32} x_2 + t_{33} x_3$$

$$\vec{i} = t_{11} \vec{t}_1 + t_{12} \vec{t}_2 + t_{13} \vec{t}_3$$

$$a_1 = t_{11} a_x + t_{21} a_y + t_{31} a_z$$

$$x_1 = t_{11} x + t_{21} y + t_{31} z$$

$$\vec{j} = t_{21} \vec{t}_1 + t_{22} \vec{t}_2 + t_{23} \vec{t}_3$$

$$a_2 = t_{12} a_x + t_{22} a_y + t_{32} a_z$$

$$x_2 = t_{12} x + t_{22} y + t_{32} z$$

$$\vec{k} = t_{31} \vec{t}_1 + t_{32} \vec{t}_2 + t_{33} \vec{t}_3$$

$$a_3 = t_{13} a_x + t_{23} a_y + t_{33} a_z$$

$$x_3 = t_{13} x + t_{23} y + t_{33} z$$

$$\nabla f = \text{grad} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

Рассмотрим $\frac{\partial f}{\partial x_1} \vec{t}_1 + \frac{\partial f}{\partial x_2} \vec{t}_2 + \frac{\partial f}{\partial x_3} \vec{t}_3$

$$\frac{\partial f}{\partial x_1} = t_{11} \frac{\partial f}{\partial x} + t_{21} \frac{\partial f}{\partial y} + t_{31} \frac{\partial f}{\partial z}$$

координаты $\text{grad} f$ не-изменяются как

$$\frac{\partial f}{\partial x_2} = t_{12} \frac{\partial f}{\partial x} + t_{22} \frac{\partial f}{\partial y} + t_{32} \frac{\partial f}{\partial z}$$

к-той мере

$$\frac{\partial f}{\partial x_3} = t_{13} \frac{\partial f}{\partial x} + t_{23} \frac{\partial f}{\partial y} + t_{33} \frac{\partial f}{\partial z}$$

вектора, т.е. $\text{grad} f$ -инвар.

$$\left(\begin{aligned} \frac{\partial}{\partial x_1} &= t_{11} \frac{\partial}{\partial x} + t_{21} \frac{\partial}{\partial y} + t_{31} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_2} &= t_{12} \frac{\partial}{\partial x} + t_{22} \frac{\partial}{\partial y} + t_{32} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_3} &= t_{13} \frac{\partial}{\partial x} + t_{23} \frac{\partial}{\partial y} + t_{33} \frac{\partial}{\partial z} \end{aligned} \right)$$

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= t_{11} \frac{\partial f}{\partial x} + t_{21} \frac{\partial f}{\partial y} + t_{31} \frac{\partial f}{\partial z} && \text{координаты} \\ \frac{\partial f}{\partial x_2} &= t_{12} \frac{\partial f}{\partial x} + t_{22} \frac{\partial f}{\partial y} + t_{32} \frac{\partial f}{\partial z} && \text{град } f \text{ ме-} \\ \frac{\partial f}{\partial x_3} &= t_{13} \frac{\partial f}{\partial x} + t_{23} \frac{\partial f}{\partial y} + t_{33} \frac{\partial f}{\partial z} && \text{нутася как} \\ &&& \text{к-ти метода} \\ &&& \text{вектора, т.е.} \\ &&& \text{град } f \text{-инвар.} \end{aligned} \quad \left(\begin{aligned} \frac{\partial}{\partial x_1} &= t_{11} \frac{\partial}{\partial x} + t_{21} \frac{\partial}{\partial y} + t_{31} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_2} &= t_{12} \frac{\partial}{\partial x} + t_{22} \frac{\partial}{\partial y} + t_{32} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x_3} &= t_{13} \frac{\partial}{\partial x} + t_{23} \frac{\partial}{\partial y} + t_{33} \frac{\partial}{\partial z} \end{aligned} \right)$$

$$\vec{u} = P\vec{i} + Q\vec{j} + R\vec{k} = P_1\vec{i}_1 + P_2\vec{i}_2 + P_3\vec{i}_3$$

$$\begin{aligned} P_1 &= t_{11}P + t_{21}Q + t_{31}R \\ P_2 &= t_{12}P + t_{22}Q + t_{32}R \\ P_3 &= t_{13}P + t_{23}Q + t_{33}R \end{aligned}$$

$$\begin{aligned} &\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \frac{\partial P_3}{\partial x_3} = \\ &= \left[t_{11} \left(t_{11} \frac{\partial P}{\partial x} + t_{21} \frac{\partial P}{\partial y} + t_{31} \frac{\partial P}{\partial z} \right) + t_{21} \left(t_{11} \frac{\partial Q}{\partial x} + t_{21} \frac{\partial Q}{\partial y} + t_{31} \frac{\partial Q}{\partial z} \right) + t_{31} \left(t_{11} \frac{\partial R}{\partial x} + t_{21} \frac{\partial R}{\partial y} + t_{31} \frac{\partial R}{\partial z} \right) \right] + \\ &+ \left[t_{12} \left(t_{12} \frac{\partial P}{\partial x} + t_{22} \frac{\partial P}{\partial y} + t_{32} \frac{\partial P}{\partial z} \right) + t_{22} \left(t_{12} \frac{\partial Q}{\partial x} + t_{22} \frac{\partial Q}{\partial y} + t_{32} \frac{\partial Q}{\partial z} \right) + t_{32} \left(t_{12} \frac{\partial R}{\partial x} + t_{22} \frac{\partial R}{\partial y} + t_{32} \frac{\partial R}{\partial z} \right) \right] + \\ &+ \left[t_{13} \left(t_{13} \frac{\partial P}{\partial x} + t_{23} \frac{\partial P}{\partial y} + t_{33} \frac{\partial P}{\partial z} \right) + t_{23} \left(t_{13} \frac{\partial Q}{\partial x} + t_{23} \frac{\partial Q}{\partial y} + t_{33} \frac{\partial Q}{\partial z} \right) + t_{33} \left(t_{13} \frac{\partial R}{\partial x} + t_{23} \frac{\partial R}{\partial y} + t_{33} \frac{\partial R}{\partial z} \right) \right] = \\ &= (t_{11}^2 + t_{12}^2 + t_{13}^2) \frac{\partial P}{\partial x} + (t_{11}t_{21} + t_{12}t_{22} + t_{13}t_{23}) \frac{\partial P}{\partial y} + (t_{11}t_{31} + t_{12}t_{32} + t_{13}t_{33}) \frac{\partial P}{\partial z} + \\ &+ (t_{21}t_{11} + t_{22}t_{12} + t_{23}t_{13}) \frac{\partial Q}{\partial x} + (t_{21}^2 + t_{22}^2 + t_{23}^2) \frac{\partial Q}{\partial y} + (t_{21}t_{31} + t_{22}t_{32} + t_{23}t_{33}) \frac{\partial Q}{\partial z} + \\ &+ (t_{31}t_{11} + t_{32}t_{12} + t_{33}t_{13}) \frac{\partial R}{\partial x} + (t_{31}t_{21} + t_{32}t_{22} + t_{33}t_{23}) \frac{\partial R}{\partial y} + (t_{31}^2 + t_{32}^2 + t_{33}^2) \frac{\partial R}{\partial z} = \\ &= (\text{T-опер. матрица}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{div } \vec{u} \end{aligned}$$

Рассмотрим вектор

$$\begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ P_1 & P_2 & P_3 \end{vmatrix} =$$

$$\begin{vmatrix} t_{11}\vec{i} + t_{21}\vec{j} + t_{31}\vec{k} & t_{12}\vec{i} + t_{22}\vec{j} + t_{32}\vec{k} & t_{13}\vec{i} + t_{23}\vec{j} + t_{33}\vec{k} \\ t_{11}\frac{\partial}{\partial x} + t_{21}\frac{\partial}{\partial y} + t_{31}\frac{\partial}{\partial z} & t_{12}\frac{\partial}{\partial x} + t_{22}\frac{\partial}{\partial y} + t_{32}\frac{\partial}{\partial z} & t_{13}\frac{\partial}{\partial x} + t_{23}\frac{\partial}{\partial y} + t_{33}\frac{\partial}{\partial z} \\ t_{11}P + t_{21}Q + t_{31}R & t_{12}P + t_{22}Q + t_{32}R & t_{13}P + t_{23}Q + t_{33}R \end{vmatrix} =$$

$$= \det \left\{ \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \right\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \cdot \det T = \text{rot } \vec{U}$$

Итак, grad, div, rot не забываются и опр. преоб. с метр. к-т