Graph theory

Irina Prosvirnina

- Definitions and examples
- Paths and cycles

Definitions and examples

- Although generally regarded as one of the more modern branches of mathematics, **graph theory** actually dates back to 1736.

In that year Leonhard Euler published the first paper on what is now called graph theory. In the paper, Euler developed a theory which solved the so-called Königsberg Bridge problem.

Definitions and examples

Euler (1707 – 1783) was born in Switzerland and spent most of his long life in Russia (St Petersburg) and Prussia (Berlin).

He was the most prolific mathematician of all time, his collected works filling more than 70 volumes.

Definitions and examples

Like many of the very great mathematicians of his era, Euler contributed to almost every branch of pure and applied mathematics.

He is also responsible, more than any other person, for much of the mathematical notation in use today.

Definitions and examples

- What is a ‘graph’? Intuitively, a graph is simply a collection of points, called ‘vertices’, and a collection of lines, called ‘edges’, each of which joins either a pair of points or a single point to itself.
- A familiar example, which serves as a useful analogy, is a road map which shows towns as vertices and the roads joining them as edges.

Definitions and examples

Definition 1

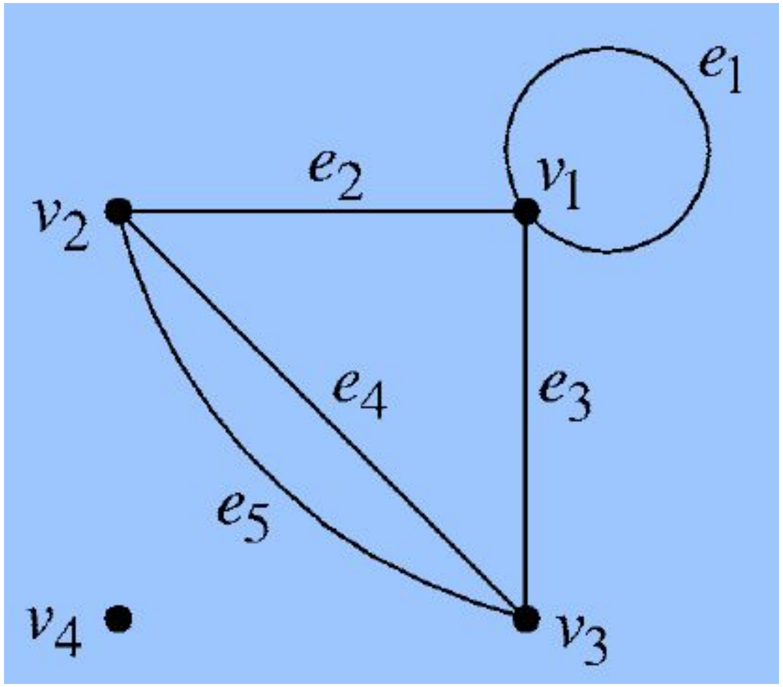
An **undirected graph** comprises:

- a finite non-empty set V of **vertices**,
- a finite set E of **edges**, and
- a function $\delta : E \rightarrow \mathcal{P}(V)$ such that, for every edge e, $\delta(e)$ is a one- or two-element subset of V.

The edge e is said to **join** the element(s) of $\delta(e)$.

An undirected graph is **simple** if there are no loops and multiple edges.

Definitions and examples



- Consider, for example, the graph Γ represented in the figure. Clearly Γ has vertex set $\{v_1, v_2, v_3\ , v_4\}$ and edges set $\{e_1, e_2, e_3, e_4, e_5\}$.

The function $\delta : E \rightarrow \mathcal{P}(V)$ is defined by

$\delta : e_1 \longmapsto \{v_1\}$

$\delta : e_2 \longmapsto \{v_1, v_2\}$

$\delta : e_3 \longmapsto \{v_1, v_3\}$

$\delta : e_4 \longmapsto \{v_2, v_3\}$

$\delta : e_5 \longmapsto \{v_2, v_3\}$.

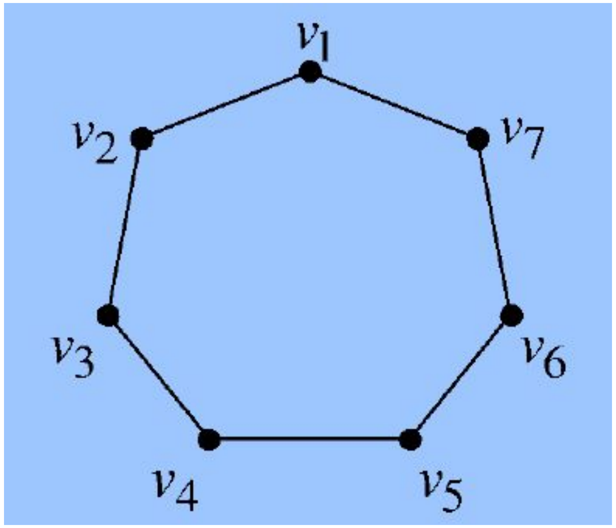
Definitions and examples

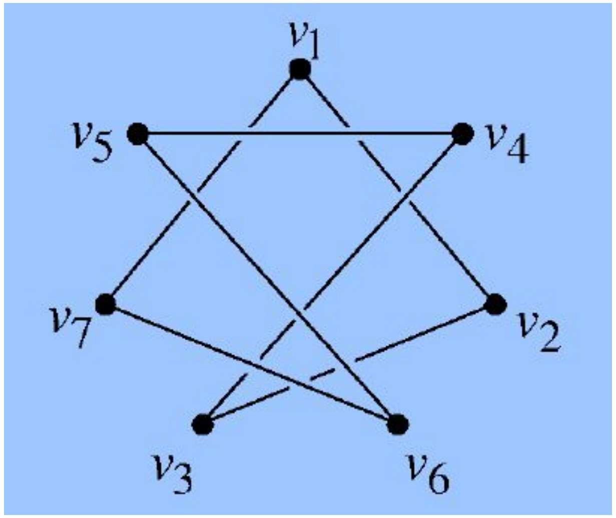
- We should emphasize that a graph and a diagram representing it are not the same thing.

A given graph may be represented by two diagrams which appear very different.

For instance, the two diagrams in the figure represent the same graph as can be observed by writing down the function

$\delta : E \rightarrow \mathcal{P}(V)$





Definitions and examples

Definition 2

- A pair of vertices v and w are **adjacent** if there exists an edge joining them. In this case we say both v and w are **incident** to e and also that e is **incident** to v and to w.
- The edges $\{e_1, e_2, \ldots, e_n\}$ are **adjacent** if they have at least one vertex in common.

Definitions and examples

Definition 2

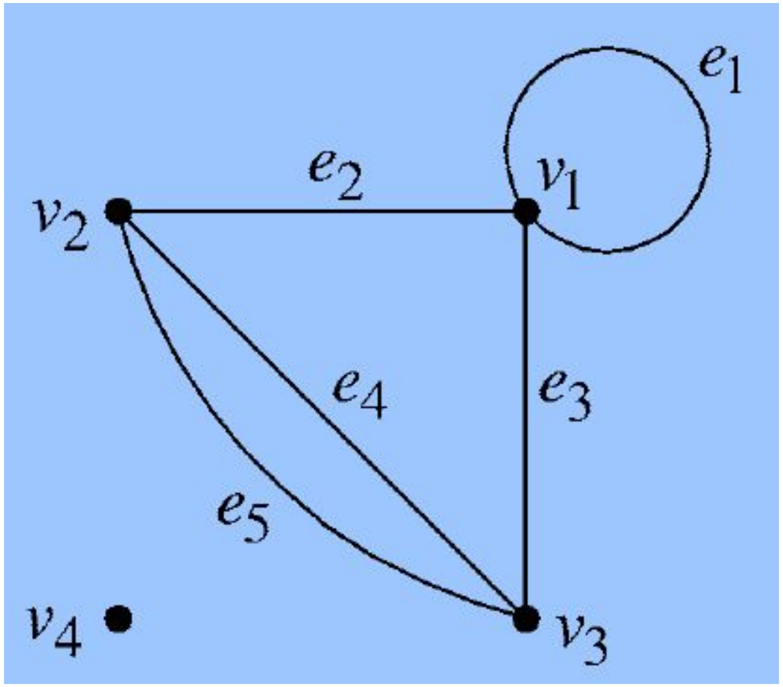
- The **degree** or **valency,** deg(v), of a vertex v is the number of edges which are incident to v. (Unless stated otherwise, a loop joining v to itself counts two towards the degree of v.)
- A graph in which every vertex has the same degree r is called **regular** (with degree r) or simply r-**regular**.

Definitions and examples

Definition 2

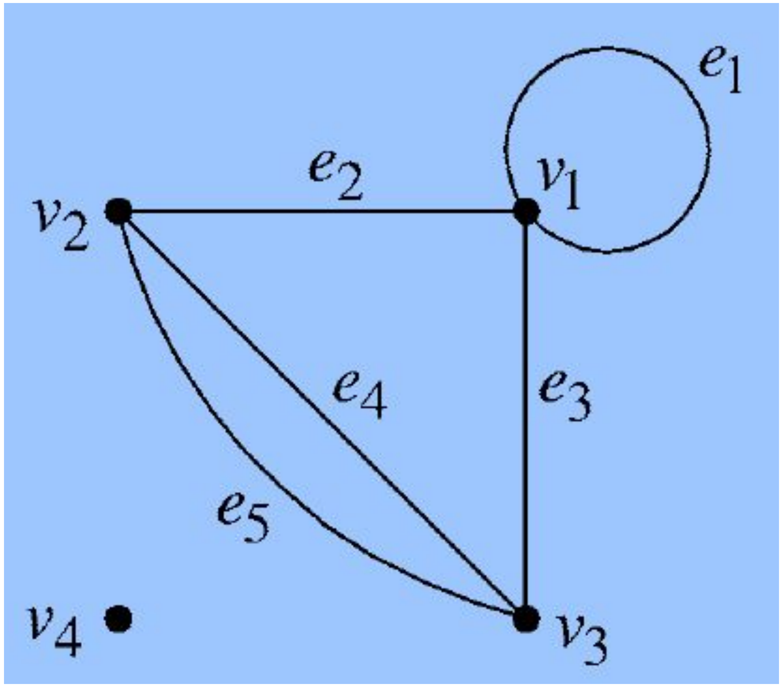
- The **degree sequence** of a graph is the sequence of its vertex degrees arranged in non-decreasing order.

Definitions and examples



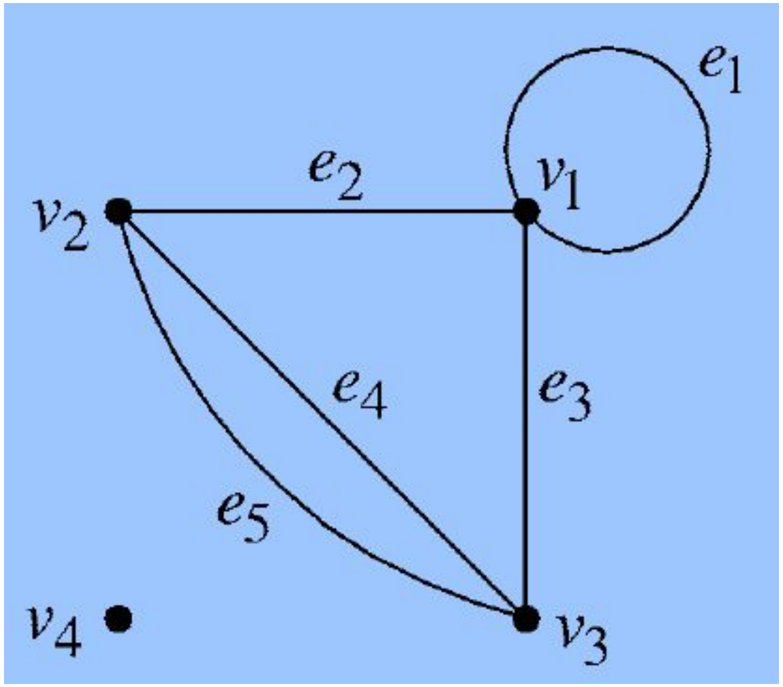
- The vertices v_1 and v_2 are adjacent, because the edge e_2 joins them.
- Similarly v_1 and v_3 are adjacent, as are v_2 and v_3.
- The vertex v_4 is adjacent to no other vertex.

Definitions and examples



- Edges e_1, e_2 and e_3 are adjacent, since they all meet at vertex v_1.
- Similarly e_2, e_4, e_5 are adjacent, as are e_3, e_4, e_5.

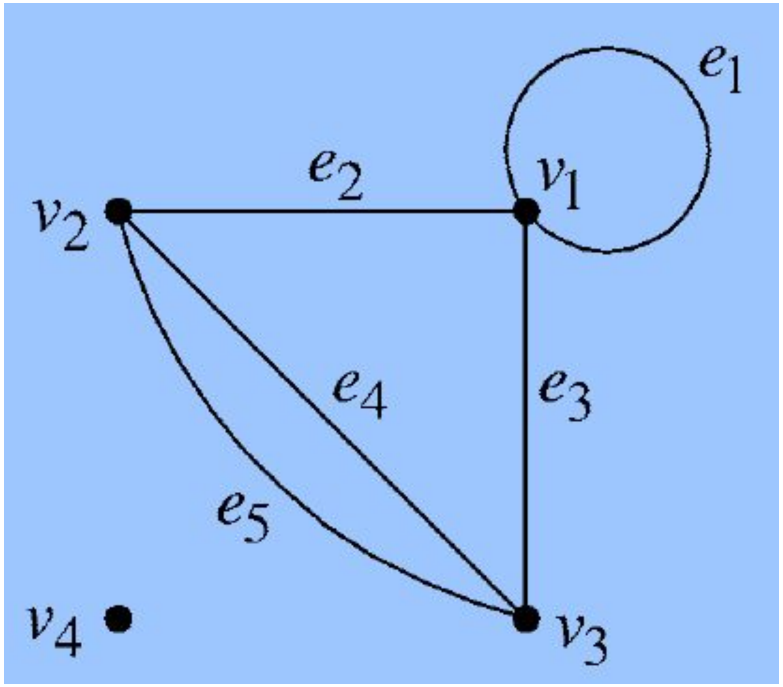
Definitions and examples



The degrees of the four vertices are given in the following table.

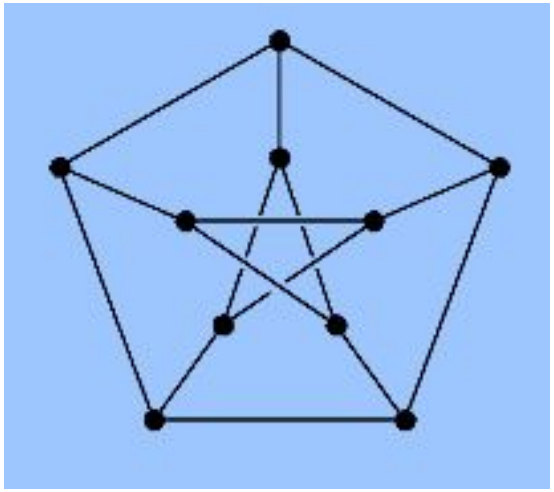
Vertex	Degree
	4
	3
	3
	0

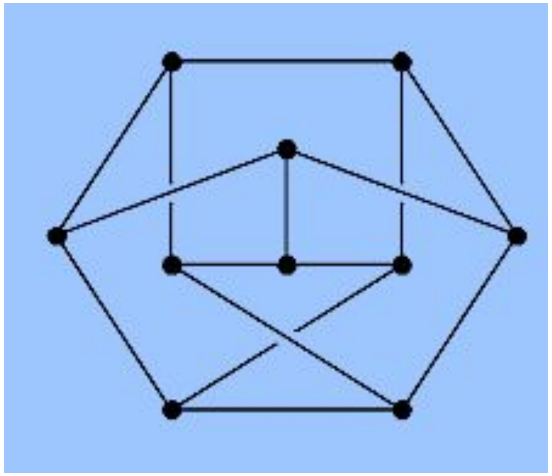
Definitions and examples



- The degree sequence of the graph is $(0, 3, 3, 4)$.

Definitions and examples





- A well known 3-regular simple graph is Peterson’s graph. Two diagrams representing this graph are given in the figure.
- In drawing diagrams of graphs we only allow edges to meet at vertices. It is not always possible to draw diagrams in the plane satisfying this property, so we may need to indicate that one edge passes underneath another as we have done in the figure.

Definitions and examples

Definition 3

- A **null graph** (or **totally disconnected graph**) is one whose edge set is empty. (Pictorially, a null graph is just a collection of points.)
- A **complete graph** is a simple graph in which every pair of distinct vertices is joined by an edge.
- A **bipartite graph** is a graph where the vertex set has a partition $\{V_1, V_2\}$ such that every edge joins a vertex of V_1 to a vertex of V_2.
- A **complete bipartite graph** is a bipartite graph such that every vertex of V_1 is joined to every vertex of V_2 by a unique edge.

Definitions and examples

<u>Example 1</u>

- Since a complete graph is simple there are no loops and each pair of distinct vertices is joined by a unique edge. Clearly a complete graph is uniquely specified by the number of its vertices.

Definitions and examples

Example 1

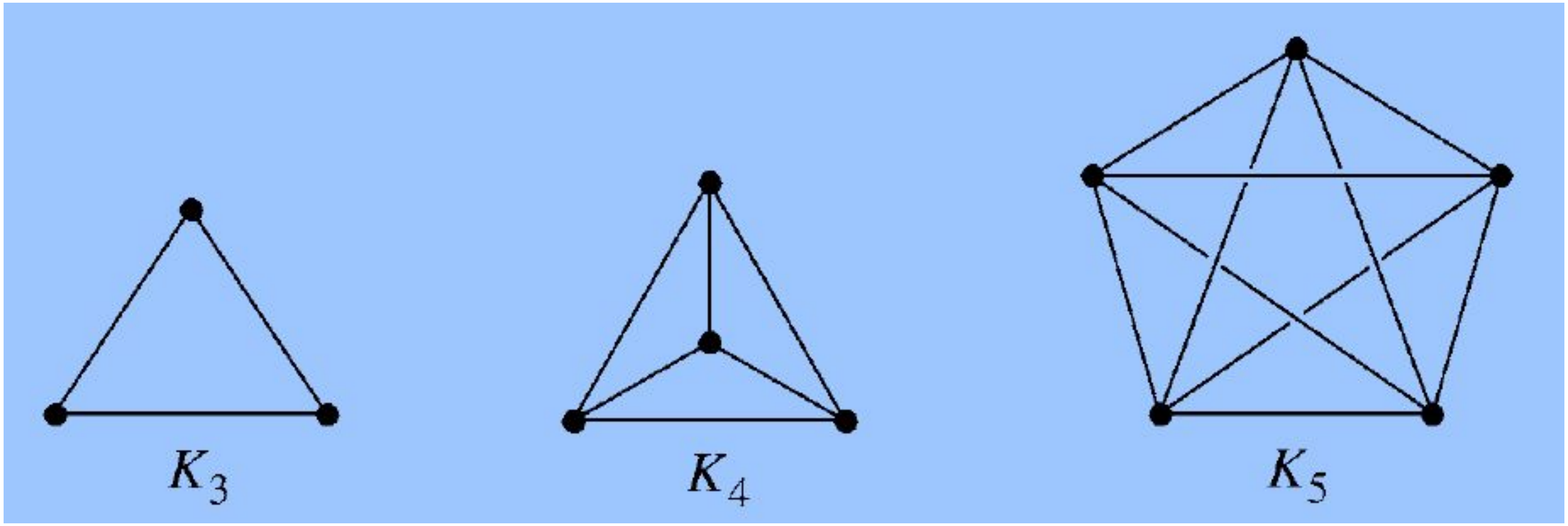
- The complete graph K_n with n vertices can be described as follows.

  It has vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_{ij}: 1 \le i < j \le n\}$ with the function δ given by $\delta(e_{ij}) = \{v_i, v_j\}$.
- The graph K_n is clearly regular with degree $n - 1$, since every vertex is connected, by a unique edge, to each of the other $n - 1$ vertices.

Definitions and examples

Example 1

- The complete graphs with three, four and five vertices are illustrated in the figure.



Definitions and examples

Example 2

- Let Γ be a bipartite graph where the vertex set V has the partition $\{V_1, V_2\}$.

  Note that Γ need not be simple. All that is required is that each edge must join a vertex of V_1 to a vertex of V_2. Given $v_1 \in V_1$ and $v_2 \in V_2$, there may be more than one edge joining them or no edge joining them.

  Clearly, though, there are no loops in Γ.

Definitions and examples

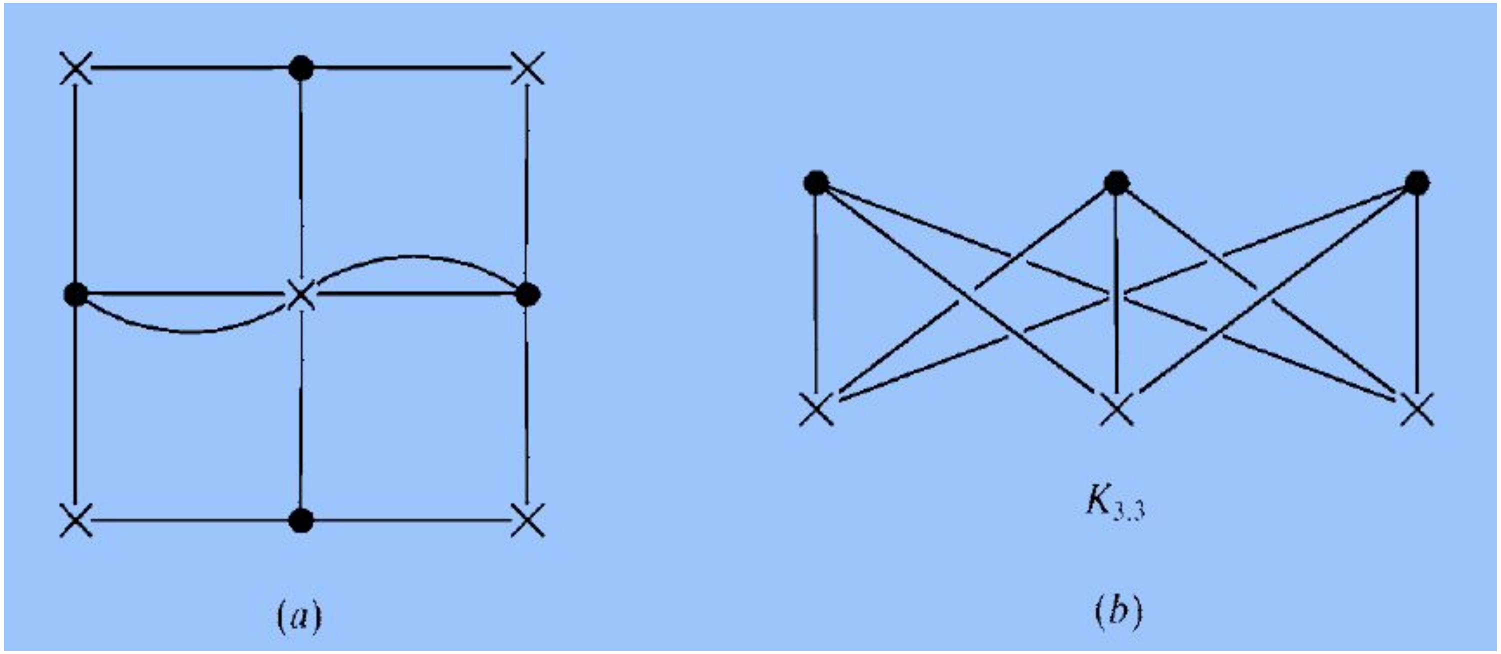
Example 2

- A complete bipartite graph is completely specified by $|V_1|$ and $|V_2|$. The **complete bipartite graph on n and m vertices**, denoted $K_{n,m}$, has $|V_1| = n$ and $|V_2| = m$. It is necessarily simple.

Definitions and examples

Example 2

- The figure shows two bipartite graphs. In each case the vertices of V_1 are indicated by full circles and the vertices of V_2 by crosses. The graph in (b) is the complete bipartite graph, $K_{3,3}$.



Definitions and examples

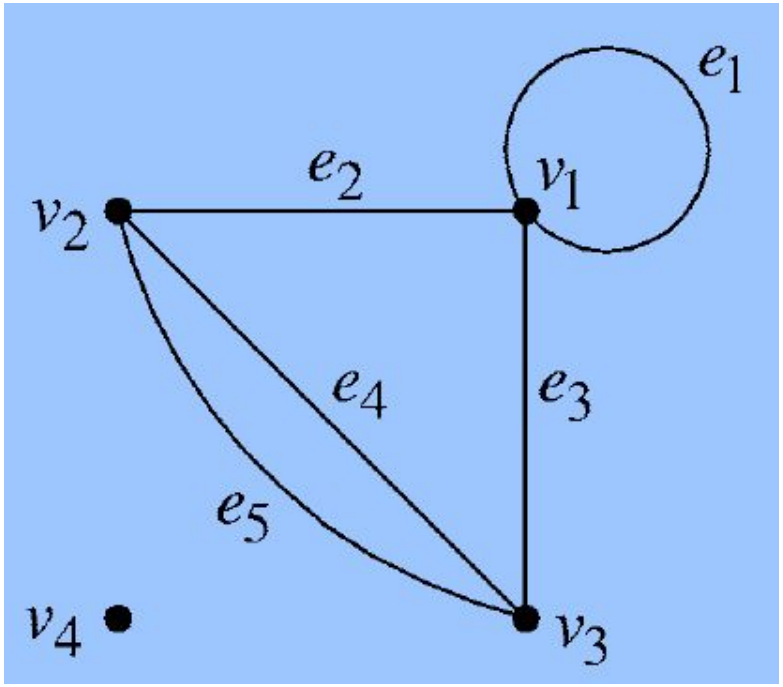
Definition 4

Let Γ be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$. The **adjacency matrix** of Γ is the $n \times n$ matrix $A = A(\Gamma)$ such that a_{ij} is the number of distinct edges joining v_i and v_j.

Definitions and examples

- The adjacency matrix is necessarily **symmetric** as the number of edges joining v_i and v_j is the same as the number joining v_j and v_i.
- The degree of vertex v_i is easily determined from the adjacency matrix.
- If there are no loops at v_i then its degree is the sum of the entries in the ith column (or ith row) of the matrix.
- Since every loop counts twice in the degree, when summing the entries in the ith column (or ith row) the diagonal element a_{ii} must be doubled to obtain the degree of v_i.

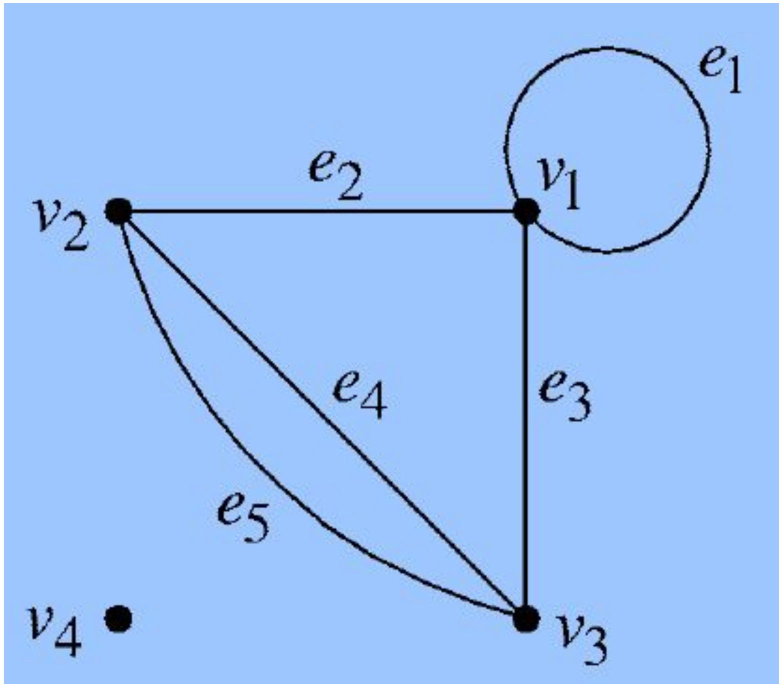
Definitions and examples



- The following is the adjacency matrix A of the graph represented in the figure

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definitions and examples



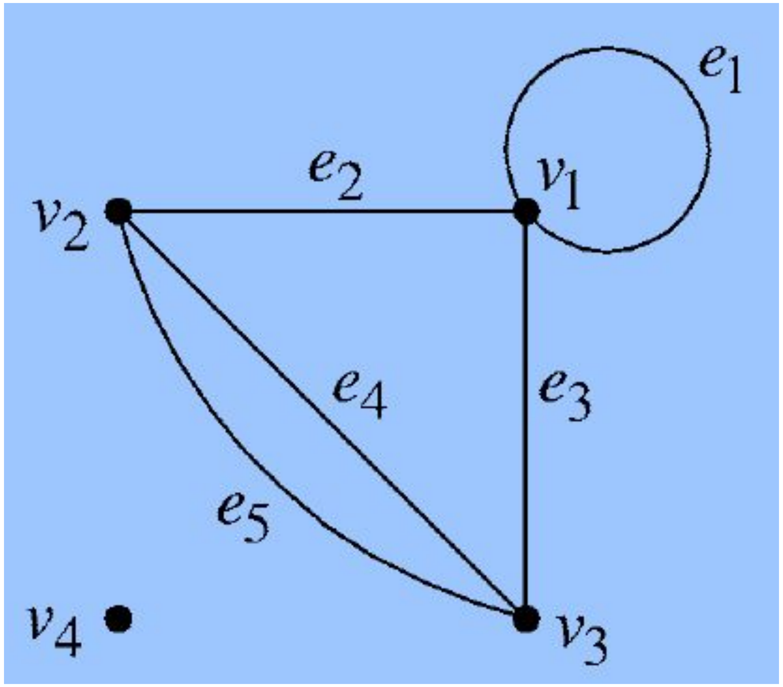
- $$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that

$V = \{v_1, v_2, v_3, v_4\}$

and the rows and columns of A refer to the vertices in the order listed.

Definitions and examples

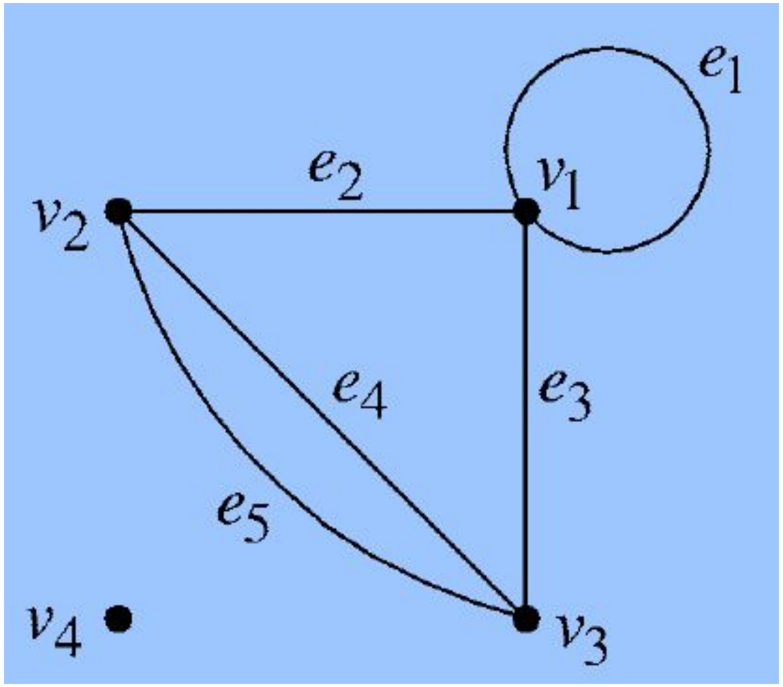


- $$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two properties of the graph are immediately apparent from the matrix.

Firstly, by considering the leading diagonal we note that there is only one loop – from v_1 to itself.

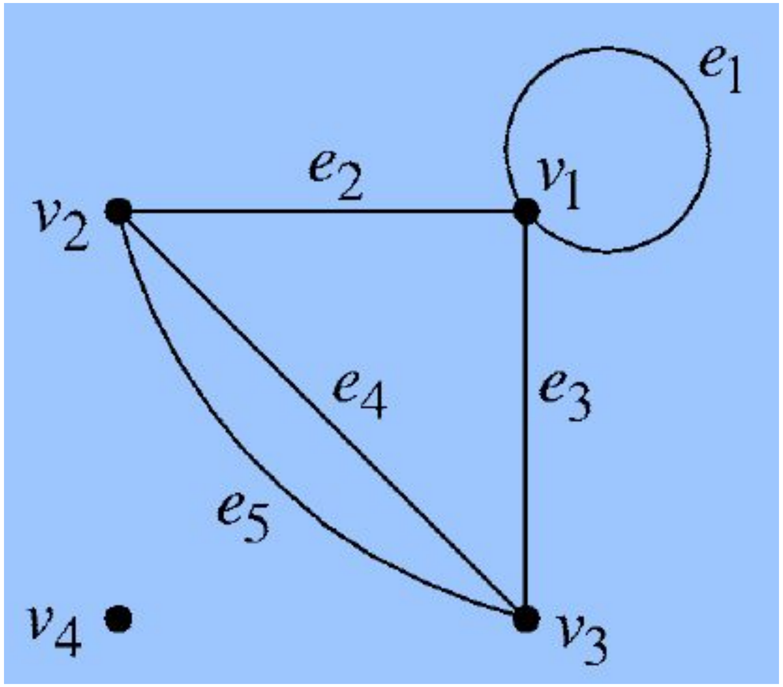
Definitions and examples



- $$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Secondly, the last row (or column) of zeros indicates that v_4 is an **isolated vertex** connected to no vertices at all (including itself).

Definitions and examples



- $$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The degrees of the vertices are easily calculated from the matrix as follows:

$\deg(v_1) = 2 \times 1 + 1 + 1 = 4$

$\deg(v_2) = 1 + 2 = 3$

$\deg(v_3) = 1 + 2 = 3$

$\deg(v_4) = 0.$

Definitions and examples

Example 3

The null graph with n vertices has the $n \times n$ zero matrix $O_{n \times n}$ as its adjacency matrix, since there are no edges whatsoever.

Definitions and examples

<u>Example 4</u>

A complete graph has adjacency matrix with zeros along the leading diagonal (since there are no loops) and ones everywhere else (since every vertex is joined to every other by a unique edge).

Definitions and examples

Definition 5

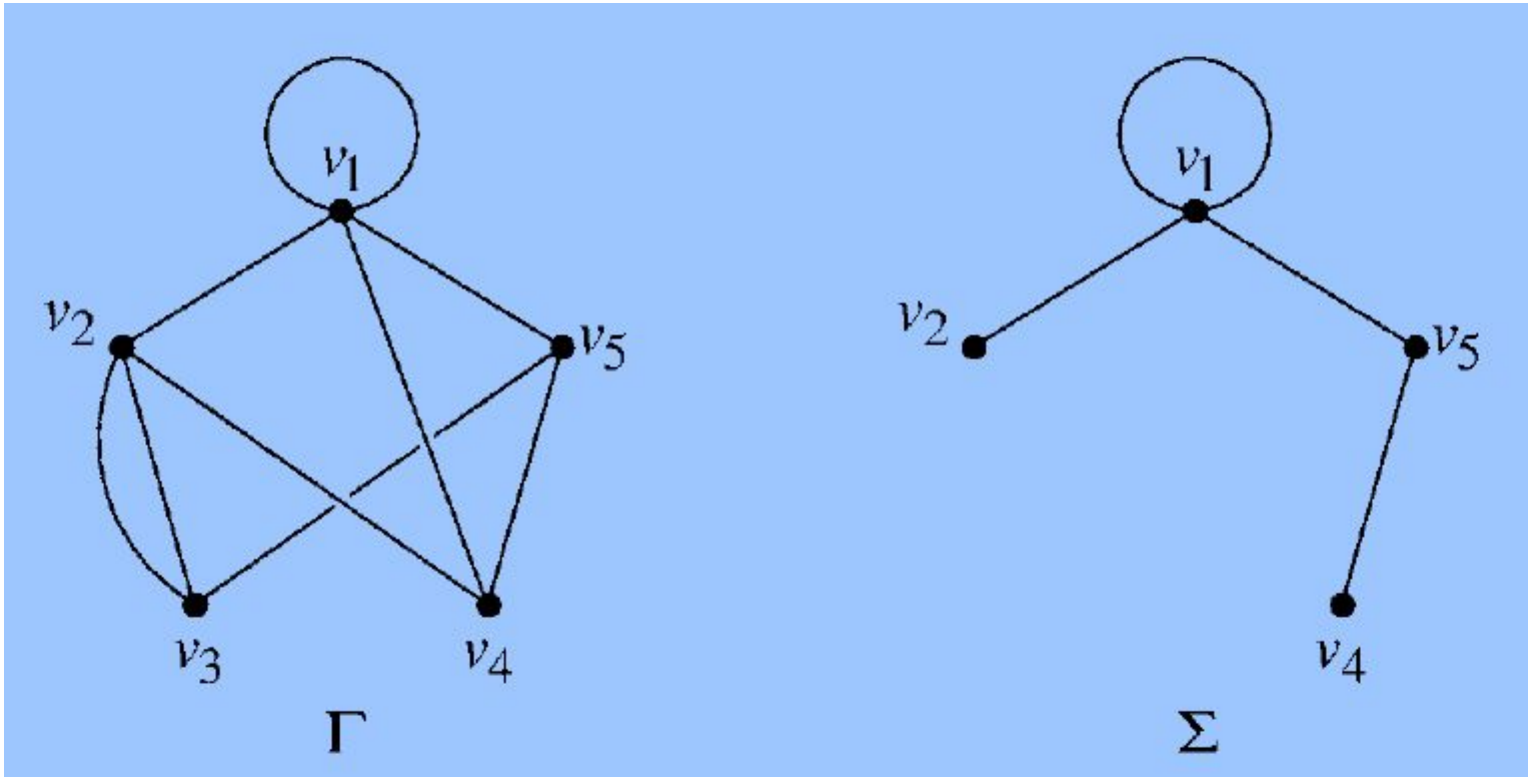
A graph Σ is a **subgraph** of the graph Γ, denoted $\Sigma \leq \Gamma$, if $V_\Sigma \subseteq V_\Gamma$, $E_\Sigma \subseteq E_\Gamma$ and $\delta_\Sigma(e) = \delta_\Gamma(e)$, for every edge e of Σ.

The condition that $\delta_\Sigma(e) = \delta_\Gamma(e)$, for every edge e of Σ, just means that the edges of the subgraph Σ must join the same vertices as they do in Γ. Intuitively, Σ is a subgraph of Γ if we can obtain a diagram for Σ by erasing some of the vertices and/or edges from a diagram of Γ. Of course, if we erase a vertex we must also erase all edges incident to it.

Definitions and examples

<u>Example 5</u>

We can regard Σ as a subgraph of Γ.



Paths and cycles

- Using the analogy of a road map, we can consider various types of ‘journeys’ in a graph.
- For instance, if the graph actually represents a network of roads connecting various towns, one question we might ask is: is there a journey, beginning and ending at the same town, which visits every other town just once without traversing the same road more than once?
- As usual, we begin with some definitions.

Paths and cycles

Definition 6

- An **edge sequence of length n** in a graph Γ is a sequence of (not necessarily distinct) edges $e_1, e_2, \dots, e_n$ such that e_i and e_{i+1} are adjacent for $i = 1, 2, \dots, n-1$. The edge sequence determines a sequence of vertices (again, not necessarily distinct) $v_0, v_1, v_2, \dots, v_{n-1}, v_n$ where $\delta(e_i) = \{v_{i-1}, v_i\}$. We say v_0 is the **initial vertex** and v_n the **final vertex** of the edge sequence.

Paths and cycles

Definition 6

- A **path** is an edge sequence in which all the edges are distinct. If in addition all the vertices are distinct (except possibly $v_0 = v_n$) the path is called **simple**.

Paths and cycles

Definition 6

- An edge sequence is **closed** if $v_0 = v_n$. A closed simple path containing at least one edge is called a **cycle** or a **circuit**.

Paths and cycles

An edge sequence is any finite sequence of edges which can be traced on the diagram of the graph without removing pen from paper. It may repeat edges, go round loops several times, etc.

Paths and cycles

Edge sequences are too general to be of very much use which is why we have defined paths.

Paths and cycles

In a path we are not allowed to ‘travel along’ the same edge more than once.

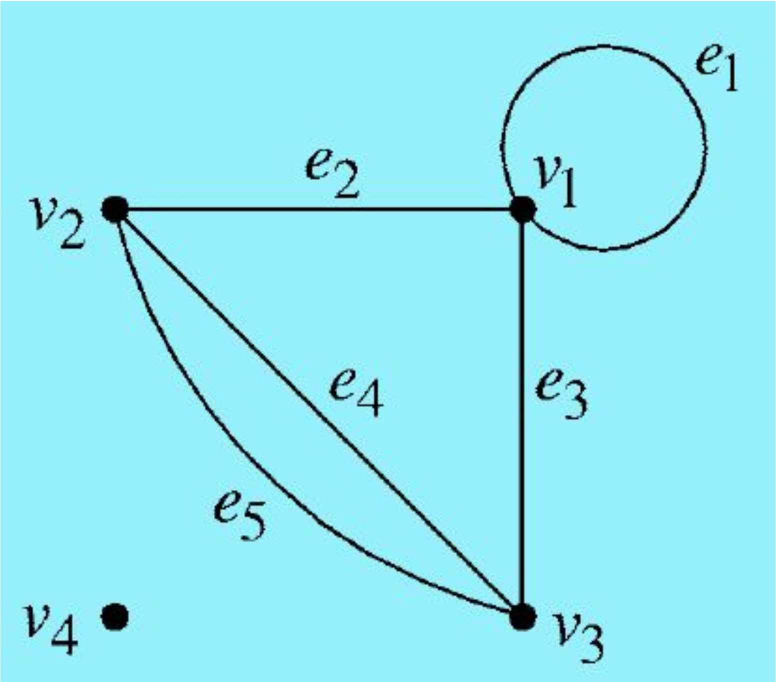
Paths and cycles

If, in addition, we do not ‘visit’ the same vertex more than once (which rules out loops), then the path is simple.

Paths and cycles

The edge sequence or path is closed if we begin and end the ‘journey’ at the same place.

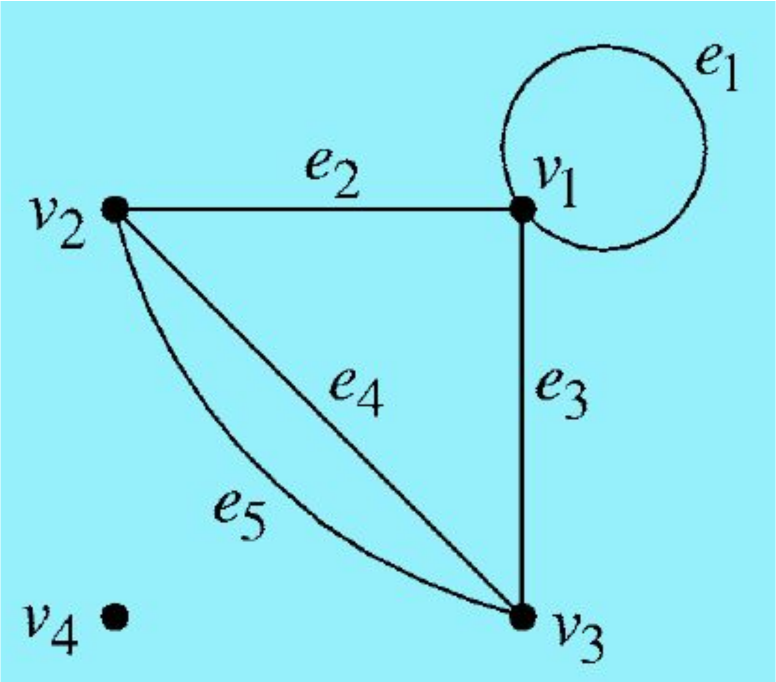
Paths and cycles



- Let Γ be the graph represented in the figure; examples of edge sequences in Γ are:

1) e_1, e_3, e_4, e_5, e_3;
2) e_3, e_3;
3) e_2, e_3, e_4;
4) e_4, e_3;
5) e_4, e_5, e_2.

Paths and cycles

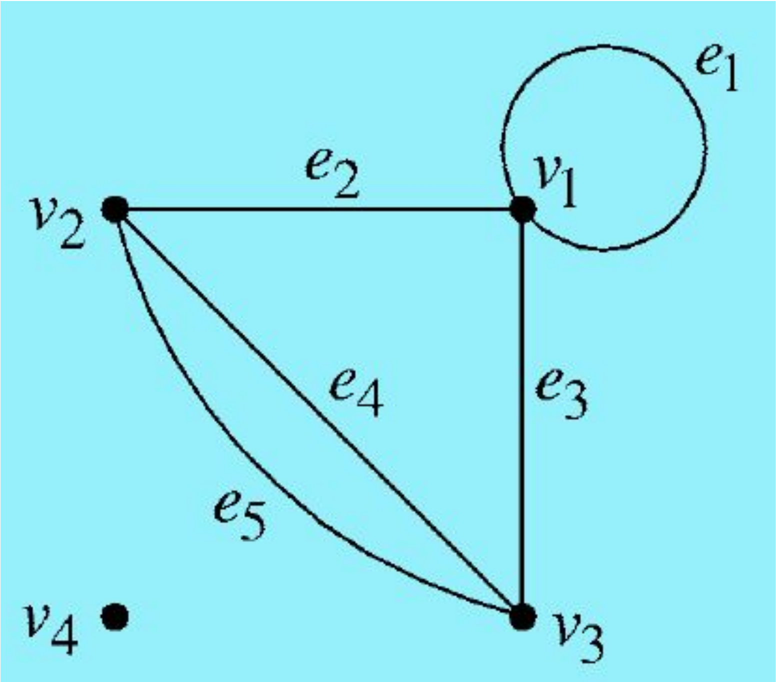


1) e_1, e_3, e_4, e_5, e_3

Sequence 1) is a closed edge sequence beginning and ending at v_1: it determines the vertex sequence $v_1, v_1, v_3, v_2, v_3, v_1$.

This edge sequence is not a path because the edge e_3 is traversed twice.

Paths and cycles



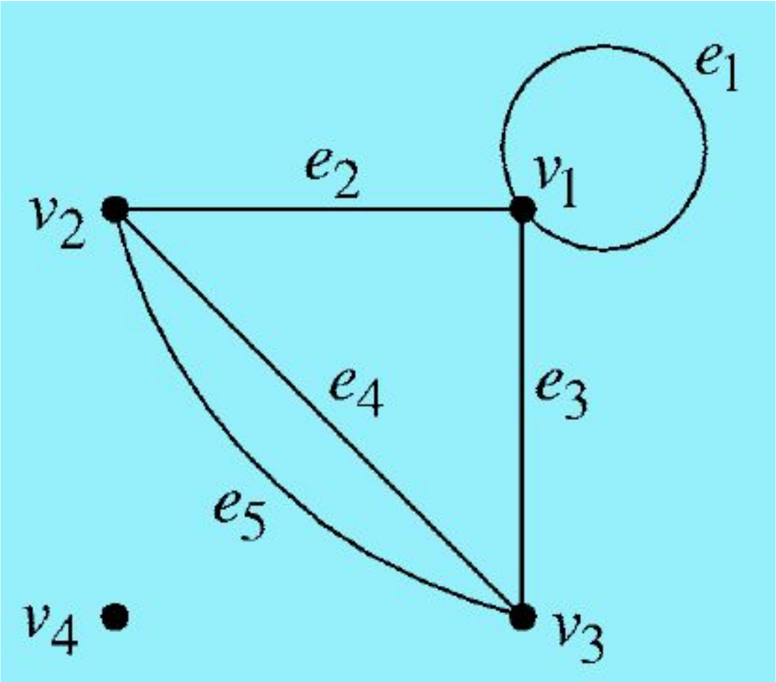
2) e_3, e_3

Sequence 2) is also closed, but it is ambiguous whether it begins (and ends) at v_1 or v_3. The vertex sequence could be either v_1, v_3, v_1 or v_3, v_1, v_3.

This ambiguity will always occur in an edge sequence of the form $e_i, e_i, \ldots, e_i$ where e_i is not a loop.

Again, it is not a path.

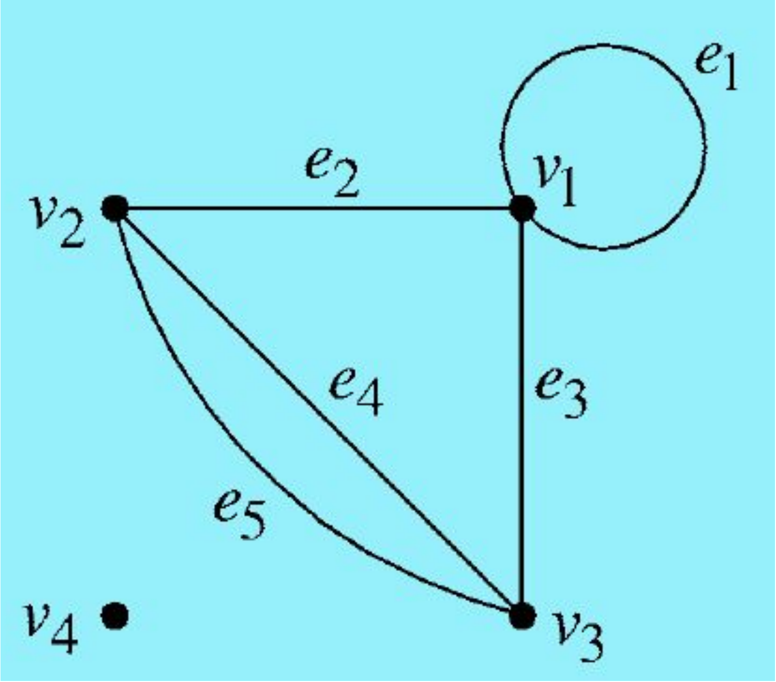
Paths and cycles



3) e_2, e_3, e_4

Sequence 3) is a cycle: it begins and ends at v_2 and no edge or vertex (except v_2 itself) is repeated.

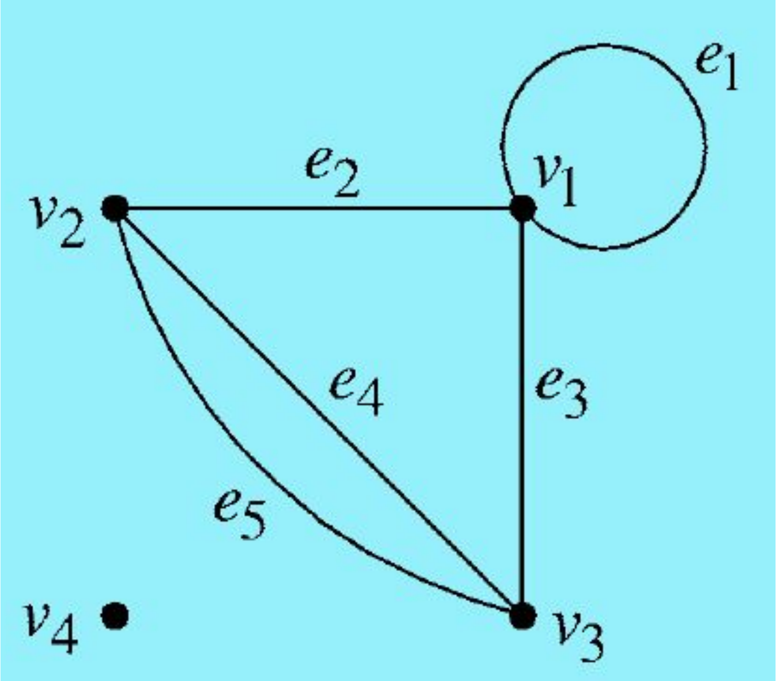
Paths and cycles



4) e_4, e_3

Sequence 4) is a simple path from v_2 to v_1.

Paths and cycles



5) e_4, e_5, e_2

Sequence 5) is a path with initial and final vertices v_2, v_1 respectively.

It is not a simple path because vertex v_2 appears twice in the associated vertex sequence.

Paths and cycles

In an intuitively obvious sense, some graphs are ‘all in one piece’ and others are made up of several pieces.

We can use paths to make this idea more precise.

Paths and cycles

Definition 7

A graph is **connected** if, given any pair of distinct vertices, there exists a path connecting them.

Paths and cycles

An arbitrary graph naturally splits up into a number of connected subgraphs, called its **(connected) components**.

The components can be defined formally as maximal connected subgraphs.

Paths and cycles

- This means that Γ_1 is a component of Γ if it is a connected subgraph of Γ and it is not itself a proper subgraph of any other ***connected*** subgraph of Γ.

This second condition is what we mean by the term maximal; it says that if Σ is a connected subgraph such that $\Gamma_1 \leq \Sigma$, then $\Sigma = \Gamma_1$ so there is no ***connected*** subgraph of Γ which is 'bigger' than Γ_1.

Paths and cycles

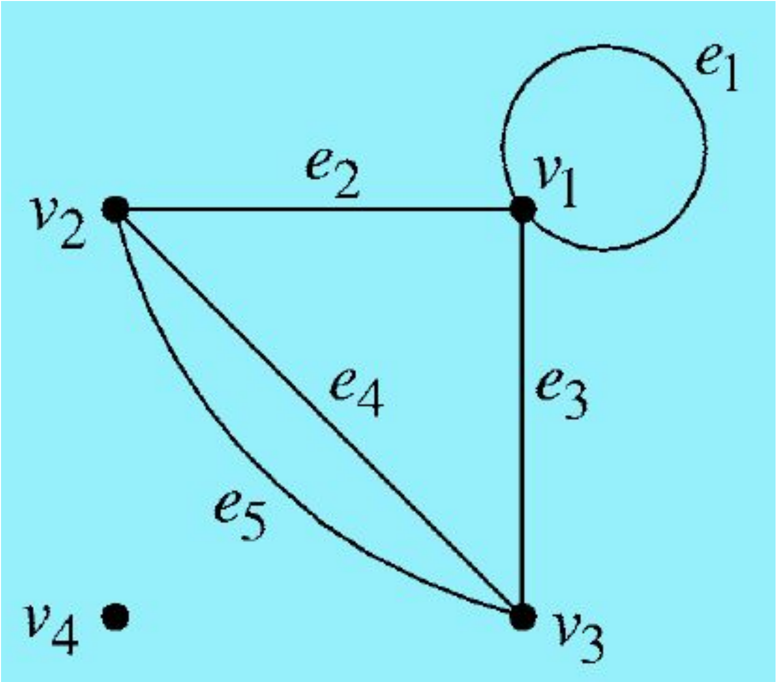
The components of a graph are just its connected ‘pieces’.

In particular, a connected graph has only one component.

Decomposing a graph into its components can be very useful.

It is usually simpler to prove results about connected graphs and properties of arbitrary graphs can frequently then be deduced by considering each component in turn.

Paths and cycles



<u>Example 6</u>

The graph illustrated in the figure has two components, one of which is the null graph with vertex set $\{v_4\}$.

Paths and cycles

Example 7

Frequently it is clear from a diagram of Γ how many components it has. Sometimes, however, we need to examine the diagram more closely. For instance, both graphs illustrated in the figure have two components, although this is not instantly apparent for the graph (b).