Topics overview

Credit and exam

Mathematical models

- A mathematical model is a description of a system or problem using mathematical concepts, tools and language.
- Mathematical model is a function, an equation, inequations, or system of equations or inequations, which represents certain aspects of the physical system or problem modelled.
- Ideally, by the application of the appropriate techniques the solution obtained from the model should also be the solution to the system problem.

Abstract algebra

Algebraic structures

- Group, Abelian group
- *Field*
- *Ring*
- Vector space
  - Vector space over a field F $(V, F, +, \times)$
    - set of vectors V together with a set of scalars F and two operations
    - vector addition: $V + V \rightarrow V$
    - scalar multiplication: $F \times V \rightarrow V$
    - axioms

Linear algebra

- Vector space
- Vectors
  - Components (coordinates)
  - Basic operations
  - Linear combination of vectors
  - Linearly dependent or independent vectors
  - Basic types of vectors

Vector spaces

- Generators
- Basis
- Basis extension
  - Steiner’s theorem

Matrices

- Type of matrix
- Matrix addition
- Matrix multiplication
- Scalar multiplication of matrix
- Inversion of square matrix
- Rank of matrix

System of linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$x_1 . \boldsymbol{a}_1 + x_2 . \boldsymbol{a}_2 + \ldots + x_n . \boldsymbol{a}_n = \boldsymbol{b}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Equivalent system of linear equations

Solution of system of linear equations

- Gauss elimination
- Jordanian elimination
- Row echelon form
- Reduced row echelon form
    - Canonical form
- Transformation of matrix $\mathbf{A}\,|\,\mathbf{b}$ to reduced row echelon form
- Frobenius theorem

Jordanian elimination

- Elementary row (column) operation
  - Exchange the rows
  - Multiplying row by a scalar
  - Add one row to another row

- Jordanian elimination
  - Choosing **leading element – pivot** on the main diagonal
  - Replacing it by 1 *using elementary operations*
  - Other elements in **column** replace by 0 *using elementary operations*

Solubility of system of linear equations

- The system has no solution (in this case, we say that the system is overdetermined)
- The system has a single solution (the system is exactly determined)
- The system has infinitely many solutions (the system is underdetermined).
- Basic solution
- Nonbasic solution
- Parametric solution

Mathematical programming

- Optimization model

$$\min \{f(x) \mid q_i(x) \leq 0 , \quad i = 1, \ldots, m ,$$
$$x^T = (x_1, x_2, \ldots, x_n)^T \in R^n \}$$

***f(x)* and *q_i (x)* - real function of many variables and *x* – vector of variables from vector space *R^n*.**

General optimality problems

- Feasibility problem
  - The satisfiability problem, also called the feasibility problem, is just the problem of finding any feasible solution at all without regard to objective value.
- Minimum and maximum value of a function
  - The problem of finding extrema of function without regard to some constraints.

Classification of optimization models

- More then one constraint
- Number of criteria
  - Single optimization
  - Multiple optimization
- Type of criteria
  - Minimization
  - Maximization
  - Goal problem
- Type of functions
  - Linear model
  - Nonlinear model
    - Convex model
    - Nonconvex model

Linear optimization model

$$\underline{z(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x} \to MIN}$$

$$\mathbf{A}\mathbf{x} \quad \begin{matrix} \le \\ = \\ \ge \end{matrix} \quad \mathbf{b}$$

$$\mathbf{x} \ge \mathbf{0}$$

Fundamental Theorem of LP

- ***If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the corner points of the feasible region.***
- (If it occurs in more than at one corner, each convex linear combination of these corner coordinates points to optimal value too)

Fundamentals theorems

- Basic solution of system of linear equations is represented by corner points of the feasible region.
- If feasible solution of LP exists, basic feasible solution exists too.
- Optimal solution of LP problem lies on the border of convex polytop of feasible solutions.
- If optimal solution of LP exists, basic optimal solution exists too.

Terminology

- Variables
  - Decision variables
  - Slack variables
  - Artificial variables
- Constraints also called conditions or restrictions
  - Capacities or Capacity constraints
  - Requirements or Requirement constraints
  - Balance constraints
  - Definitional constraints
- Objective function also called criteria function

Terminology

- Feasible solution – feasibility region, search space, choice set
  - Basic solution
- Infeasible solution
- Optimal solution
- Alternative solution
- Suboptimal solution
- Objective function also called criteria function, cost function, energy function, or energy functional

Existence of solution

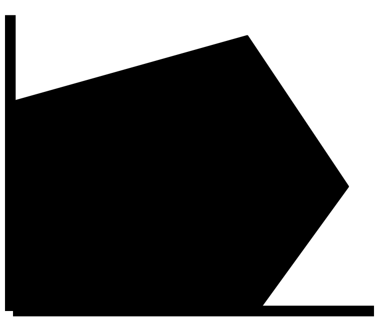
- Nonexistence of solution
  - If the feasible region is empty (that is, there are no points that satisfy all the constraints), the both the maximum value and the minimum value of the objective function do not exist.
  - If the feasible region is unbounded, and the vector of coefficients of the objective function lies in the feasible cone, then the minimum value of the objective function exists, but the maximum value does not, is unbounded.
- Existence of one or more solutions
  - If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.
  - If two or more optimal solutions exists, than infinite number of solutions exist – linear combinations of some solution

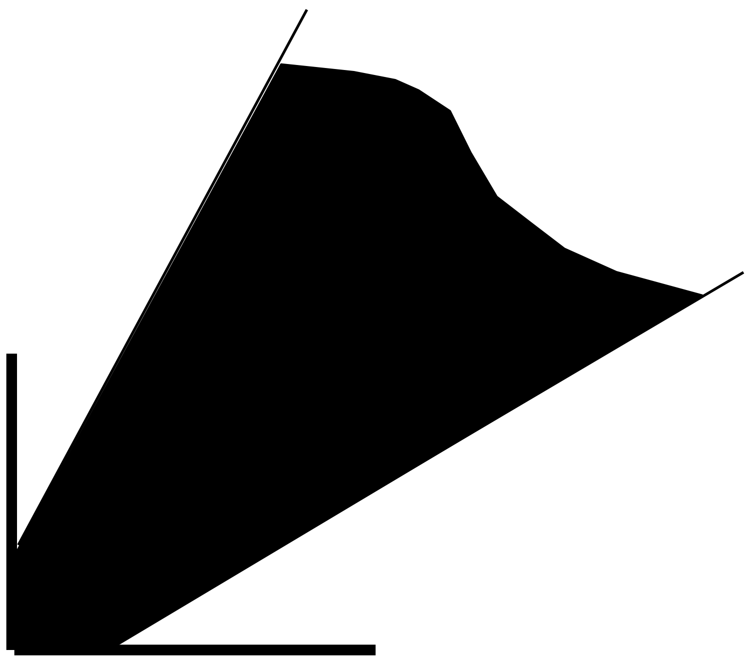
Matrices as basic vectors

- Column space of a matrix is the set of all possible linear combinations of its column vectors
  - Description of vectors (Space of solutions)
  - Description of scalars (Column space)
- Row space of a matrix is the set of all possible linear combinations of its row vectors

Graphical representation I

- Convex polytop
  - Bounded
  - Unbounded

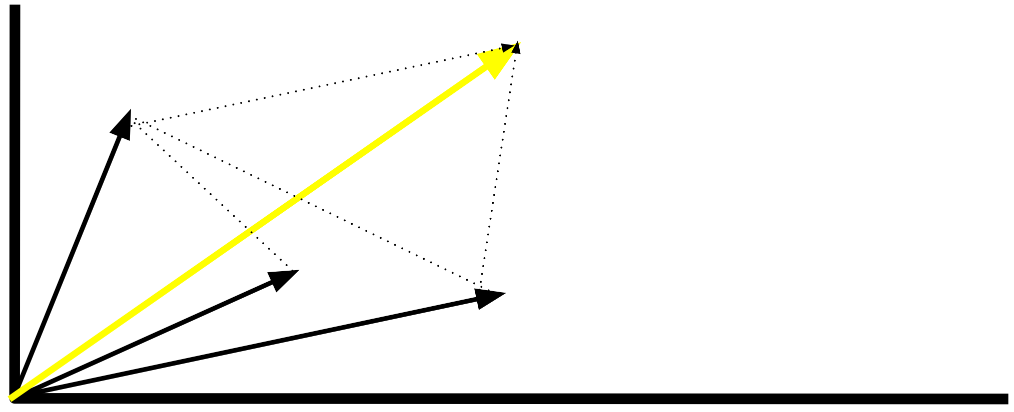




Graphical representation II

Column space of matrix of coefficients

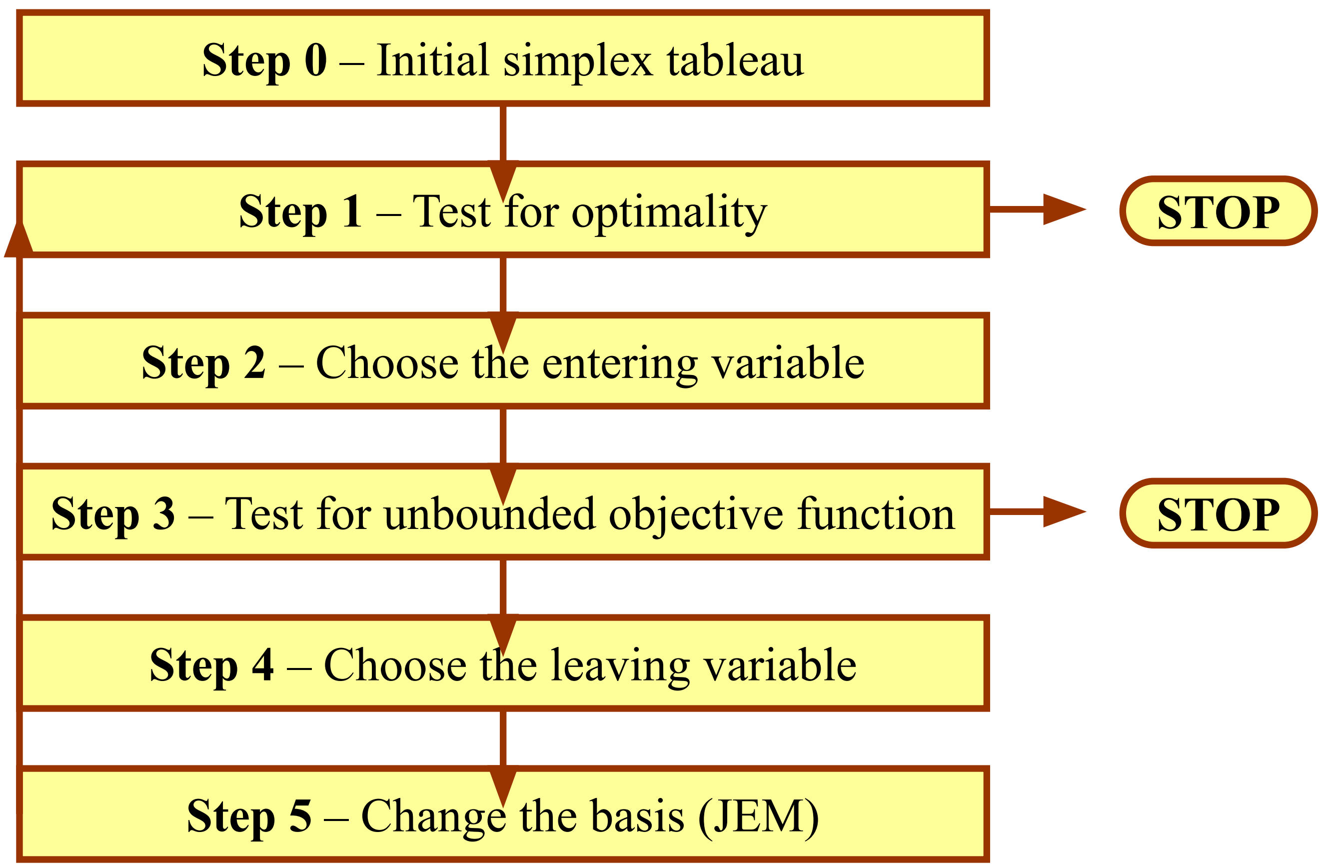
$$\sum_{j=1}^{n} a_j x_j = b$$



Simplex Method

- Simplex method
  - <u>Starts with a feasible solution</u>
  - <u>Tests whether or not it is optimum.</u>
  - <u>If not, the method proceeds a better solution</u>
- It is based on Jordanian elimination procedure.
  - It deals with equations and not with inequations

The Simplex Algorithm



The Simplex Algorithm

Converting LP into standard and canonical form

- Definition of slack variables
- Definition of artificial variables
- Big M method
- Equations, positive right hand side, canonical form

Test for optimality

- z_j is the amount of profit given by replacing some of the present basic variables mix with one unit of the column variable
- Determine the entering basic variable

Test for unbounded criterion and for feasibility

- Determine the leaving basic variable

Changing basis

- Jordanian elimination method

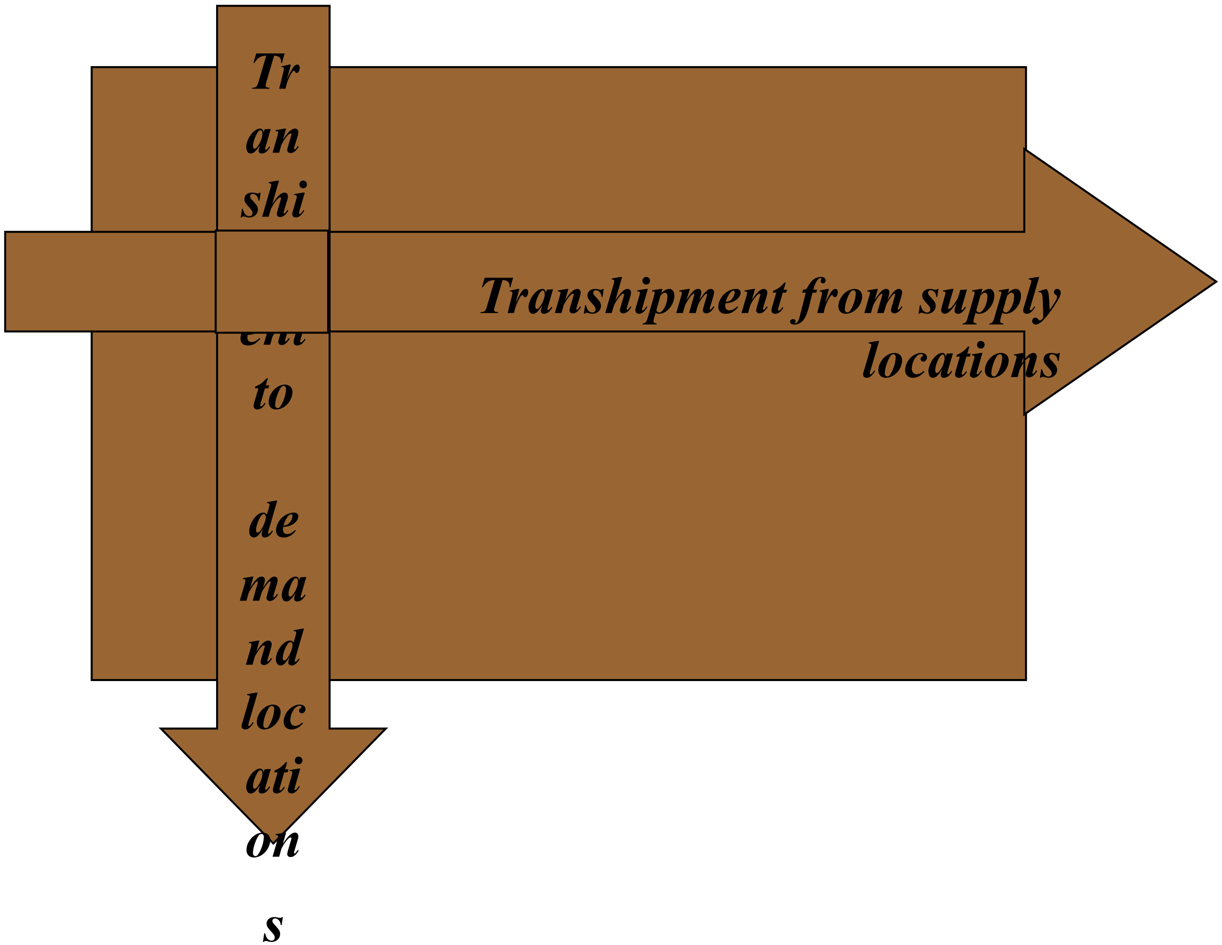
Solubility of linear model

- One optimal solution
- Infinite number of optimal solution
  - Alternate solutions - If the z_j - c_j value for one or more nonbasic variables is 0 in the optimal tableau,
- No solution
  - Unbounded Linear programs
    - If all entries in the pivot column are nonpositive, the linear program is unbounded.
  - Infeasible Linear Programs
    - If an artificial variable remains positive in the "optimal tableau," the problem is infeasible.

Simple transportation problem

- **Suppliers**, source – supply of *i-th* supplier a_i
- **Demands**, destinations – demand of *j-th* destination b_j
- **Route** (i,j) – unique connection between supplier and demand
- Unit **transportation costs** (distances) between each origin and destination – c_{ij}
- x_{ij} – number of **units shipped** from supplier *i* to demand *j*
- **Goal**: Minimize of the cost of shipping goods or maximize the profit of shipping

Transportation table



Balanced transportation model

$\Sigma_j \, x_{ij} = a_i$, $i=1,\ldots,m$

$\Sigma_i \, x_{ij} = b_j$, $j=1,\ldots,n$

$x_{ij} \geq 0$

$\Sigma_i \, \Sigma_j \, c_{ij}.x_{ij} \rightarrow \text{MIN}$

Balanced transportation system

Total supply = total demand $\Sigma_j a_i = \Sigma_j b_j$

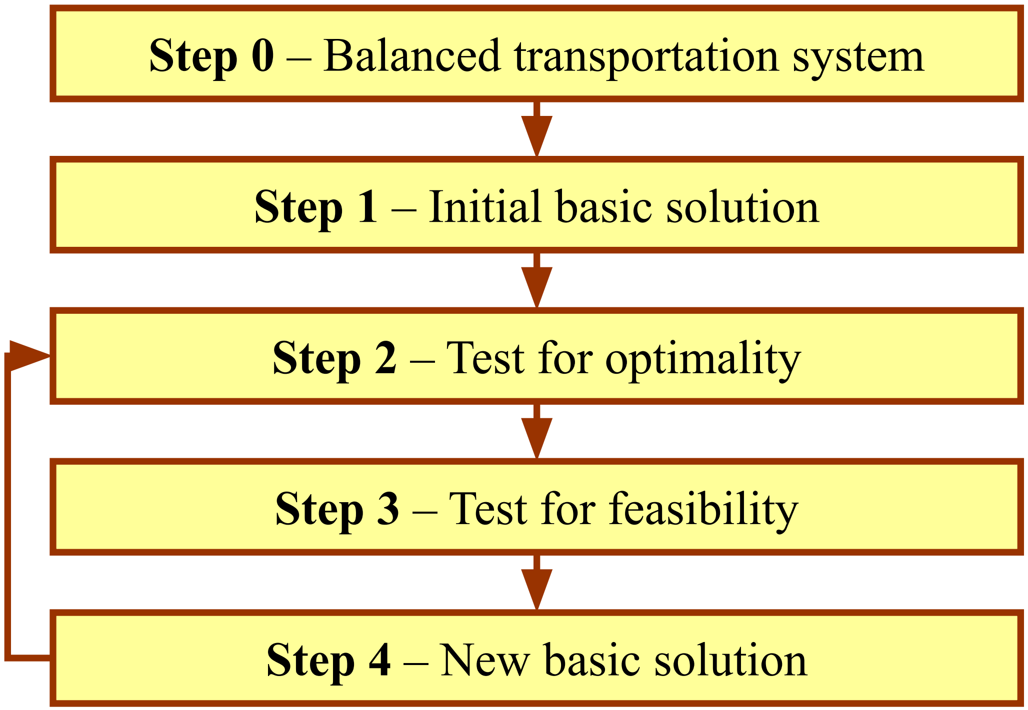
- dummy supplier
- dummy destination

- **Frobenius theorem**
  - If the system is balanced then the model is always solvable.
  - Basic solution **m+n-1 basic variables (routes)**

Solving of the TP

- Initial solution must be feasible
  - Northwest-Corner method (NWCM)
  - Least-Cost method (LCM)
  - Vogel's approximation method (VAM).
- Test for optimality
  - Stepping stone method
  - Cost of basic route mix
- Test for feasibility
  - Change of transportation plan
  - Stepping stone method

Transportation method



Degeneracy

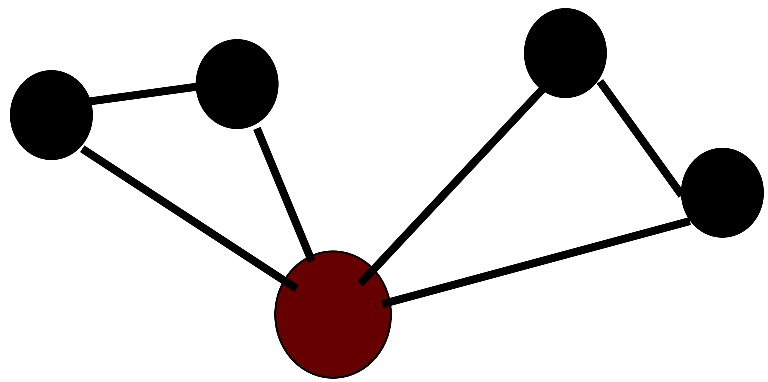
- The basic solution is degenerate if some of basic variables is equal to zero.
- Degeneracy does not need some changes in the simplex method.
- Degeneracy requires to choose the missing basic variable - to perform shipping stone method – in transportation method.
  - $(m + n – 1)$
  - Degeneracy requires to choose the missing basic variable so a closed path can be developed for all other empty cells.

Result analysis

- Optimal solution
- Alternative solution
- Suboptimal solution
  - Perspective routes
  - Routes substitution
  - Possible shipped amount

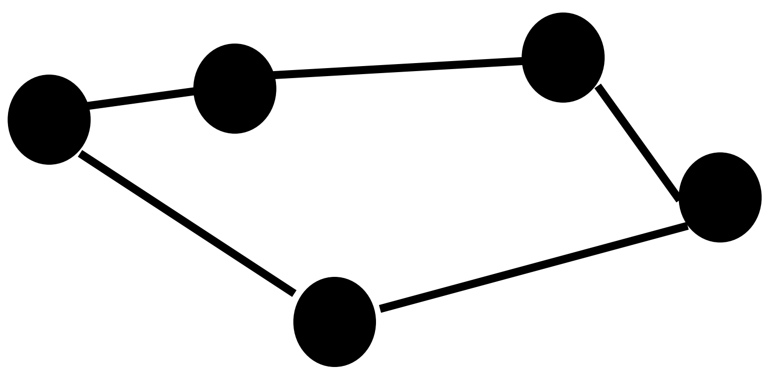
Vehicle routing problem

- Given a list of cities and their pairwise distances
- The task is to find a multi cycle tour, which return to the central city and visit each city exactly once
  - Central point
  - All other point only once
  - Constraints for subcycles – time, vehicle capacity ….



Travelling salesman problem

- Given a list of cities and their pairwise distances
- The task is to find a shortest possible tour that visits each city exactly once



Solving of TSP

- Try all permutations of points
  - N! possibilities

- Principle: adding of branches to pass all points before all points will be visited
- Selection of branches according to the distance – current advantage can be disadvantage in the future

- The nearest neighbour algorithm
- Vogel's Approximation Method

Vehicle routing problem

- Majer‘s method
  - Central point
  - Selecting the most distant point from the central point
  - Then adding the nearest points subject to constraints
  - Construction of the first cycle by VAM or NNM

  - Again selecting the most distant point from the remaining ….

  - The method finishes if all points (except central) are in one of subroutes.

Game

- Model of conflict or competition
- Cooperative, non-cooperative games
- Antagonistic – non-antagonistic game
- Time – simultaneous and sequential game
- Repetition
- Game - play - strategy – move
- **Model elements**
  - Players, strategies, payoffs

Solution of game

- Each player tries to maximize his welfare at the expense of the others.
- Result – **The best (optimal) strategies** which gives to each player the best outcome according to the other players and their strategies
  - Maximal possible win of each player
- **Value of game** – players' result, expected payoff

Model of game

- Tree (extensive) form of model
  - Game tree (decision tree - moves)
- Normal form of model
  - List of players, list of strategy spaces, and list of payoff functions
  - Payoff matrix (decision matrix)

Matrix game

- Two-person game
- Finite number of strategies for each player
- Zero-sum game
  - Sum of payoffs for both players is zero
  - Outcom of one player is a loss of the other player

- Matrix form of game model

Pure and mixed strategy

- Pure strategy
  - One best strategy
  - How to find it – saddle point
- Mixed strategy
  - Probability of strategy realization or frequency of strategy usage
  - How to find it – linear optimization model

Matrix game solution

Theorem

- The optimal pure strategies exist in the matrix game, if and only if the game has the saddle point.

The Minimax Theorem.

- *For every finite two-person zero-sum game,*
- *(1) there is a number w, called the value of the game,*
- *(2) there is a mixed strategy for Player A such that his average gain is at least w no matter what B does, and*
- *(3) there is a mixed strategy for Player B such that his average loss is at most w no matter what A does.*

Decision model

- Model elements
  - Decision alternatives
  - States of nature
  - Decision matrix (table) – payoffs associated with each combination of alternatives and events
  - Decision tree

- Decision criterion
- Certainty, risk, uncertainty

Solution of decision problems

- Selection of the dominating alternative
- Selection of the best alternative
- Selection of the alternative according to the highest utility

Selection of the dominating alternative

- Outcome dominance: a_I dominates a_K

$$\boxed{\min_{j=1,\ldots,n} v_{Ij} \geq \max_{j=1,\ldots,n} v_{Kj}}$$

- Event dominance: a_I dominates a_K

$$\boxed{v_{Ij} \geq v_{Kj} \quad \forall j, j = 1,\ldots,n}$$

- Probabilistic dominance: a_I dominates a_K

$$\boxed{P(v_I \geq x) \geq P(v_K \geq x)}$$

Selection of the best alternative

- Decision-making under certainty
- Decision-making under uncertainty
  - Maximax rule
  - Wald criterion - maximin rule
  - Savage criterion - minimax regret rule
  - Laplace criterion - principle of insufficient reason
  - Hurwicz criterion
- Decision-making under risk
  - EMV criterion - expected monetary value criterion
  - EOL criterion - expected opportunity loss criterion

Multiple Objective Decision Making

- Infinite Number of Alternatives
- At least two criteria
- Example – Linear multi criteria programming

$$
\begin{array}{c}
y_1(\mathbf{x}) = \mathbf{c}_1^{\mathrm{T}}\mathbf{x} \rightarrow MAX \\
\vdots \\
y_k(\mathbf{x}) = \mathbf{c}_k^{\mathrm{T}}\mathbf{x} \rightarrow MIN \\
\hline
\mathbf{Ax} \leq \mathbf{b} \\
\mathbf{x} \geq \mathbf{0}
\end{array}
$$

Multiple Attribute Decision Making

- Finite Number of Alternatives
- Evaluation of all alternatives with respect to all attributes

$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathbf{f_1} & \mathbf{f_2} & \mathbf{\ldots} & \mathbf{f_k} \end{array} \\
\begin{array}{c} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_p} \end{array} &
\begin{pmatrix}
y_{11} & y_{12} & \cdots & y_{1k} \\
y_{21} & y_{22} & \cdots & y_{2k} \\
\cdots & \cdots & \cdots & \cdots \\
y_{p1} & y_{p2} & \cdots & y_{pk}
\end{pmatrix}
\end{array}
$$

Basic terms

- Ideal alternative
- Nadir alternative

- Dominating and dominated alternative
- The best alternative – preferred alternative
- Pareto efficient alternative

The aim of MADM

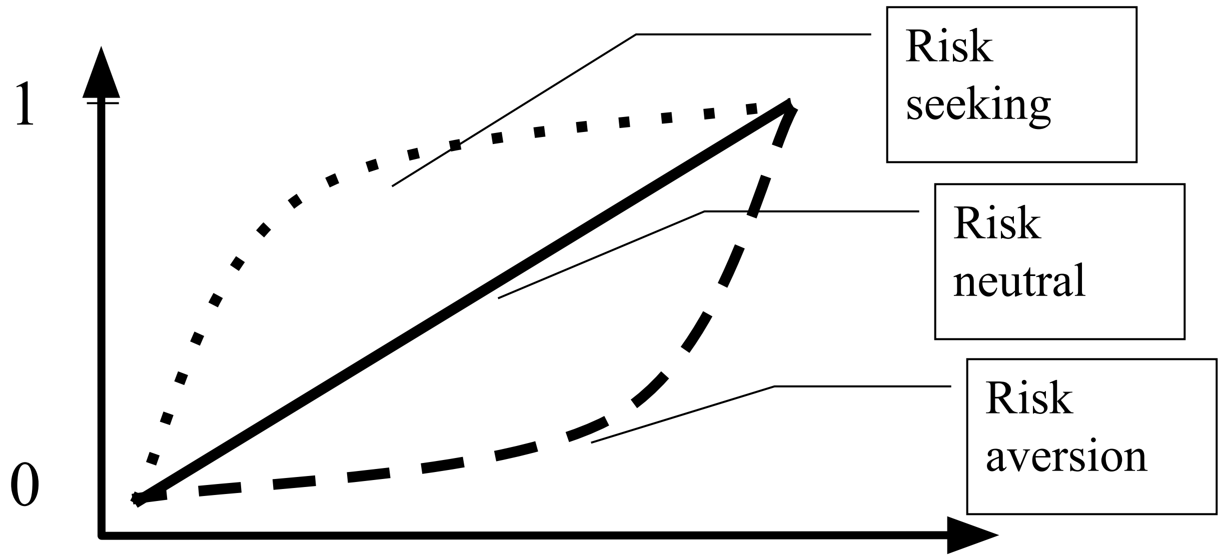
- Selection of the best alternatives (one or more)
- Dichotomizing into the efficient and non efficient alternatives
- Ranking of all alternatives
- Noncompensatory model
  - Not permit trade-offs between attributes
- Compensatory Model
  - Permit trade-offs between attributes

Utility, utility function

- **Utility** is a measure of satisfaction
- All attribute values can be expresed by utility value
- Partial utility values can be combined into global utility value
- Risk seeking, risk neutral and risk aversion decision maker

Utility function

- A utility function represents a preference relation
- Mapping of attribute values into interval $\langle 0, 1 \rangle$



Informations

- Inter and intra attribute comparisons
  - Criteria preferences
  - Alternatives preferences
    - Not necessary in numerical form

  - No preference information given
  - Nominal information – *standard level of attribute*
  - Ordinal information – *qualitative - ordering*
  - Cardinal information - *quantitative*

Methods for assesing information

- Sequence Method
  - Criteria/alternatives are arranged according their importance to a sequence from most to least important.
- Scoring Method
  - Each criterion/alternative isevaluated by certain number of points from chosen scale.
- Pairwise comparison
  - The judgement of the relative importance of each pair of criteria

MADM methods

- Scoring or sequence methods
- Standard level methods
- Simple additive weighting method
  - *Attributes must be measured in the* ***same scale***

Credit and Exam

- **Credit specifications**
  - Selftests in Moodle – 60 % of points
  - Resit credit test – 60 % of points

- **Exam specifications**
  - Exam written test – 60 % of points
    - Theory – 2 questions (15 points each)
    - Small example (30 points)
    - Practical application (40 points)
  - Oral examination (immediately after test)