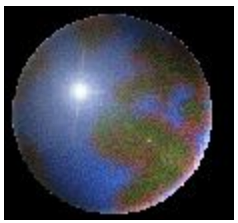
Linear Algebra



Chapter 3

Determinants

3.1 Introduction to Determinants

Definition

The **determinant** of a 2 × 2 matrix A is denoted **$|A|$** and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2 × 2 matrix is given by *the different of the products of the two diagonals* of the matrix.

The notation **det(A)** is also used for the determinant of A.

Example 1

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2\times 1) - (4\times(-3)) = 2 + 12 = 14$$

Definition

Let A be a square matrix.

The **minor** of the element a_{ij} is denoted M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of A.

The **cofactor** of a_{ij} is denoted C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that $C_{ij} = M_{ij}$ *or* $-M_{ij}$.

Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

Minor of a_{11} : $M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times (-2)) = 3$

Cofactor of a_{11} : $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

Minor of a_{32} : $M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1 \times 2) - (3 \times 4) = -10$

Cofactor of a_{32} : $C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$

Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

$$\text{If } A \text{ is } 3\times 3, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4\times 4, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\vdots$$

$$\text{If } A \text{ is } n\times n, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of $|A|$.

Example 3

Evaluate the determinant of the following matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$

Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of <u>any row or column</u> and their cofactors.

ith row expansion: $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

jth column expansion: $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= -3\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= -3[(2\times 1) - (-1\times 2)] + 0[(1\times 1) - (-1\times 4)] - 1[(1\times 2) - (2\times 4)]$$

$$= -12 + 0 + 6 = -6$$

Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\ &= 3\begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\ &= 3(2)\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3-2) = 6 \end{aligned}$$

Example 6

Solve the following equation for the variable x.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution

Expand the determinant to get the equation

$$x(x-2)-(x+1)(-1)=7$$

Proceed to simplify this equation and solve for x.

$$x^2-2x+x+1=7$$

$$x^2-x-6=0$$

$$(x+2)(x-3)=0$$

$$x=-2 \text{ or } 3$$

There are two solutions to this equation, $x = -2$ or 3.

Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |\mathrm{A}| = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

Homework

- Exercises will be given by the teachers of the practical classes.

3.2 Properties of Determinants

Theorem 3.2

Let A be an $n \times n$ matrix and c be a nonzero scalar.

(a) If $A \underset{cRk}{\approx} B$ then $|B| = c|A|$.

(b) If $A \underset{Ri \leftrightarrow Rj}{\approx} B$ then $|B| = -|A|$.

(c) If $A \underset{Ri + cRj}{\approx} B$ then $|B| = |A|$.

Proof (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$
$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \ldots + ca_{kn}C_{kn}$$
$$\therefore |B| = c|A|.$$

Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \underset{C2+2C3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$

Example 2

If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

(a) $B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix}$ (b) $B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix}$ (c) $B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$

Solution

(a) $A \underset{3C2}{\approx} B_1$ Thus $|B_1| = 3|A| = 36.$

(b) $A \underset{R2 \leftrightarrow R3}{\approx} B_2$ Thus $|B_2| = -|A| = -12.$

(c) $A \underset{R3+2R1}{\approx} B_3$ Thus $|B_3| = |A| = 12.$

Definition

A square matrix A is said to be **singular** if $|A|=0$.

A is **nonsingular** if $|A|\neq 0$.

Theorem 3.3

Let A be a square matrix. A is singular if

(a) all the elements of a row (column) are zero.

(b) two rows (columns) are equal.

(c) two rows (columns) are proportional. (i.e., $Ri=cRj$)

Proof

(a) Let all elements of the kth row of A be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

(c) If $Ri=cRj$, then $A \underset{Ri-cRj}{\approx} B$, row i of B is [0 0 … 0].

$\Rightarrow |A|=|B|=0$

Example 3

Show that the following matrices are singular.

$$\text{(a) } A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad \text{(b) } B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution

(a) All the elements in column 2 of A are zero. Thus $|A| = 0$.

(b) Row 2 and row 3 are proportional. Thus $|B| = 0$.

Theorem 3.4

Let A and B be $n \times n$ matrices and c be a nonzero scalar.

(a) $|cA| = c^n|A|$.

(b) $|AB| = |A||B|$.

(c) $|A^t| = |A|$.

(d) $\left|A^{-1}\right| = \dfrac{1}{|A|}$ (assuming A^{-1} exists)

Proof

(a)

$$A \underset{cR1,\, cR2,\, \ldots,\, cRn}{\approx} cA \quad \Rightarrow \quad |cA| = c^n|A|$$

(d)

$$|A| \cdot \left|A^{-1}\right| = \left|A \cdot A^{-1}\right| = |I| = 1 \quad \Rightarrow \quad \left|A^{-1}\right| = \frac{1}{|A|}$$

Example 4

If A is a 2×2 matrix with $|A| = 4$, use Theorem 3.4 to compute the following determinants.

(a) $|3A|$ (b) $|A^2|$ (c) $|5A^tA^{-1}|$, assuming A^{-1} exists

Solution

(a) $|3A| = (3^2)|A| = 9 \times 4 = 36.$

(b) $|A^2| = |AA| = |A|\,|A| = 4 \times 4 = 16.$

(c) $|5A^tA^{-1}| = (5^2)|A^tA^{-1}| = 25|A^t||A^{-1}| = 25|A|\frac{1}{|A|} = 25.$

Example 5

Prove that $|A^{-1}A^tA| = |A|$

Solution

$$\left|A^{-1}A^tA\right| = \left|(A^{-1}A^t)A\right| = \left|A^{-1}A^t\right|\left|A\right| = \left|A^{-1}\right|\left|A^t\right|\left|A\right| = \frac{1}{|A|}\left|A\right|\left|A\right| = \left|A\right|$$

Example 6

Prove that if A and B are square matrices of the same size, with A being singular, then AB is also singular. Is the converse true?

Solution

$(\Rightarrow)$

$$|A| = 0 \quad \Rightarrow \quad |AB| = |A||B| = 0$$

Thus the matrix AB is singular.

$(\Leftarrow)$

$$|AB| = 0 \Rightarrow |A||B| = 0 \Rightarrow |A| = 0 \text{ or } |B| = 0$$

Thus AB being singular implies that either A or B is singular.

The inverse is not true.

Homework

- Exercises will be given by the teachers of the practical classes.

Exercise 11

Prove the following identity without evaluating the determinants.

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & c & e \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} b & d & f \\ p & q & r \\ u & v & w \end{vmatrix}$$

Solution

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = (a+b)\begin{vmatrix} q & r \\ v & w \end{vmatrix} - (c+d)\begin{vmatrix} p & r \\ u & w \end{vmatrix} + (e+f)\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

3.3 Numerical Evaluation of a Determinant

Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

upper – triangular   lower – triangular

Numerical Evaluation of a Determinant

Theorem 3.5

The determinant of a triangular matrix is the product of its diagonal elements.

Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22} \cdots a_{nn}$$

Example 1

Let $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$, find $|A|$.

Sol. $|A| = 2 \times 3 \times (-5) = -30.$

Numerical Evaluation of a Determinant

Example 2

Evaluation the determinant. $\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$

Solution (elementary row operations)

$$\begin{vmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{vmatrix} \begin{matrix} = \\ \mathrm{R2+R1} \\ \mathrm{R3+(-2)R1} \end{matrix} \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$$

$$\begin{matrix} = \\ \mathrm{R3}+R2 \end{matrix} \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 13 \end{vmatrix} = 2\times(-1)\times 13 = -26$$

Example 3

Evaluation the determinant. $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix} \begin{matrix} = \\ R2+(-2)R1 \\ R3+(-1)R1 \\ R4+R1 \end{matrix} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 2 \end{vmatrix}$$

$$\begin{matrix} = \\ R4+2R3 \end{matrix} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= 1 \times (-1) \times (-2) \times 6 = 12$$

Example 4

Evaluation the determinant. $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1}}{=} \begin{vmatrix} 1 & -2 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$\underset{R2 \leftrightarrow R3}{=} (-1)\begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (-1) \times 1 \times 2 \times (-1) = 2$$

Example 5

Evaluation the determinant. $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} \begin{matrix} = \\ R2+R1 \\ R3+(-2)R1 \\ R4+(-6)R1 \end{matrix} \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & -11 \end{vmatrix} = 0$$

diagonal element is zero and all elements below this diagonal element are zero.

3.4 Determinants, Matrix Inverse, and Systems of Linear Equations

Definition

Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}.

The matrix whose (i, j)th element is C_{ij} is called the **matrix of cofactors** of A.

The transpose of this matrix is called the **adjoint** of A and is denoted adj(A).

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^t$$

adjoint matrix

Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution The cofactors of A are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14 \quad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3 \quad C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9 \quad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \quad C_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1 \quad C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors of A is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

Theorem 3.6

Let A be a square matrix with $|A| \neq 0$. A is invertible with

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Proof

Consider the matrix product $A \cdot \text{adj}(A)$. The (i, j)th element of this product is

$$\begin{aligned}(i, j)\text{th element} &= (\text{row } i \text{ of } A) \times (\text{column } j \text{ of adj}(A)) \\ &= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix} \\ &= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}\end{aligned}$$

Proof of Theorem 3.6

If $i = j$, $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |A|$.

If $i \neq j$, let $A \underset{Rj \text{ is replaced by } Ri}{\Longrightarrow} B$.

Matrices A and B have the same cofactors $C_{j1}, C_{j2}, \ldots, C_{jn}$.

So $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |B| = 0$.

row i = row j in B

Therefore $(i.j)\text{th element} = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\therefore A \cdot \text{adj}(A) = |A| I_n$

Since $|A| \neq 0$, $A\left(\frac{1}{|A|}\text{adj}(A)\right) = I_n$

Similarly, $\left(\frac{1}{|A|}\text{adj}(A)\right)A = I_n$. Thus $A^{-1} = \frac{1}{|A|}\text{adj}(A)$

Theorem 3.7

A square matrix A is invertible if and only if $|A| \neq 0$.

Proof

($\Rightarrow$) Assume that A is invertible.

$$\Rightarrow AA^{-1} = I_n.$$

$$\Rightarrow |AA^{-1}| = |I_n|.$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

($\Leftarrow$) Theorem 3.6 tells us that if $|A| \neq 0$, then A is invertible.

$$A^{-1} \textit{ exists if and only if } |A| \neq 0.$$

Example 2

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

Solution

$|A| = 5 \neq 0.$ A is invertible.

$|B| = 0.$ B is singular. The inverse does not exist.

$|C| = 0.$ C is singular. The inverse does not exist.

$|D| = 2 \neq 0.$ D is invertible.

Example 3

Use the formula for the inverse of a matrix to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution

$|A| = 25$, so the inverse of A exists. We found adj(A) in Example 1

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{25}\begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$

Homework

- Exercises will be given by the teachers of the practical classes.

- ***Exercise***

  Show that if $A = A^{-1}$, then $|A| = \pm 1$.

  Show that if $A^t = A^{-1}$, then $|A| = \pm 1$.

Theorem 3.8

Let $AX = B$ be a system of n linear equations in n variables.
(1) If $|A| \neq 0$, there is a unique solution.
(2) If $|A| = 0$, there may be many or no solutions.

Proof

(1) If $|A| \neq 0$

$\Rightarrow A^{-1}$ exists (Thm 3.7)

$\Rightarrow$ there is then a unique solution given by $X = A^{-1}B$ (Thm 2.9).

(2) If $|A| = 0$

$\Rightarrow$ since $A \approx \ldots \approx$ C implies that if $|A| \neq 0$ then $|C| \neq 0$ (Thm 3.2).

$\Rightarrow$ the reduced echelon form of A is not I_n.

$\Rightarrow$ The solution to the system $AX = B$ is not unique.

$\Rightarrow$ many or no solutions.

Example 4

Determine whether or not the following system of equations has an unique solution.

$$\begin{aligned} 3x_1 + 3x_2 - 2x_3 &= 2 \\ 4x_1 + x_2 + 3x_3 &= -5 \\ 7x_1 + 4x_2 + x_3 &= 9 \end{aligned}$$

Solution

Since

$$\begin{vmatrix} 3 & 3 & -2 \\ 4 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix} = 0$$

Thus the system does not have an unique solution.

Theorem 3.9 Cramer's Rule

Let $AX = B$ be a system of n linear equations in n variables such that $|A| \neq 0$. The system has a unique solution given by

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ \ldots,\ x_n = \frac{|A_n|}{|A|}$$

Where A_i is the matrix obtained by replacing column i of A with B.

Proof

$|A| \neq 0 \Rightarrow$ the solution to $AX = B$ is unique and is given by

$$X = A^{-1}B$$

$$= \frac{1}{|A|}\operatorname{adj}(A)B$$

Proof of Cramer's Rule

x_i, the ith element of X, is given by

$$x_i = \frac{1}{|A|}[\text{row } i \text{ of adj}(A)] \times B$$

$$= \frac{1}{|A|}\begin{bmatrix} C_{1i} C_{2i} \cdots C_{ni} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{|A|}(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni})$$

the cofactor expansion of $|A_i|$ in terms of the ith column

Thus $x_i = \frac{|A_i|}{|A|}$

Example 5

Solving the following system of equations using Cramer's rule.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= -2 \\ 2x_1 + 5x_2 + x_3 &= -5 \\ x_1 + 2x_2 + 3x_3 &= 6 \end{aligned}$$

Solution

The matrix of coefficients A and column matrix of constants B are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

It is found that $|A| = -3 \neq 0$. Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$

Giving $|A_1| = -3, |A_2| = 6, |A_3| = -9$

Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1,\quad x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2,\quad x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is $x_1 = 1, x_2 = -2, x_3 = 3.$

Example 6

Determine values of λ for which the following system of equations has nontrivial solutions.Find the solutions for each value of λ.

$$(\lambda+2)x_1+(\lambda+4)x_2=0$$
$$2x_1+(\lambda+1)x_2=0$$

Solution

homogeneous system

$\Rightarrow x_1 = 0,\ x_2 = 0$ is the trivial solution.

$\Rightarrow$ nontrivial solutions exist $\Rightarrow$ many solutions

$$\Rightarrow \begin{vmatrix} \lambda+2 & \lambda+4 \\ 2 & \lambda+1 \end{vmatrix} = 0$$

$\Rightarrow (\lambda+2)(\lambda+1)-2(\lambda+4)=0 \Rightarrow \lambda^2+\lambda-6=0 \Rightarrow (\lambda-2)(\lambda+3)=0$

$\Rightarrow \lambda = -3$ or $\lambda = 2$.

λ = – 3 results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

This system has many solutions, $x_1 = r$, $x_2 = r$.

λ = 2 results in the system

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions, $x_1 = -3r/2$, $x_2 = r$.

Homework

- Exercises will be given by the teachers of the practical classes.