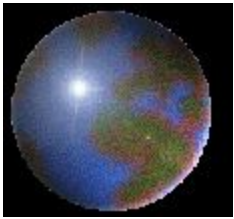
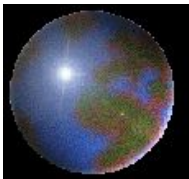


# Linear Algebra



## ***Chapter 3*** ***Determinants***



## 3.1 Introduction to Determinants

### Definition

The **determinant** of a  $2 \times 2$  matrix  $A$  is denoted  $|A|$  and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

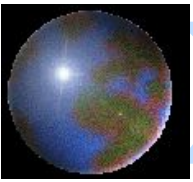
Observe that the determinant of a  $2 \times 2$  matrix is given by *the different of the products of the two diagonals* of the matrix.

The notation **det**( $A$ ) is also used for the determinant of  $A$ .

### Example 1

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$



## Definition

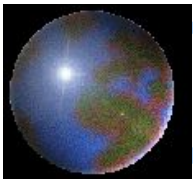
Let  $A$  be a square matrix.

The **minor** of the element  $a_{ij}$  is denoted  $M_{ij}$  and is the determinant of the matrix that remains after deleting row  $i$  and column  $j$  of  $A$ .

The **cofactor** of  $a_{ij}$  is denoted  $C_{ij}$  and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that  $C_{ij} = M_{ij}$  or  $-M_{ij}$ .



## Example 2

Determine the minors and cofactors of the elements  $a_{11}$  and  $a_{32}$  of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

### Solution

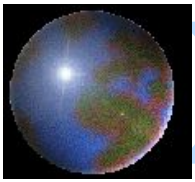
$$\text{Minor of } a_{11} : M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times (-2)) = 3$$

$$\text{Cofactor of } a_{11} : C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$\text{Minor of } a_{32} : M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1 \times 2) - (3 \times 4) = -10$$

$$\text{Cofactor of } a_{32} : C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$





## Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

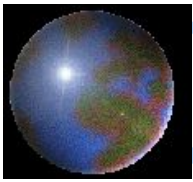
$$\text{If } A \text{ is } 3 \times 3, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4 \times 4, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

⊠

$$\text{If } A \text{ is } n \times n, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \dots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of  $|A|$ .



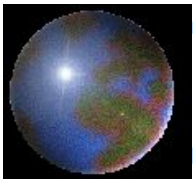
## Example 3

Evaluate the determinant of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

### Solution

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$



# Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

*i*th row expansion:  $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$

*j*th column expansion:  $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$

## Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

### Solution

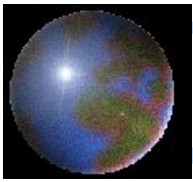
$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= -3 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= -3[(2 \times 1) - (-1 \times 2)] + 0[(1 \times 1) - (-1 \times 4)] - 1[(1 \times 2) - (2 \times 4)]$$

$$= -12 + 0 + 6 = -6$$





## Example 5

Evaluate the determinant of the following  $4 \times 4$  matrix.

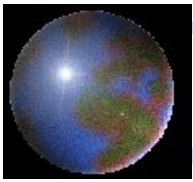
$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

### Solution

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\ &= 3 \begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\ &= 3(2) \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3 - 2) = 6 \end{aligned}$$







## Example 6

Solve the following equation for the variable  $x$ .

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

### Solution

Expand the determinant to get the equation

$$x(x-2) - (x+1)(-1) = 7$$

Proceed to simplify this equation and solve for  $x$ .

$$x^2 - 2x + x + 1 = 7$$

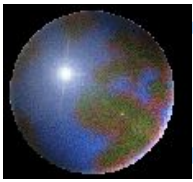
$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } 3$$

There are two solutions to this equation,  $x = -2$  or  $3$ .





# Computing Determinants of $2 \times 2$ and $3 \times 3$ Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

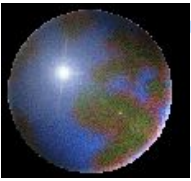
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

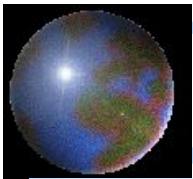
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)



# *Homework*

- Exercises will be given by the teachers of the practical classes.



## 3.2 Properties of Determinants

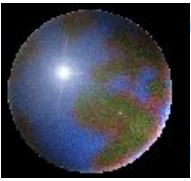
### Theorem 3.2

Let  $A$  be an  $n \times n$  matrix and  $c$  be a nonzero scalar.

- (a) If  $A \underset{cRk}{\approx} B$  then  $|B| = c|A|$ .
- (b) If  $A \underset{Ri \leftrightarrow Rj}{\approx} B$  then  $|B| = -|A|$ .
- (c) If  $A \underset{Ri+cRj}{\approx} B$  then  $|B| = |A|$ .

### Proof (a)

$$\begin{aligned}|A| &= a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn} \\|B| &= ca_{k1}C_{k1} + ca_{k2}C_{k2} + \dots + ca_{kn}C_{kn} \\ \therefore |B| &= c|A|.\end{aligned}$$

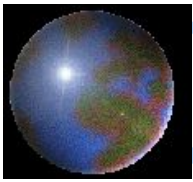


# Example 1

Evaluate the determinant  $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$ .

## Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \stackrel{C_2+2C_3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3) \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$



## Example 2

If  $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -2 & -4 \\ -2 & -4 & 10 \end{bmatrix}$  is known.

Evaluate the determinants of the following matrices.

$$(a) B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix} \quad (b) B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix} \quad (c) B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$$

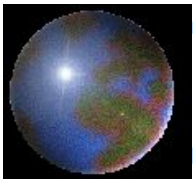
### Solution

(a)  $A \underset{3C2}{\approx} B_1$  Thus  $|B_1| = 3|A| = 36$ .

(b)  $A \underset{R2 \leftrightarrow R3}{\approx} B_2$  Thus  $|B_2| = -|A| = -12$ .

(c)  $A \underset{R3+2R1}{\approx} B_3$  Thus  $|B_3| = |A| = 12$ .





## Definition

A square matrix  $A$  is said to be **singular** if  $|A|=0$ .

$A$  is **nonsingular** if  $|A|\neq 0$ .

### Theorem 3.3

Let  $A$  be a square matrix.  $A$  is singular if

- (a) all the elements of a row (column) are zero.
- (b) two rows (columns) are equal.
- (c) two rows (columns) are proportional. (i.e.,  $R_i=cR_j$ )

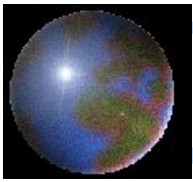
### Proof

(a) Let all elements of the  $k$ th row of  $A$  be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \dots + 0C_{kn} = 0$$

(c) If  $R_i=cR_j$ , then  $A \underset{R_i-cR_j}{\approx} B$ , row  $i$  of  $B$  is  $[0 \ 0 \ \dots \ 0]$ .

$$\Rightarrow |A|=|B|=0$$



## Example 3

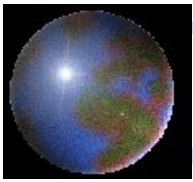
Show that the following matrices are singular.

$$(a) A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b) B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

### Solution

- (a) All the elements in column 2 of  $A$  are zero. Thus  $|A| = 0$ .
- (b) Row 2 and row 3 are proportional. Thus  $|B| = 0$ .





## Theorem 3.4

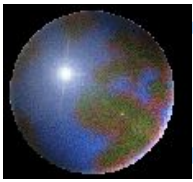
Let  $A$  and  $B$  be  $n \times n$  matrices and  $c$  be a nonzero scalar.

- (a)  $|cA| = c^n |A|$ .
- (b)  $|AB| = |A||B|$ .
- (c)  $|A^t| = |A|$ .
- (d)  $|A^{-1}| = \frac{1}{|A|}$  (assuming  $A^{-1}$  exists)

### Proof

$$(a) \quad A \underset{cR1, cR2, \dots, cRn}{\approx} cA \Rightarrow |cA| = c^n |A|$$

$$(d) \quad |A| \cdot |A^{-1}| = |A \cdot A^{-1}| = |I| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$$



## Example 4

If  $A$  is a  $2 \times 2$  matrix with  $|A| = 4$ , use Theorem 3.4 to compute the following determinants.

- (a)  $|3A|$                       (b)  $|A^2|$                       (c)  $|5A^tA^{-1}|$ , assuming  $A^{-1}$  exists

### Solution

(a)  $|3A| = (3^2)|A| = 9 \times 4 = 36.$

(b)  $|A^2| = |AA| = |A| |A| = 4 \times 4 = 16.$

(c)  $|5A^tA^{-1}| = (5^2)|A^tA^{-1}| = 25|A^t||A^{-1}| = 25|A| \frac{1}{|A|} = 25.$

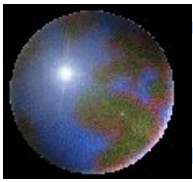
### Example 5

Prove that  $|A^{-1}A^tA| = |A|$



### Solution

$$|A^{-1}A^tA| = |(A^{-1}A^t)A| = |A^{-1}A^t||A| = |A^{-1}||A^t||A| = \frac{1}{|A|}|A||A| = |A|$$



## Example 6

Prove that if  $A$  and  $B$  are square matrices of the same size, with  $A$  being singular, then  $AB$  is also singular. Is the converse true?

### Solution

( $\Rightarrow$ )

$$|A| = 0 \quad \Rightarrow \quad |AB| = |A||B| = 0$$

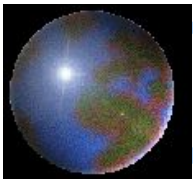
Thus the matrix  $AB$  is singular.

( $\Leftarrow$ )

$$|AB| = 0 \Rightarrow |A||B| = 0 \Rightarrow |A| = 0 \text{ or } |B| = 0$$

Thus  $AB$  being singular implies that either  $A$  or  $B$  is singular.

The inverse is not true.



# Homework

- Exercises will be given by the teachers of the practical classes.

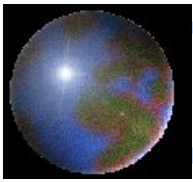
## Exercise 11

Prove the following identity without evaluating the determinants.

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & c & e \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} b & d & f \\ p & q & r \\ u & v & w \end{vmatrix}$$

**Solution**

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = (a+b) \begin{vmatrix} q & r \\ v & w \end{vmatrix} - (c+d) \begin{vmatrix} p & r \\ u & w \end{vmatrix} + (e+f) \begin{vmatrix} p & q \\ u & v \end{vmatrix}$$



## 3.3 Numerical Evaluation of a Determinant

### Definition

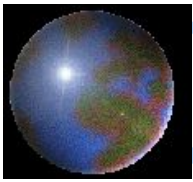
A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

upper – triangular

lower – triangular



# Numerical Evaluation of a Determinant

## Theorem 3.5

The determinant of a triangular matrix is the product of its diagonal elements.

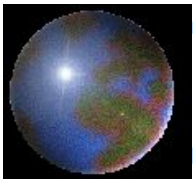
## Proof

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & a_{34} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = \dots = a_{11} a_{22} \dots a_{nn}$$

## Example 1

Let  $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$ , find  $|A|$ .

Sol.  $|A| = 2 \times 3 \times (-5) = -30$ .



# Numerical Evaluation of a Determinant

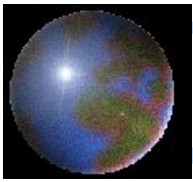
## Example 2

Evaluation the determinant.  $\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$

**Solution** (elementary row operations)

$$\begin{vmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{vmatrix} \begin{array}{l} = \\ \text{R2} + \text{R1} \\ \text{R3} + (-2)\text{R1} \end{array} \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 13 \end{vmatrix} \begin{array}{l} = 2 \times (-1) \times 13 = -26 \\ \text{R3} + \text{R2} \end{array}$$



# Example 3

Evaluation the determinant.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$$

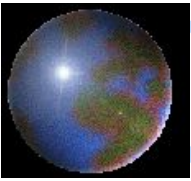
## Solution

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix} \begin{array}{l} = \\ R2 + (-2)R1 \\ R3 + (-1)R1 \\ R4 + R1 \end{array} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} \begin{array}{l} \\ \\ R4 + 2R3 \end{array}$$

$$= 1 \times (-1) \times (-2) \times 6 = 12$$





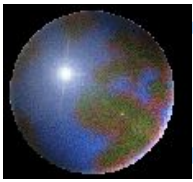
# Example 4

Evaluation the determinant.  $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

## Solution

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix} &= \begin{vmatrix} 1 & -2 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix} \begin{array}{l} \text{R2} + \text{R1} \\ \text{R3} + (-2)\text{R1} \end{array} \\ &= \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix} \begin{array}{l} \\ \text{R2} \leftrightarrow \text{R3} \end{array} \\ &= (-1) \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix} \end{aligned}$$
$$= (-1) \times 1 \times 2 \times (-1) = 2$$





# Example 5

Evaluation the determinant.

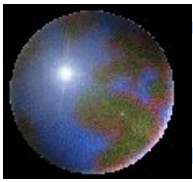
$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$$

## Solution

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} \begin{array}{l} \\ R2 + R1 \\ R3 + (-2)R1 \\ R4 + (-6)R1 \end{array} = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & -11 \end{vmatrix} = 0$$

diagonal element is zero and all elements below this diagonal element are zero.





## 3.4 Determinants, Matrix Inverse, and Systems of Linear Equations

### Definition

Let  $A$  be an  $n \times n$  matrix and  $C_{ij}$  be the cofactor of  $a_{ij}$ .

The matrix whose  $(i, j)$ th element is  $C_{ij}$  is called the **matrix of cofactors** of  $A$ .

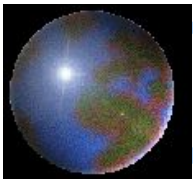
The transpose of this matrix is called the **adjoint** of  $A$  and is denoted  $\text{adj}(A)$ .

$$\begin{bmatrix} C_{11} & C_{12} & \boxtimes & C_{1n} \\ C_{21} & C_{22} & \boxtimes & C_{2n} \\ \boxtimes & \boxtimes & & \boxtimes \\ C_{n1} & C_{n2} & \boxtimes & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \boxtimes & C_{1n} \\ C_{21} & C_{22} & \boxtimes & C_{2n} \\ \boxtimes & \boxtimes & & \boxtimes \\ C_{n1} & C_{n2} & \boxtimes & C_{nn} \end{bmatrix}^t$$

adjoint matrix



# Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix  $A$ .

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

**Solution** The cofactors of  $A$  are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14 \quad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3 \quad C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9 \quad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \quad C_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

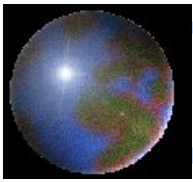
$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1 \quad C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors

of  $A$  is 
$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of  $A$  is

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$



## Theorem 3.6

Let  $A$  be a square matrix with  $|A| \neq 0$ .  $A$  is invertible with

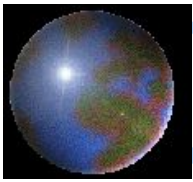
$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

### Proof

Consider the matrix product  $A \cdot \text{adj}(A)$ . The  $(i, j)$ th element of this product is

$(i, j)$ th element = (row  $i$  of  $A$ )  $\times$  (column  $j$  of  $\text{adj}(A)$ )

$$\begin{aligned} &= [a_{i1} \quad a_{i2} \quad \boxtimes \quad a_{in}] \begin{bmatrix} C_{j1} \\ C_{j2} \\ \boxtimes \\ C_{jn} \end{bmatrix} \\ &= a_{i1}C_{j1} + a_{i2}C_{j2} + \boxtimes + a_{in}C_{jn} \end{aligned}$$



# Proof of Theorem 3.6

If  $i = j$ ,  $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |A|$ .

If  $i \neq j$ , let  $A \Rightarrow B$ .  
 $R_j$  is replaced by  $R_i$

Matrices A and B have the same cofactors  $C_{j1}, C_{j2}, \dots, C_{jn}$ .

So  $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |B| = 0$ .

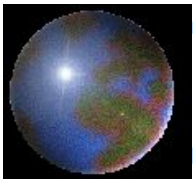
row  $i =$  row  $j$  in  $B$

Therefore  $(i, j)$ th element =  $\begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \therefore A \cdot \text{adj}(A) = |A|I_n$

Since  $|A| \neq 0$ ,  $A \left( \frac{1}{|A|} \text{adj}(A) \right) = I_n$

Similarly,  $\left( \frac{1}{|A|} \text{adj}(A) \right) A = I_n$ .

Thus  $A^{-1} = \frac{1}{|A|} \text{adj}(A)$



## Theorem 3.7

A square matrix  $A$  is invertible if and only if  $|A| \neq 0$ .

### Proof

( $\Rightarrow$ ) Assume that  $A$  is invertible.

$$\Rightarrow AA^{-1} = I_n.$$

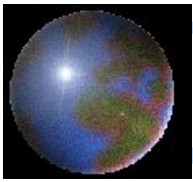
$$\Rightarrow |AA^{-1}| = |I_n|.$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

( $\Leftarrow$ ) Theorem 3.6 tells us that if  $|A| \neq 0$ , then  $A$  is invertible.

*$A^{-1}$  exists if and only if  $|A| \neq 0$ .*



## Example 2

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

### Solution

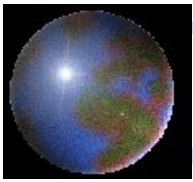
$|A| = 5 \neq 0$ .  $A$  is invertible.

$|B| = 0$ .  $B$  is singular. The inverse does not exist.

$|C| = 0$ .  $C$  is singular. The inverse does not exist.

$|D| = 2 \neq 0$ .  $D$  is invertible.





## Example 3

Use the formula for the inverse of a matrix to compute the inverse of the matrix

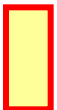
$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

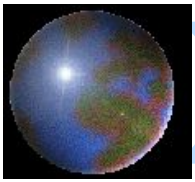
### Solution

$|A| = 25$ , so the inverse of  $A$  exists. We found  $\text{adj}(A)$  in Example 1

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{25} \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$





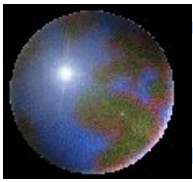
# Homework

- Exercises will be given by the teachers of the practical classes.

- *Exercise*

Show that if  $A = A^{-1}$ , then  $|A| = \pm 1$ .

Show that if  $A^t = A^{-1}$ , then  $|A| = \pm 1$ .



## Theorem 3.8

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables.

(1) If  $|A| \neq 0$ , there is a unique solution.

(2) If  $|A| = 0$ , there may be many or no solutions.

### Proof

(1) If  $|A| \neq 0$

$\Rightarrow A^{-1}$  exists (Thm 3.7)

$\Rightarrow$  there is then a unique solution given by  $X = A^{-1}B$  (Thm 2.9).

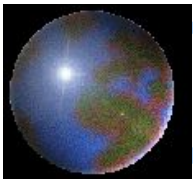
(2) If  $|A| = 0$

$\Rightarrow$  since  $A \approx \dots \approx C$  implies that if  $|A| \neq 0$  then  $|C| \neq 0$  (Thm 3.2).

$\Rightarrow$  the reduced echelon form of  $A$  is not  $I_n$ .

$\Rightarrow$  The solution to the system  $AX = B$  is not unique.

$\Rightarrow$  many or no solutions.



## Example 4

Determine whether or not the following system of equations has an unique solution.

$$3x_1 + 3x_2 - 2x_3 = 2$$

$$4x_1 + x_2 + 3x_3 = -5$$

$$7x_1 + 4x_2 + x_3 = 9$$

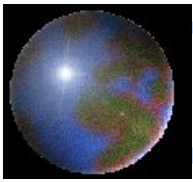
### Solution

Since

$$\begin{vmatrix} 3 & 3 & -2 \\ 4 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix} = 0$$

Thus the system does not have an unique solution.





## Theorem 3.9 Cramer's Rule

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables such that  $|A| \neq 0$ . The system has a unique solution given by

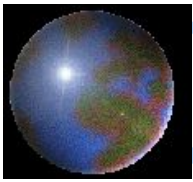
$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

Where  $A_i$  is the matrix obtained by replacing column  $i$  of  $A$  with  $B$ .

### Proof

$|A| \neq 0 \Rightarrow$  the solution to  $AX = B$  is unique and is given by

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{|A|} \text{adj}(A)B \end{aligned}$$



# Proof of Cramer's Rule

$x_i$ , the  $i$ th element of  $X$ , is given by

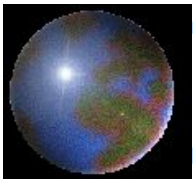
$$x_i = \frac{1}{|A|} [\text{row } i \text{ of } \text{adj}(A)] \times B$$

$$= \frac{1}{|A|} [C_{1i} \ C_{2i} \ \dots \ C_{ni}] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{|A|} (b_1 C_{1i} + b_2 C_{2i} + \dots + b_n C_{ni})$$

the cofactor expansion of  $|A_i|$   
in terms of the  $i$ th column

Thus  $x_i = \frac{|A_i|}{|A|}$



## Example 5

Solving the following system of equations using Cramer's rule.

$$x_1 + 3x_2 + x_3 = -2$$

$$2x_1 + 5x_2 + x_3 = -5$$

$$x_1 + 2x_2 + 3x_3 = 6$$

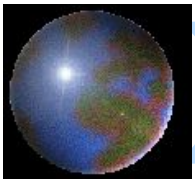
### Solution

The matrix of coefficients  $A$  and column matrix of constants  $B$  are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

It is found that  $|A| = -3 \neq 0$ . Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$



Giving  $|A_1| = -3, |A_2| = 6, |A_3| = -9$

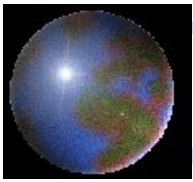
Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2, \quad x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is  $x_1 = 1, x_2 = -2, x_3 = 3$ .







## Example 6

Determine values of  $\lambda$  for which the following system of equations has nontrivial solutions. Find the solutions for each value of  $\lambda$ .

$$(\lambda + 2)x_1 + (\lambda + 4)x_2 = 0$$

$$2x_1 + (\lambda + 1)x_2 = 0$$

### Solution

homogeneous system

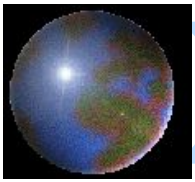
$\Rightarrow x_1 = 0, x_2 = 0$  is the trivial solution.

$\Rightarrow$  nontrivial solutions exist  $\Rightarrow$  many solutions

$$\Rightarrow \begin{vmatrix} \lambda + 2 & \lambda + 4 \\ 2 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda + 2)(\lambda + 1) - 2(\lambda + 4) = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = 2.$$



$\lambda = -3$  results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

This system has many solutions,  $x_1 = r, x_2 = r$ .

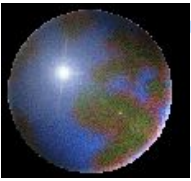
$\lambda = 2$  results in the system

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions,  $x_1 = -3r/2, x_2 = r$ .





# *Homework*

- Exercises will be given by the teachers of the practical classes.