Discrete Mathematics

PROBABILITY-II

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"The Consequences of an Act Affect the Probability of its Occurring Again!"
- B. F. Skinner -

Recap

1. Why should we learn Probability?

2. Formulating questions in terms of probability

3. Building the probability model
   - Four-step Method

4. Uniform sample spaces

5. Counting

Today’s Objectives

1. Counting subsets of a set

2. Conditional Probability

3. Independence

4. Total Probability Theorem

5. Baye’s theorem

6. Random variables

Counting Subsets of a Set

$\binom{n}{k}$ =The number of k−element subsets of an n element set.

Is read as "n choose k"

Why Count Subsets of Set?

- **Example:**

  Suppose we select 5 cards at random from a deck of 52 cards.

  What is the probability that we will end up having a full house?

  Doing this using the possibility tree will take some effort.

Counting Subsets of a Set

$$\binom{n}{k} = ?$$

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$$n(n-1)(n-2)\ldots(n-k+1)=\frac{n!}{(n-k)!}$$

Counting Subsets of a Set

- To solve for this, let's start by looking at two ways of constructing an **ordered** sequence of k distinct items!

1. Pick an element one at a time

   $$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

2. Choose k elements and order them

Counting Subsets of a Set

- To solve for this, let's start by looking at two ways of constructing an **ordered** sequence of k distinct items!

1. Pick an element one at a time

$$n(n-1)(n-2)\ldots(n-k+1)=\frac{n!}{(n-k)!}$$

2. Choose k elements and order them

$$\binom{n}{k}k!$$

Counting Subsets of a Set

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  ❖ Thus

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

Counting Subsets of a Set

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$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

Hence

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Conditional Probability

- **An Interesting Kind of Probability Question**

- "After this lecture, when I go to UI canteen for lunch, what is the probability that today they will be serving biryani (my favorite food)?

Biryani ☺

Conditional Probability

- Of course, the vast majority of the food that the cafeteria prepares is NEITHER delicious NOR is it ever biryani (low probability).
- But they do cook dishes that contain rice, so now the question is "what's the probability that food from UI is delicious given that it contains rice?"
- This is called **"Conditional Probability"**

Conditional Probability

- What is the probability that it will rain this afternoon, given that it is cloudy this morning?

- What is the probability that two rolled dice sum to 10, given that both are odd?

Written as

- **P(A|B)** – denotes the probability of event A, given that event B happens.

Conditional Probability

- **So, how to answer the “Food Court” question?**

Conditional Probability

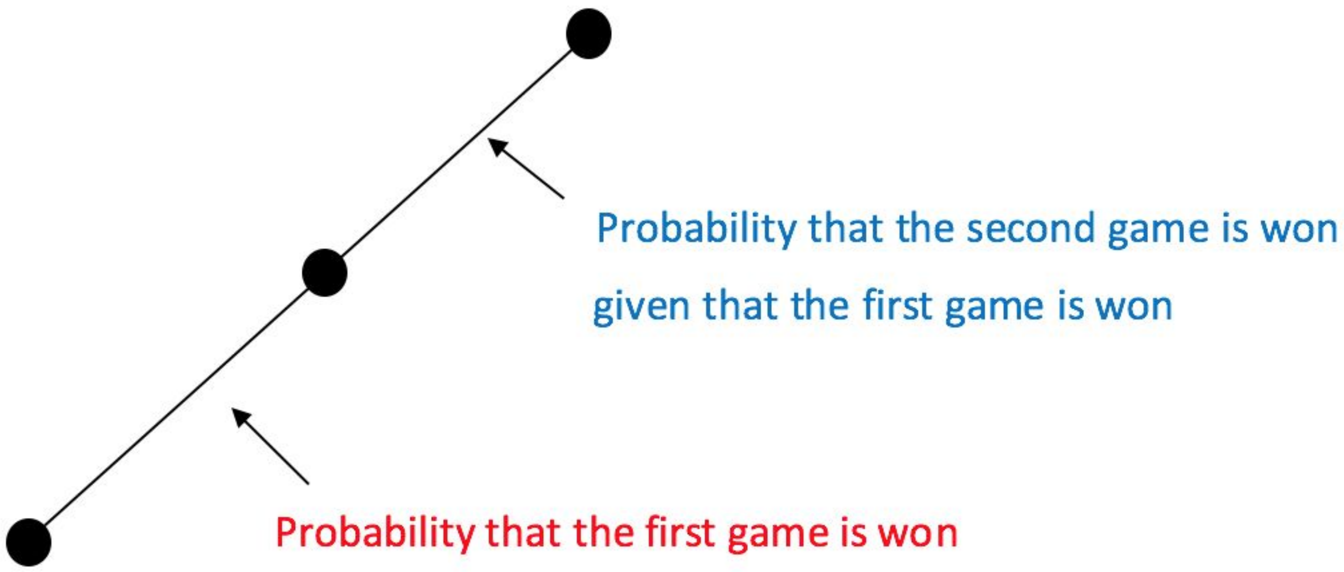
$$P(A|B) = \frac{\mathrm{P}\,(A \cap B)}{\mathrm{P}(B)}$$

Why Do Tree Diagrams Work?

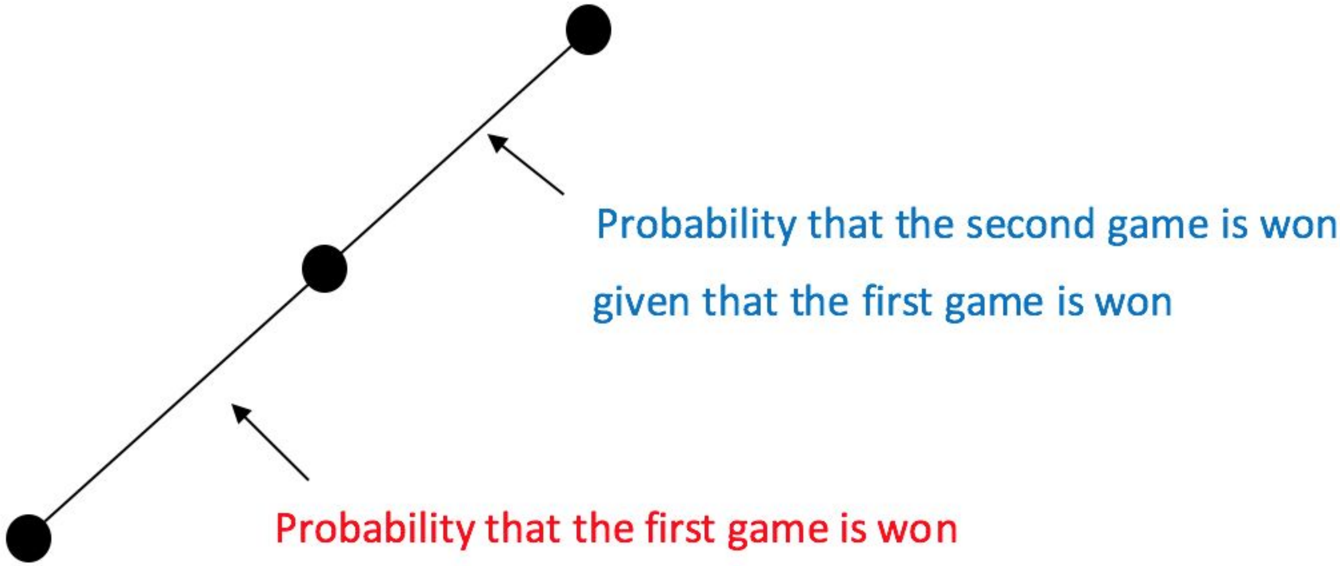
- We have solved multiple probability problems using tree diagrams
- Let's think for a moment about "why do tree diagrams work?"
- The answer involves conditional probabilities
- In fact, the probabilities that we have been recording on the edges of a tree diagram are conditional probabilities
- More generally, on each edge of a tree diagram, we record that the probability that the experiment proceeds along that part, given that it reaches the parent vertex

Why Do Tree Diagrams Work?

Let's look the upper most edges of the probability tree for the previous example!

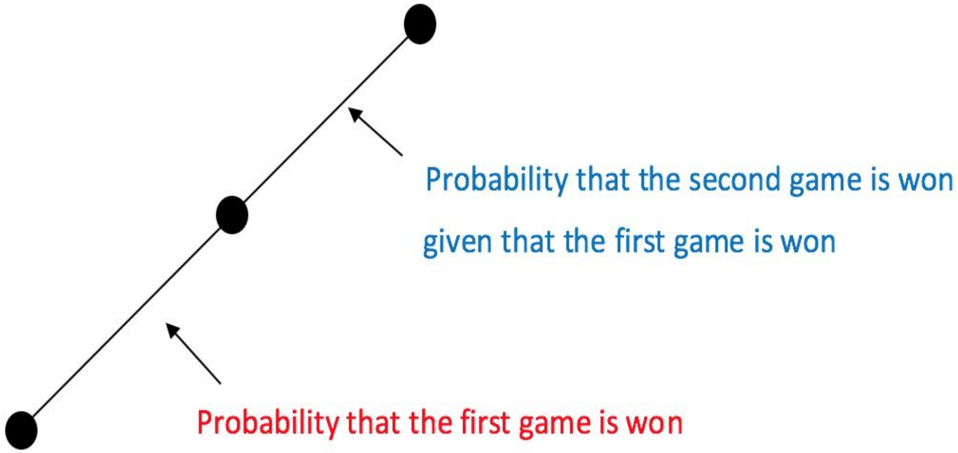


Why Do Tree Diagrams Work?



$$\mathrm{P}(W1W2) = \mathrm{P}(W1 \cap W2) = \frac{1}{2} . \frac{2}{3} = \frac{1}{3}$$

Why Do Tree Diagrams Work?



$$P(WW) = \frac{1}{2} . \frac{2}{3} = \frac{1}{3}$$

$$\mathbf{P}(\boldsymbol{win\ first\ game} \cap \boldsymbol{win\ second\ game})$$

$$= \mathbf{P}\ (\boldsymbol{win\ first\ game}) . \boldsymbol{P}(\boldsymbol{win\ second\ game} \mid \boldsymbol{win\ first\ game})$$

Why Do Tree Diagrams Work?

Rule (Product Rule for 2 Events). *If* $\Pr[E_1] \neq 0$, *then:*

$$\Pr[E_1 \cap E_2] = \Pr[E_1] \cdot \Pr\left[E_2 \mid E_1\right].$$

Rule (Product Rule for n Events).

$$\Pr[E_1 \cap E_2 \cap \ldots \cap E_n] = \Pr[E_1] \cdot \Pr\left[E_2 \mid E_1\right] \cdot \Pr\left[E_3 \mid E_1 \cap E_2\right] \cdots \cdot \Pr\left[E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1}\right]$$

provided that

$$\Pr[E_1 \cap E_2 \cap \cdots \cap E_{n-1}] \neq 0.$$

"So the *Product Rule* is the formal justification for *multiplying edge probabilities* in a probability tree to get outcome probabilities"

Independence

- Intuitively, two events A and B are independent if knowing that A happens does not affect the probability that B happens

- Thus

$$\boldsymbol{P(B|A) = P(B)}$$

Independence

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- Now, we already know that

$$P(B \cap A) = P(A)P(B|A)$$

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- Now, we already know that

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- Putting two together

$$\boxed{P(B \cap A) = P(A)P(B)}$$

Independence

- Why use this definition instead of the intuitive one?

$$P(B \cap A) = P(A)P(B)$$

Independence

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Because it is symmetric in the roles of A and B

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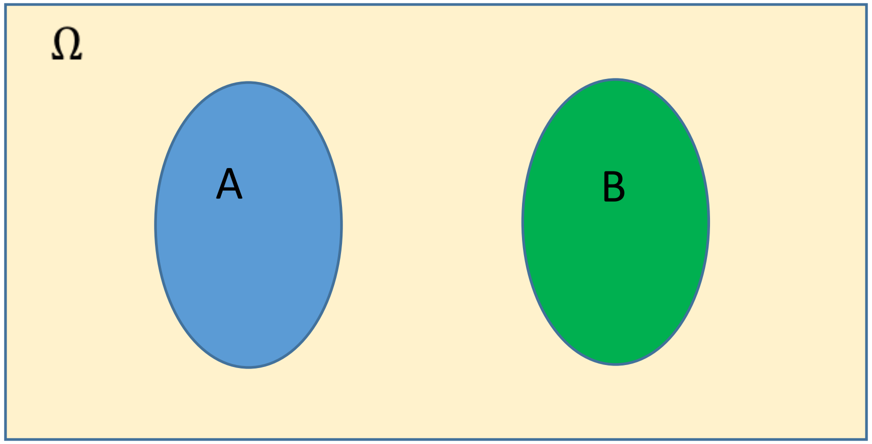
$$P(B \cap A) = P(A)P(B)$$

Because it is symmetric in the roles of A and B

$$\Longrightarrow P(A|B) = P(A)$$

What Independence Really Means?

- Are these events independent?



$P(A) > 0 \; and \; P(B) > 0$

What Independence Really Means?

- Thus being dependent is completely different from being disjoint!

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- Two events are independent, if the occurrence of one does not change our belief about the occurrence of the other.

- Typically the case when the two events are determined by two physically distinct and non-interacting processes.
  - Getting heads in a coin toss and snowing outside

Independence---Cont.

- Generally, independence is an assumption that we assume when modeling a phenomenon.
- The reason we so-often assume statistical independence is not because of its real-world accuracy
- It is because of its armchair appeal: It makes the math easy

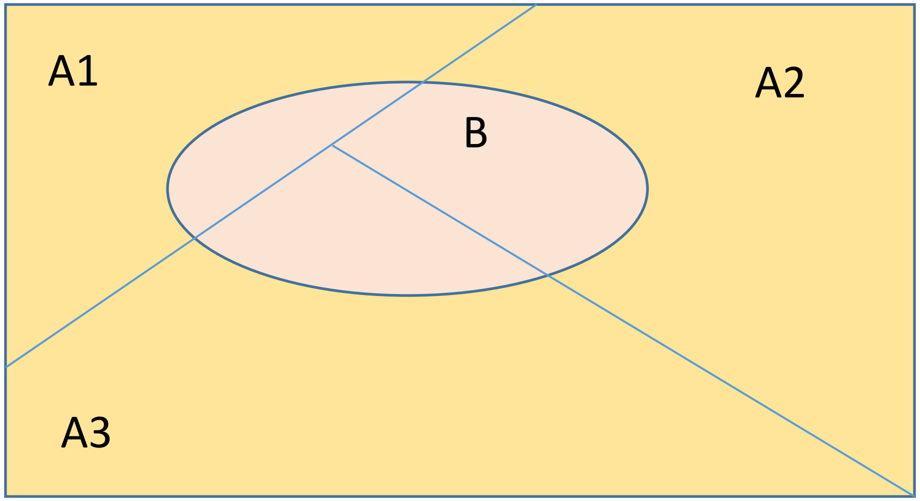
How does it do that?

- By splitting a compound probability into a product of individual probabilities.

(Note for TAs: Include example of Independence assumption in tutorials)

Total Probability Theorem

- Take a look at the figure below



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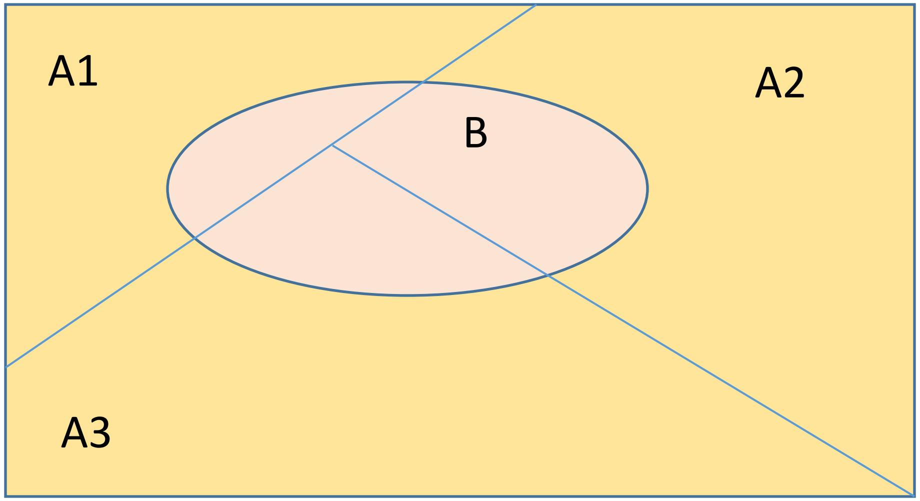

$P(B)$?

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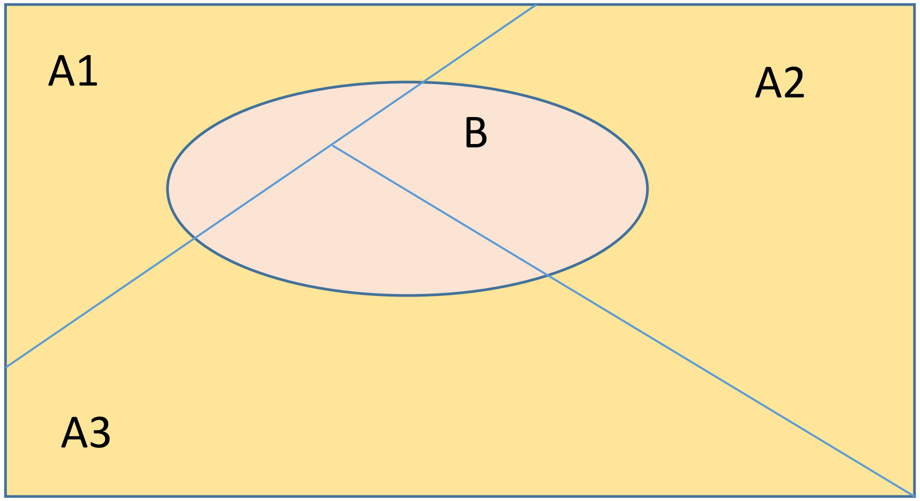

$$P(B) = P(B \cap A1) + P(B \cap A2) + P(B \cap A3)$$

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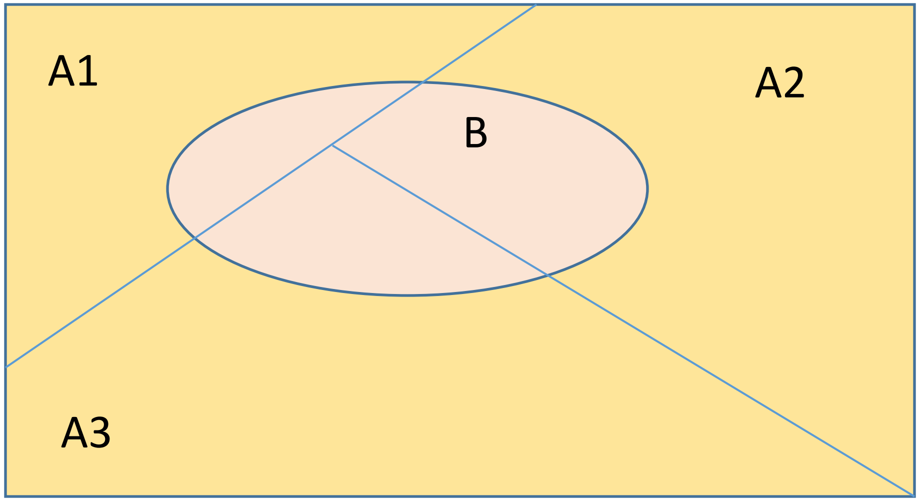

$$P(B) = P(B \cap A1) + P(B \cap A2) + P(B \cap A3)$$

$$= P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3)$$

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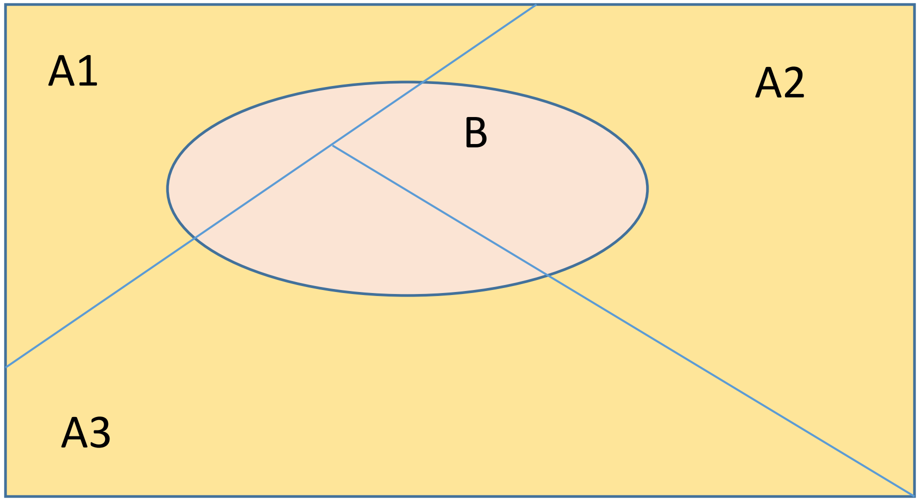

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$$= P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3)$$

$$= \sum_i P(Ai)P(B|Ai)$$

Total Probability Theorem

- Where do we use it?
  - ❖ Baye’s Theorem!

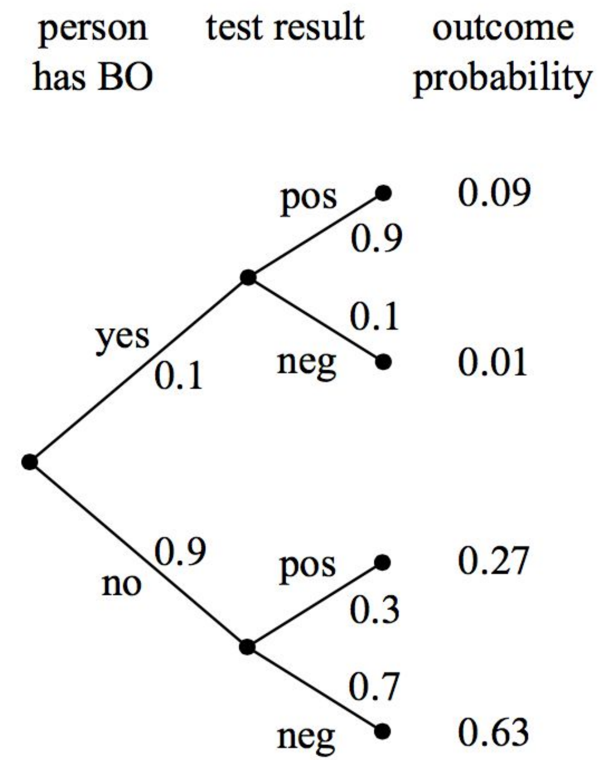
Medical Testing Problem

- Let's assume a "not-so-perfect" test for a medical condition called BO suffered by 10% of the population
- The test is not-so-perfect because
  - **90%** of the tests come positive if you have BO
  - **70%** of the tests come negative if you don't have BO
- If we randomly test a person for BO, and if the test comes positive, what is the probability that the person has BO.

Probability Tree

A: The test came positive

B: The person has BO



BO is suffered by 10% of the population

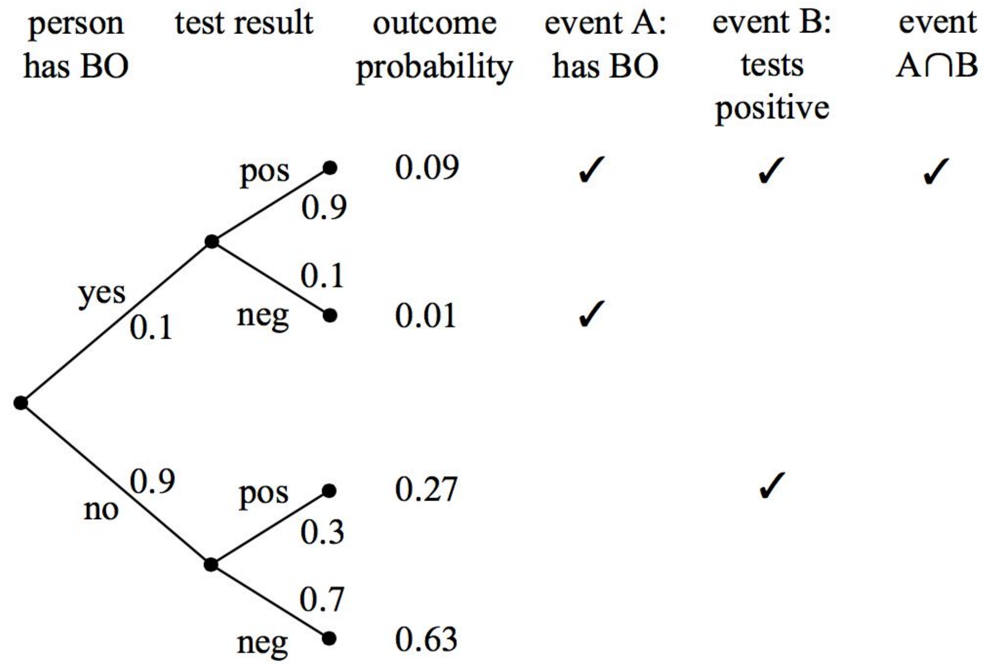
If someone has BO, there is a 90% chance that the test will be positive

If someone does not have the condition, there is a 70% chance that the test will be negative.

Probability Tree

A: The test is positive

B: The person has BO



$$P(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$$

$$P(B|A) = \frac{0.09}{0.09 + 0.27} = \frac{1}{4}$$

Conditional Probability Tree---Cont.

- **Surprising, Right!**

  - So if the test comes out positive, the person has only 25% chance of having the diseases

  - Conclusion:

    - Tests are flawed

    - Tests give test probabilities not the real probabilities

Bayes Theorem

- **How to correct for such Flawed Tests**

- **Bayes Theorem**
  - It lets you relate the test probabilities with the real probabilities.
  - More specifically, it lets you relate $\boldsymbol{P(A|B)}$ with $\boldsymbol{P(B|A)}$.
  - **What is $\boldsymbol{P(B|A)}$?**

Bayes Theorem---Cont.

- **A Posteriori Probabilities**
  - A conditional probability in reverse $\boldsymbol{P(B|A)}$ is called a posteriori probability.
  - You can understand this by considering that event B precedes event A in time.

Bayes Theorem---Cont.

- **A Posteriori Probabilities**

- **For example:**

  - The probability that it was cloudy this morning, given that it rained in the afternoon.

- Mathematically speaking, there is no difference between a posteriori probability and a conditional probability.

Flawed Test

- **Coming Back to Flawed Test**

- ***Let***

A: The test came positive

B: Person has BO

Flawed Test

- **Then**
    - $\boldsymbol{P(A|B)}$ means the chance that indicator **A** (a person's test came positive) happened given that the event **B** occurred (the person has the disease).

Flawed Test

- $\boldsymbol{P(A|B)}$ means the chance that indicator **A** (a person's test came positive) happened given that the event **B** occurred (the person has the disease).

- $\boldsymbol{P(B|A)}$ means the probability that event **B** (a person having disease) happened given the indicator **A** (the person's test came positive)

Flawed Test

- $P(A|B)$ means the chance that indicator **A** (a person's test came positive) happened given that the event **B** occurred (the person has the disease).

- $P(B|A)$ means the probability that event **B** (a person having disease) happened given the indicator **A** (the person's test came positive)

$$P(B|A) = \frac{P(A|B) . \mathrm{P}(B)}{\mathrm{P}(A)}$$

Flawed Test

$$\boldsymbol{P(B|A) = \frac{P(A|B)\,.\,\mathrm{P}(B)}{\mathrm{P}(A)}}$$

$$P(Has\ BO|Pos\ Test) = \frac{P(Pos\ Test|Has\ BO)}{P(Pos\ test)}$$

Random Variables

- So far, we focused on probabilities of events.
- For example,
  - The probability that someone wins the Monty Hall Game
  - The probability that someone has a rare medical condition given that he/she tests positive

Random Variables

- But most often, we are interested in knowing more than this.

- For example,
  - ❖ How many players must play Monty Hall Game before one of them finally wins?

  - ❖ How long will a weather certain condition last?

  - ❖ How long will I loose gambling with a strange coin all night?

- To be able to answer such questions, we have to learn about "Random Variables"

Random Variables---Cont.

- "Random Variables" are nothing but "functions"
- A *random variable* *R* on a probability space is a function whose domain is the sample space and whose range is a set of Real numbers.

Random Variables---Cont.

- “Random Variables” are nothing but “functions”
- A *random variable R* on a probability space is a function whose domain is the sample space and whose range is a set of Real numbers.
- Let’s look at this example!
  - Tossing three independent coins and noting
    - C: the number of heads that appear
    - M: 1 if all are heads or tails, 0 otherwise
- If we look closely, we will see that C and M are in fact functions that map every outcome of the experiment to a number.

Random Variables---Cont.

- **Example ---Cont.**

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$C(HHH) = 3$ $C(THH) = 2$

$C(HHT) = 2$ $C(THT) = 1$

$C(HTH) = 2$ $C(TTH) = 1$

$C(HTT) = 1$ $C(TTT) = 0$

$M(HHH) = 1$ $M(THH) = 0$

$M(HHT) = 0$ $M(THT) = 0$

$M(HTH) = 0$ $M(TTH) = 0$

$M(HTT) = 0$ $M(TTT) = 1$

- Thus **C** and **M** are random variables!

Indicator Random Variables

- Maps every outcome to either 0 or 1 – its range is {0,1} – *indicates* that a sample point has/hasn't a certain property

- M from our example: Partitions the sample space into two blocks

- Such random variables are called indicator random variables

$$\underbrace{HHH\ \ TTT}_{M=1} \qquad \underbrace{HHT\ \ HTH\ \ HTT\ \ THH\ \ THT\ \ TTH}_{M=0}$$

Random Variables and Events

- General random variables partition the sample space into several blocks).

- Note that **C** from our previous example is a general random variable

$$\underbrace{TTT}_{C=0} \quad \underbrace{TTH \;\; THT \;\; HTT}_{C=1} \quad \underbrace{THH \;\; HTH \;\; HHT}_{C=2} \quad \underbrace{HHH}_{C=3}$$

- Notice that each sample in the block has the same value for the random variable
- An equation or an inequality involving a random variable can be regarded as an event.

Random Variables and Events---Cont.

- For example

$$P(C = 2) = P(HHT, HTH, HHT)$$

$$= P(THH) + P(HTH) + P(HHT)$$

Random Variables.

- More generally, an event can be defined as

$$\{w|R(w) = x\} \; is\ the\ event\ that\ R = x$$

- And its probability can be defined as

$$\mathrm{P}(R = x) = \sum_{w|R(w)=x} \mathrm{P}(w)$$

- A random variable could be continuous or discrete

- When dealing with continuous random variables, use "integrals" instead of "summations"

Expected Value

- Weighted average of the values of a random variable
- Provides a central point for the distribution of the values of a random variable
- We can solve many problems using the notion of expected values
  - ❖ How many heads are expected to appear if a coin is tossed 100 times?
  - ❖ What is the expected number of comparisons used to find an element in a list using the linear search?

Expected Value---Cont.

$$Ex[R] ::= \sum_{w \in S} R(w) Pr[w]$$

Expected Value---Cont.

$$Ex[R] ::= \sum_{w \in S} R(w) Pr[w]$$

- For example, the expected value of a random variable with uniform distribution on $\{1, 2, \ \dots, n\}$ is

$$Ex[R_n] = \sum_{i=1}^{n} i \, . \frac{1}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

Variance

Consider the following two gambling games:

- Game A: You win \$2 with probability 2/3 and lose \$1 with probability 1/3.

- Game B: You win \$1002 with probability 2=3 and lose \$2001 with probabil- ity 1=3.

- Which game would you play?

Variance

Let's compute the expected return for both games:

Variance

- Game A: You win \$2 with probability 2/3 and lose \$1 with probability 1/3.

$$Ex[A] = 2.\frac{2}{3} + (-1).\frac{1}{3} = 1$$

Variance

- Game B: You win \$1002 with probability 2=3 and lose \$2001 with probabil- ity 1=3.

$$Ex[B] = 1002.\frac{2}{3} + (-2001).\frac{1}{3} = 1$$

Expected return is the same. Thus expected value is not enough to make the decision

Variance

The variance Var[R] of a random variable R is

$$Var[R] = Ex[(R - Ex[R])^2]$$

Variance

- Game A: You win \$2 with probability 2/3 and lose \$1 with probability 1/3.

$$A - \mathrm{Ex}[A] = \begin{cases} 1 & \text{with probability } \frac{2}{3} \\ -2 & \text{with probability } \frac{1}{3} \end{cases}$$

$$(A - \mathrm{Ex}[A])^2 = \begin{cases} 1 & \text{with probability } \frac{2}{3} \\ 4 & \text{with probability } \frac{1}{3} \end{cases}$$

$$\mathrm{Ex}[(A - \mathrm{Ex}[A])^2] = 1 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3}$$

$$\boxed{\mathrm{Var}[A] = 2.}$$

Variance

For game B

$$B - \mathrm{Ex}[B] = \begin{cases} 1001 & \text{with probability } \frac{2}{3} \\ -2002 & \text{with probability } \frac{1}{3} \end{cases}$$

$$(B - \mathrm{Ex}[B])^2 = \begin{cases} 1{,}002{,}001 & \text{with probability } \frac{2}{3} \\ 4{,}008{,}004 & \text{with probability } \frac{1}{3} \end{cases}$$

$$\mathrm{Ex}[(B - \mathrm{Ex}[B])^2] = 1{,}002{,}001 \cdot \frac{2}{3} + 4{,}008{,}004 \cdot \frac{1}{3}$$

$$\boxed{\mathrm{Var}[B] = 2{,}004{,}002.}$$

- Intuitively, this means that the payoff in Game A is usually close to the expected value of \$1, but the payoff in Game B can deviate very far from this expected value – high variance means high risk.

Standard Deviation

- Because of its definition in terms of the square of a random variable, the variance of a random variable may be very far from a typical deviation from the mean.

Standard Deviation

- For example, in Game B above, the deviation from the mean is 1001 in one outcome and -2002 in the other. But the variance is a whopping 2,004,002
- The problem is with the "***units***" of variance.
  - If a random variable is in dollars, then the expected value is also in dollars, but the variance is in ***square dollars***

Standard Deviation

- For this reason, ***standard deviation*** is often used to describe the deviation of a random variable from its expected value

$$\sigma_R = \sqrt{Var[R]} = \sqrt{Ex[(R - Ex[R])^2]}$$

- For example, the standard deviation for games A and B are

$$\sigma_A = \sqrt{Var[A]} = \sqrt{2} \approx 1.41$$
$$\sigma_B = \sqrt{Var[B]} = \sqrt{2{,}004{,}002} \approx 1416$$

Why bother squaring in the first place?