Climate tipping as a noisy bifurcation: a predictive technique

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• Part I (JMTT) Bifurcations and their precursors

• Part II (JS) Normal form estimates

#### **Dynamical System**



# ControlμResponsex

- Stable (type 1 response)
   Stable (type 2 response)
   Unstable (type 1 response)
- Unstable (type 2 response)



*µ* (solar heating)

## Instantaneous Basin loss at a Fold



### Before

After

## Introduction

- Focus on the Earth, or a relevant sub-system (Lenton).
- Regard it as a nonlinear dissipative dynamical system.
- Ignore discontinuities and memory effects.
- We have a large but finite set of ODEs and phase space.
- This large complex system has activity at many scales.

#### **Effective Noise**

Small fast action is noise to the overall dynamics (OD) Models of the OD might need added random noise Bifurcations of the OD may underlie climate tipping

### **Control Parameters**

- We may have many slowly-varying control parameters,  $\mu_i$
- But they can subsumed into a single  $\mu$  (eg. slow time)
- This limits the relevant bifurcations to those with co-dimension (CD) = 1
- We now explain the co-dimension concept, before moving on to classify the *CD* = 1 bifurcations

# Unfolding Euler's Ditchfork

A real column has imperfections. With P it does not reach pitchfork, C.

Catastrophe Theory shows that only one extra control is needed to hit C.

One such control is the side load, R. R = R\* cancels out the imperfections.

Needing 2 controls to be observable we say a pitchfork has co-dimension 2.



A climate tip from a single slow evolution must be co-dimension 1.

Co-Dimension 1 Bifurcations (we shall be listing all 18)

Bifurcations can be classified as:

- (a) Safe Bifurcations
- (b) Explosive Bifurcations
- (c) Dangerous Bifurcations





#### (a) Safe Bifurcations

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#### (a.1) Local Supercritical Bifurcations

- **1. Supercritical Hopf**
- 2. Supercritical Neimark
- **3. Supercritical Flip**

(a.2) Global Bifurcations

4. Band Merging

Point to cycle Cycle to torus

Cycle to cycle

Chaos to chaos

**SUBTLE:** continuous supercritical growth of new attractor path SAFE: no fast jump or enlargement of the attracting set DETERMINATE: single outcome even with small noise NO HYSTERESIS: path retraced on reversal of control sweep NO BASIN CHANGE: basin boundary remote from attractors NO INTERMITTENCY: in the responses of the attractors

#### (b) Explosive Bifurcations

- 5. Flow Explosion
- 6. Map Explosion
- 7. Intermittency Explosion: Flow
- 8. Intermittency Explosion: Map
- 9. Regular-Saddle Explosion
- **10. Chaotic-Saddle Explosion**

Point to cycle Cycle to torus Point to chaos Cycle to chaos Chaos to chaos Chaos to chaos



CATASTROPHIC: global events, abrupt enlargement of attracting set EXPLOSIVE: enlargement, but no jump to remote attractor DETERMINATE: with single outcome even with small noise NO HYSTERESIS: paths retraced on reversal of control sweep NO BASIN CHANGE: basin boundary remote from attractors INTERMITTENCY: lingering in old domain, flashes through the new

## **Example of an Explosive Event**

Flow-explosion transforms point attractor to a cycle

Equilibrium path has a regular saddle-node fold. Saddle outset flows around a closed loop to the node. A stable cycle is created. Initial period is infinite (critical slowing).

Precursor: same as static fold.



#### (c) Dangerous Bifurcations



**CATASTROPHIC:** blue-sky disappearance of attractor **DANGEROUS:** sudden jump to new attractor (of any type) **INDETERMINACY:** outcome can depend on global topology **HYSTERESIS:** path not reinstated on control reversal **BASIN:** tends to zero (c.2), attractor hits edge of residual basin (c.1, c.3) **NO INTERMITTENCY:** but critical slowing in global events

Jump

# BASINS (1)



# BASINS (2)

#### (b) With HYSTERESIS, blue-sky JUMPS and SHRINKING BASINS



**Dangerous Bifurcations** 

### **Precursors of our 18 bifurcations**

Precursors of codimension-one bifurcations ( local decay rate of transients -> 0		
Supercritical Hopf Supercritical Neimark Supercritical flip Band merging	S: point to cycle S: cycle to torus S: cycle to cycle S: chaos to chaos	<ul> <li>linearly with control c276a</li> <li>linearly with control</li> <li>linearly with control lingers near impinging boundary</li> </ul>
Flow explosion Map explosion Intermittency expl: flow Intermittency expl: map Regular interior crisis Chaotic interior crisis	E: point to cycle E: cycle to torus E: point to chaos E: cycle to chaos E: chaos to chaos E: chaos to chaos	<ul> <li>linearly along folding path</li> <li>linearly along folding path</li> <li>linearly with control</li> <li>as for trigger (fold, flip, Neimark) lingers near impinging saddle lingers near impinging saddle</li> </ul>
Static fold Cyclic fold Subcritical Hopf Subcritical Neimark Subcritical flip Saddle connection Regular exterior crisis Chaotic exterior crisis	D: from point D: from cycle D: from point D: from cycle D: from cycle D: from cycle D: from cycle D: from chaos D: from chaos	<ul> <li>linearly along folding path</li> <li>linearly along folding path</li> <li>linearly with control</li> <li>linearly with control</li> <li>linearly with control period of cycle tends to infinity lingers near impinging saddle lingers near impinging saddle</li> </ul>

### **INDETERMINATE JUMP**

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Indeterminacy in the Cyclic Fold (possible with a 2D outset)



## **Concluding Remarks**

- Bifurcation concepts for climate studies:
- Co-dimension-one events in dissipative systems.
- Safe, explosive and dangerous forms.
- Hysteresis and basin boundary structure
- Slowing of transients prior to an instability.

Our recent publications All can be found in Jan Sieber's Homepage http://userweb.port.ac.uk/~sieberj

- J.M.T. Thompson & J. Sieber, Predicting climate tipping points, in *Geo-Engineering Climate Change* (eds. Launder & Thompson) CUP 2010.
- J.M.T. Thompson & Jan Sieber, Climate tipping as a noisy bifurcation: a predictive technique, to appear in *IMA J. Appl. Maths*. http://arxiv.org/abs/1007.1376
- J.M.T. Thompson & Jan Sieber, Predicting climate tipping as a noisy bifurcation: a review, to appear in *Int. J. Bifurcation* & *Chaos* (this is an extended version of the top paper).