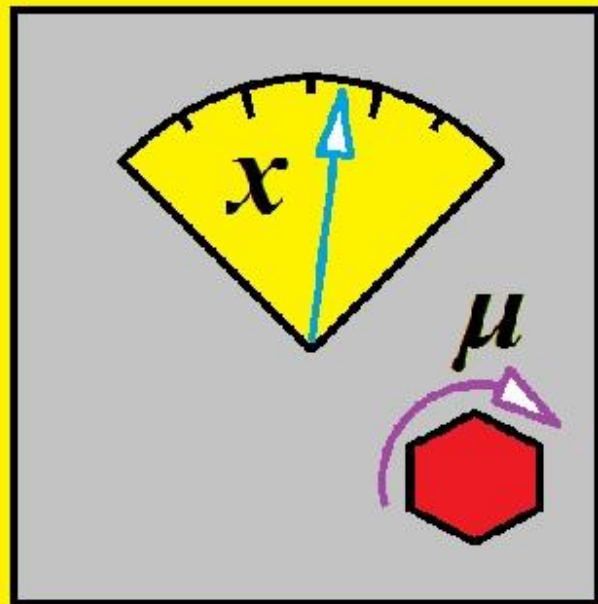


Climate tipping as a noisy bifurcation: a predictive technique

- J Michael T Thompson (DAMTP, Cambridge)
- Jan Sieber (Maths, Portsmouth)
- Part I (JMTT) Bifurcations and their precursors
- Part II (JS) Normal form estimates

Dynamical System

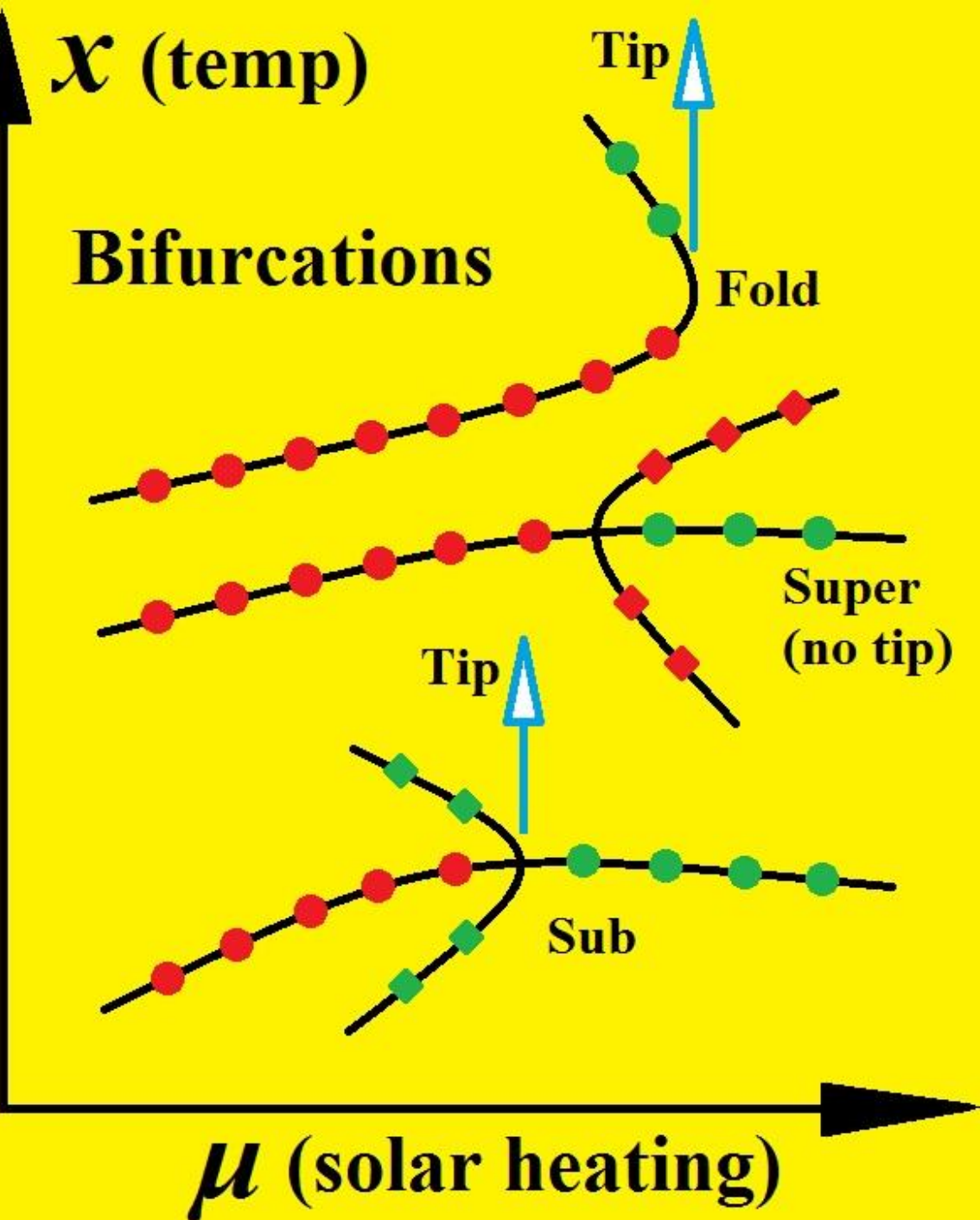


Control μ
Response x

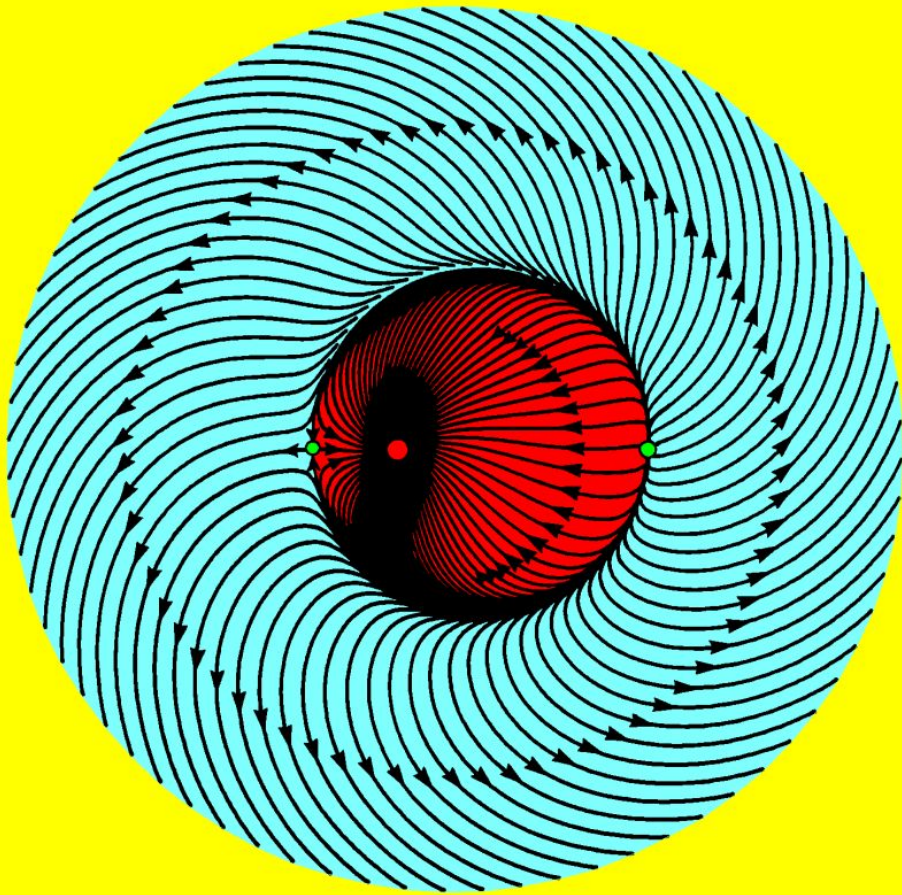
- Stable (type 1 response)
- ◆ Stable (type 2 response)
- Unstable (type 1 response)
- ◆ Unstable (type 2 response)

x (temp)

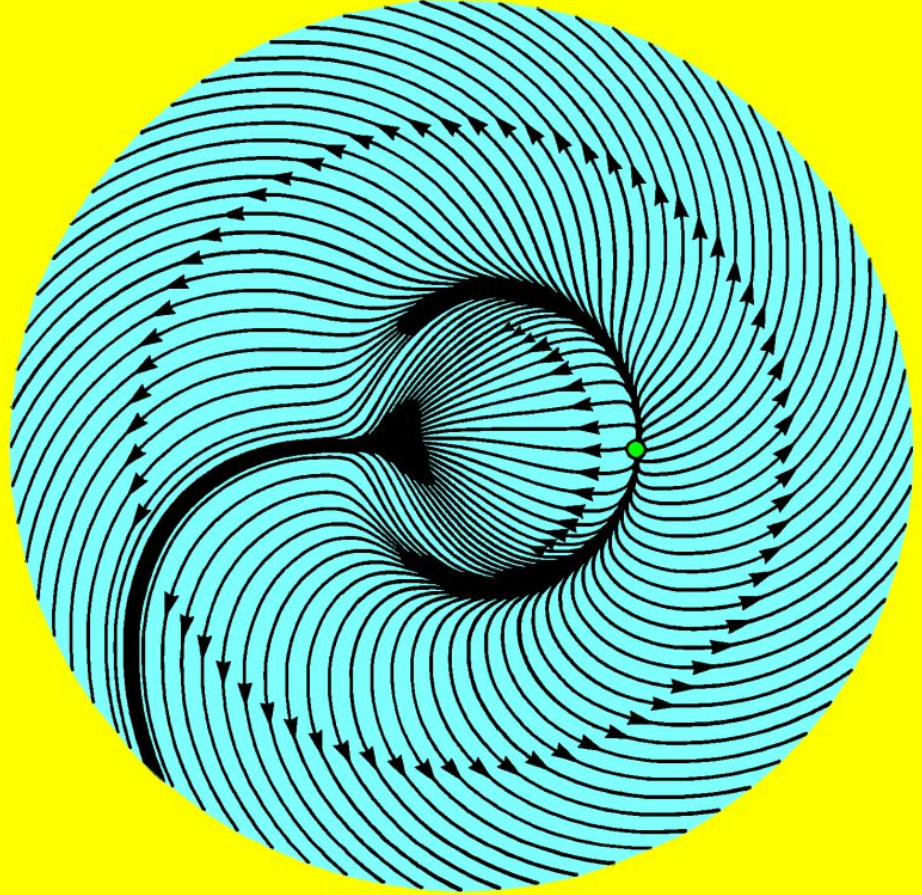
Bifurcations



Instantaneous Basin loss at a Fold



Before



After

Introduction

- Focus on the Earth, or a relevant sub-system (Lenton).
- Regard it as a nonlinear dissipative dynamical system.
- Ignore discontinuities and memory effects.
- We have a large but finite set of ODEs and phase space.
- This large complex system has activity at many scales.

Effective Noise

Small fast action is noise to the overall dynamics (OD)

Models of the OD might need added random noise

Bifurcations of the OD may underlie climate tipping

.....

.....

Control Parameters

We may have many slowly-varying control parameters, μ_i

But they can be subsumed into a single μ (eg. slow time)

This limits the relevant bifurcations to those with co-dimension (CD) = 1

We now explain the co-dimension concept, before moving on to classify the $CD = 1$ bifurcations

Unfolding Euler's Ditchfork

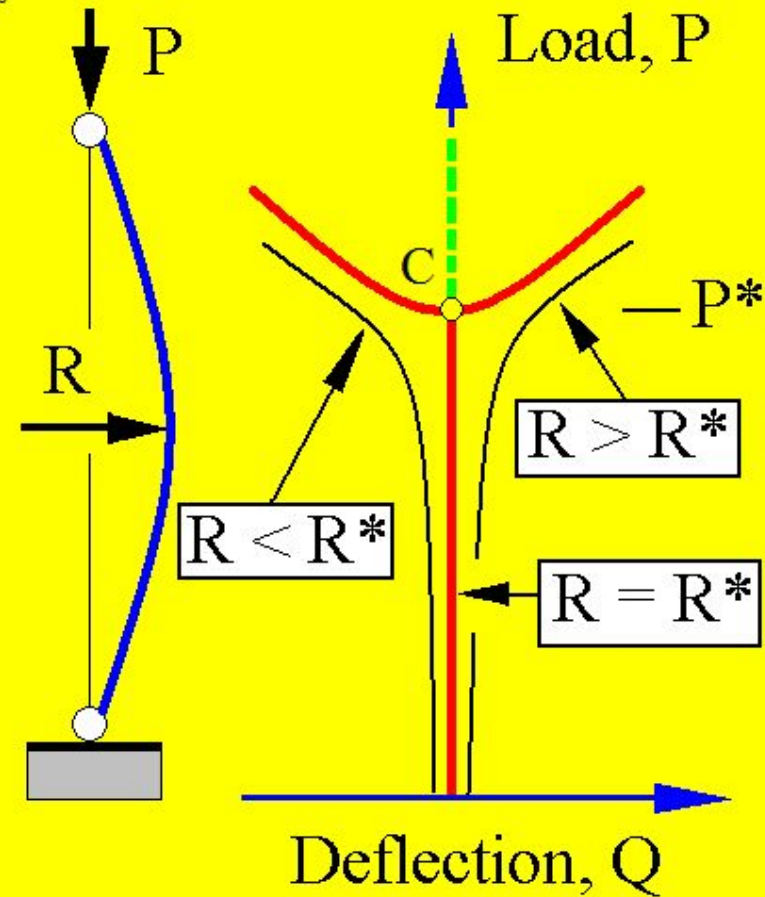
A real column has imperfections.
With P it does not reach pitchfork, C .

Catastrophe Theory shows that only
one extra control is needed to hit C .

One such control is the side load, R .
 $R = R^*$ cancels out the imperfections.

Needing 2 controls to be observable
we say a pitchfork has co-dimension 2.

A climate tip from a single slow evolution must be co-dimension 1.



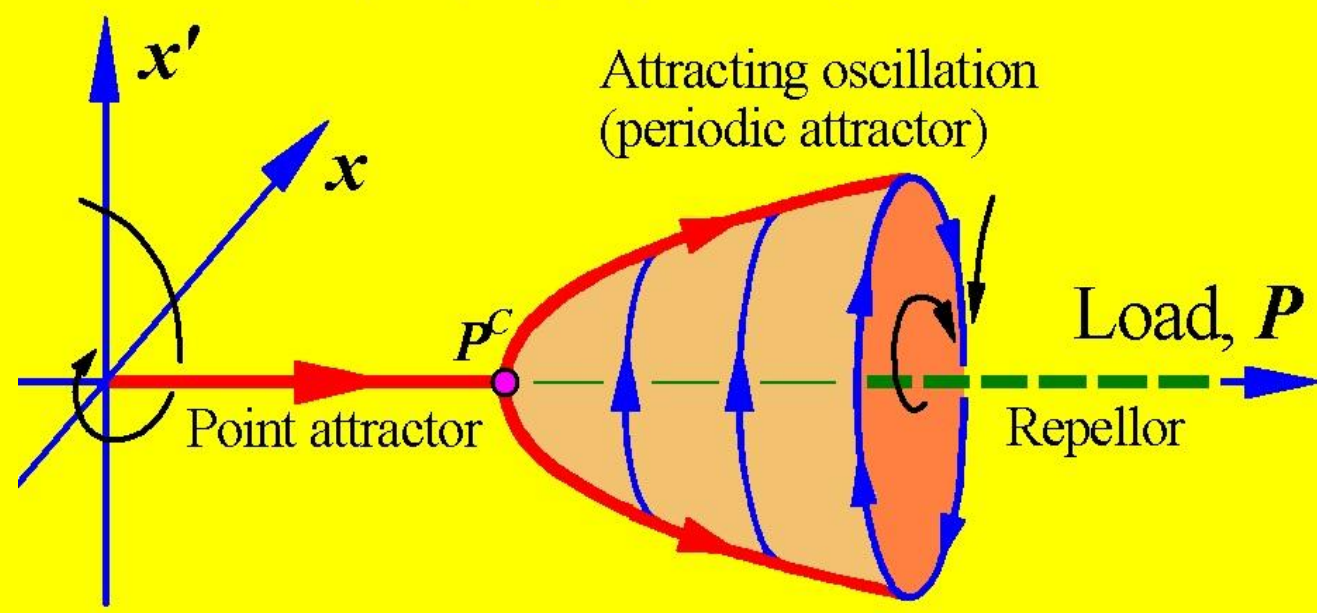
Co-Dimension 1 Bifurcations

(we shall be listing all 18)

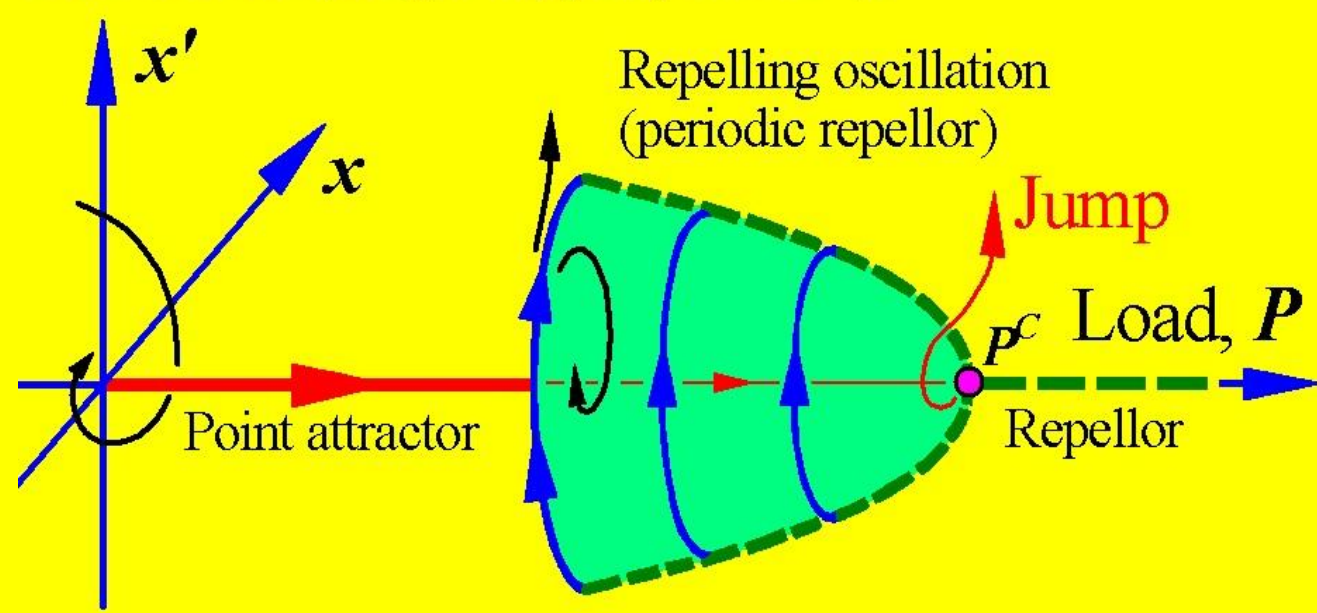
- Bifurcations can be classified as:
 - (a) Safe Bifurcations
 - (b) Explosive Bifurcations
 - (c) Dangerous Bifurcations

Safe and dangerous forms of the Hopf bifurcation

Super-critical (safe, no jump from P^C):



Sub-critical (dangerous, jump from P^C):



click

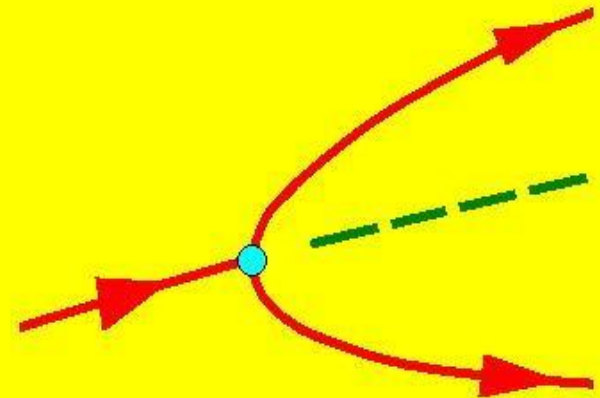
(a) Safe Bifurcations

(a.1) Local Supercritical Bifurcations

- | | |
|--------------------------|----------------|
| 1. Supercritical Hopf | Point to cycle |
| 2. Supercritical Neimark | Cycle to torus |
| 3. Supercritical Flip | Cycle to cycle |

(a.2) Global Bifurcations

- | | |
|-----------------|----------------|
| 4. Band Merging | Chaos to chaos |
|-----------------|----------------|



SUBTLE: continuous supercritical growth of new attractor path

SAFE: no fast jump or enlargement of the attracting set

DETERMINATE: single outcome even with small noise

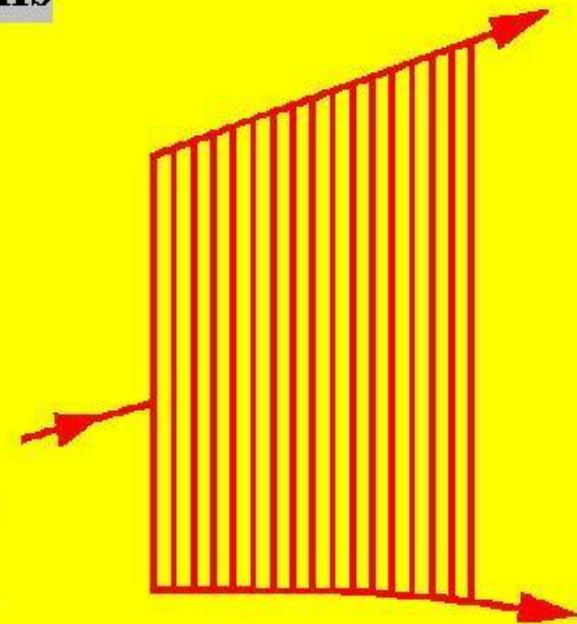
NO HYSTERESIS: path retraced on reversal of control sweep

NO BASIN CHANGE: basin boundary remote from attractors

NO INTERMITTENCY: in the responses of the attractors

(b) Explosive Bifurcations

- | | |
|----------------------------------|----------------|
| 5. Flow Explosion | Point to cycle |
| 6. Map Explosion | Cycle to torus |
| 7. Intermittency Explosion: Flow | Point to chaos |
| 8. Intermittency Explosion: Map | Cycle to chaos |
| 9. Regular-Saddle Explosion | Chaos to chaos |
| 10. Chaotic-Saddle Explosion | Chaos to chaos |



CATASTROPHIC: global events, abrupt enlargement of attracting set

EXPLOSIVE: enlargement, but no jump to remote attractor

DETERMINATE: with single outcome even with small noise

NO HYSTERESIS: paths retraced on reversal of control sweep

NO BASIN CHANGE: basin boundary remote from attractors

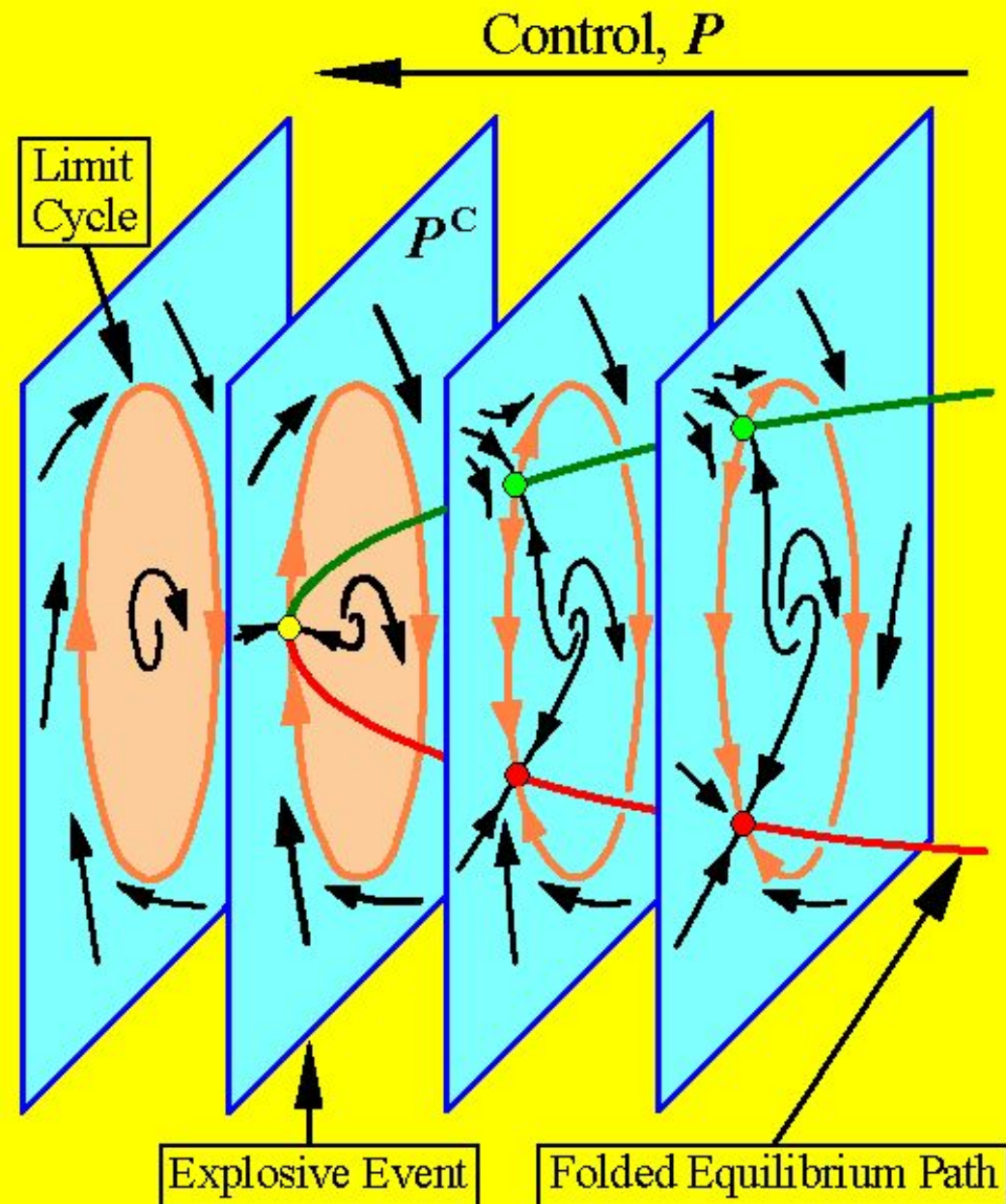
INTERMITTENCY: lingering in old domain, flashes through the new

Example of an Explosive Event

Flow-explosion transforms point attractor to a cycle

Equilibrium path has a regular saddle-node fold.
Saddle outset flows around a closed loop to the node.
A stable cycle is created.
Initial period is infinite (critical slowing).

Precursor: same as static fold.



(c) Dangerous Bifurcations

(c.1) Local Saddle-Node Bifurcations

11. Static Fold

from Point

12. Cyclic Fold

from Cycle

(c.2) Local Subcritical Bifurcations

13. Subcritical Hopf

from Point

14. Subcritical Neimark

from Cycle

15. Subcritical Flip

from Cycle

(c.3) Global Bifurcations

16. Saddle Connection

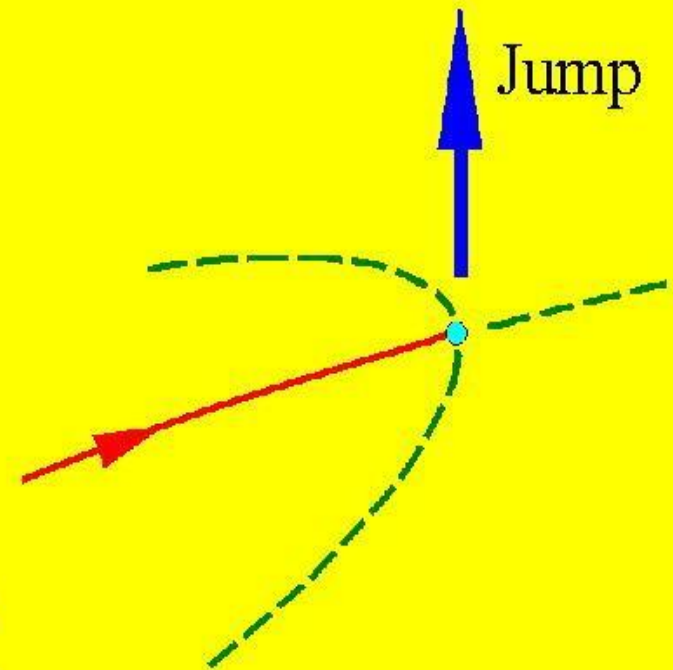
from Cycle

17. Regular-Saddle Catastrophe

from Chaos

18. Chaotic-Saddle Catastrophe

from Chaos



CATASTROPHIC: blue-sky disappearance of attractor

DANGEROUS: sudden jump to new attractor (of any type)

INDETERMINACY: outcome can depend on global topology

HYSTERESIS: path not reinstated on control reversal

BASIN: tends to zero (c.2), attractor hits edge of residual basin (c.1, c.3)

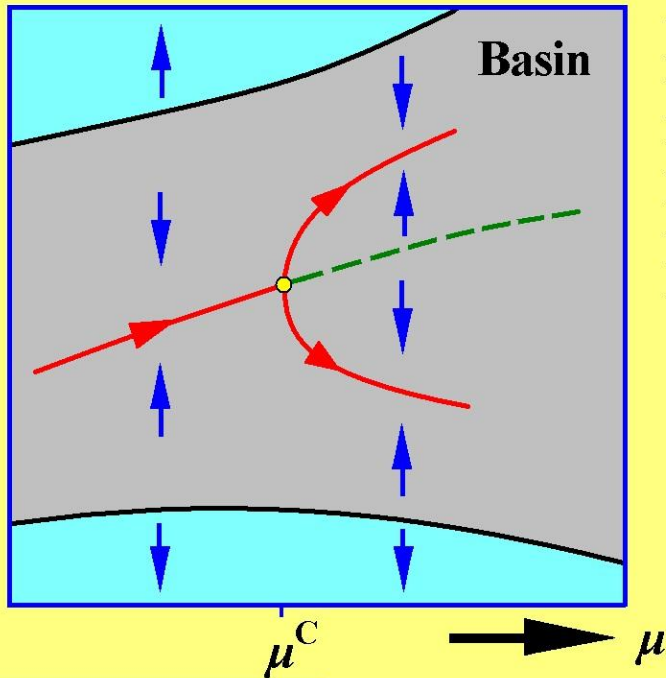
NO INTERMITTENCY: but critical slowing in global events

BASINS (1)

(a) With **NO HYSTERESIS** and **REMOTE BASIN BOUNDARIES**

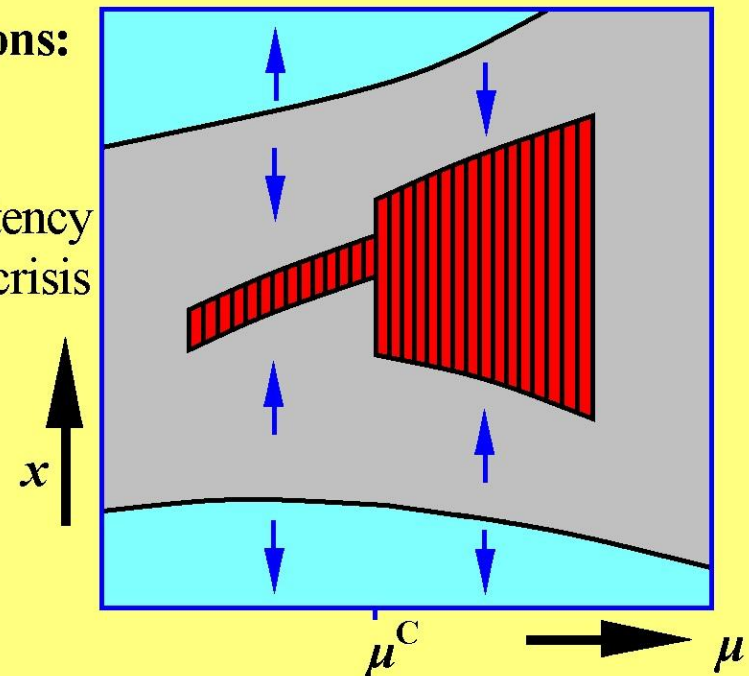
Safe Bifurcations

Supercritical:
Hopf
Flip
Neimark
Pitchfork (2μ)



Explosive Bifurcations

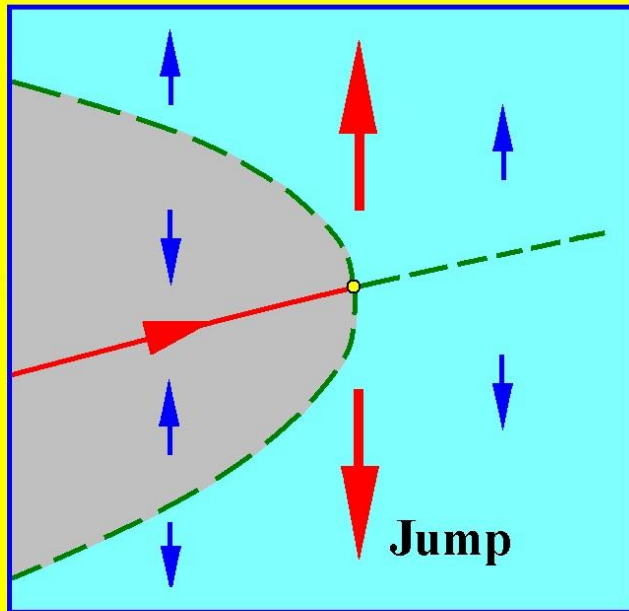
Explosions:
Flow
Map
Intermittency
Interior crisis



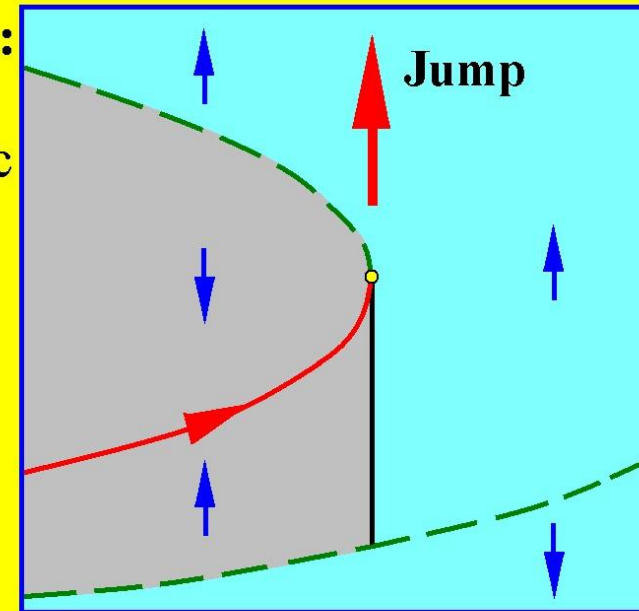
BASINS (2)

(b) With **HYSTERESIS**, blue-sky **JUMPS** and **SHRINKING BASINS**
Dangerous Bifurcations

Subcritical:
Hopf
Flip
Neimark
Pitchfork (2μ)



Folds:
Static
Cyclic



Precursors of our 18 bifurcations

Precursors of codimension-one bifurcations (● local decay rate of transients $\rightarrow 0$)

c276a

Supercritical Hopf
 Supercritical Neimark
 Supercritical flip
 Band merging

S: point to cycle
 S: cycle to torus
 S: cycle to cycle
 S: chaos to chaos

● linearly with control
 ● linearly with control
 ● linearly with control
 lingers near impinging boundary

Flow explosion
 Map explosion
 Intermittency expl: flow
 Intermittency expl: map
 Regular interior crisis
 Chaotic interior crisis

E: point to cycle
 E: cycle to torus
 E: point to chaos
 E: cycle to chaos
 E: chaos to chaos
 E: chaos to chaos

● linearly along folding path
 ● linearly along folding path
 ● linearly with control
 ● as for trigger (fold, flip, Neimark)
 lingers near impinging saddle
 lingers near impinging saddle

Static fold
 Cyclic fold
 Subcritical Hopf
 Subcritical Neimark
 Subcritical flip
 Saddle connection
 Regular exterior crisis
 Chaotic exterior crisis

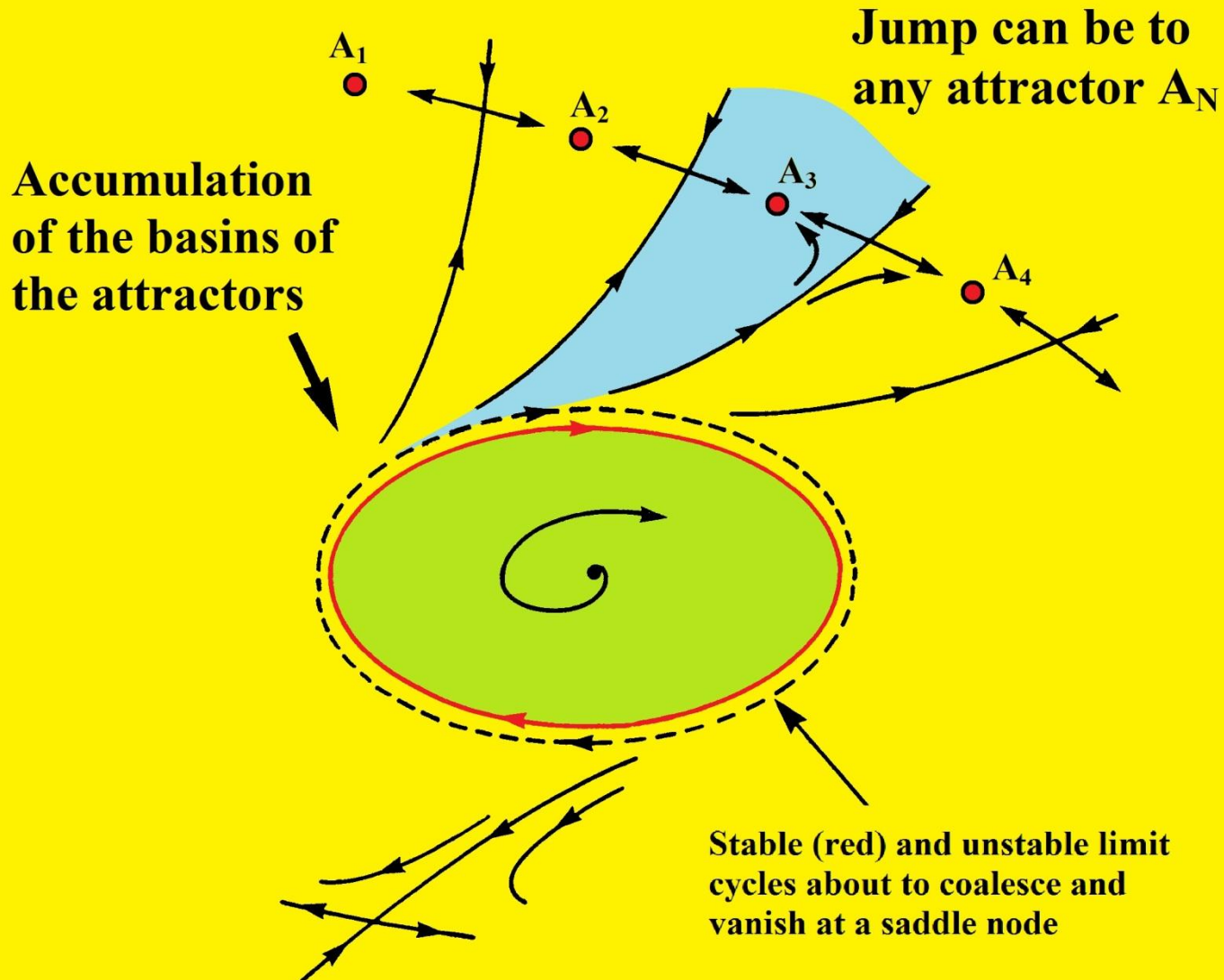
D: from point
 D: from cycle
 D: from point
 D: from cycle
 D: from cycle
 D: from cycle
 D: from cycle
 D: from chaos
 D: from chaos

● linearly along folding path
 ● linearly along folding path
 ● linearly with control
 ● linearly with control
 ● linearly with control
 period of cycle tends to infinity
 lingers near impinging saddle
 lingers near impinging saddle

INDETERMINATE JUMP

**Indeterminacy in the Cyclic Fold
(possible with a 2D outset)**

c352



Concluding Remarks

- Bifurcation concepts for climate studies:
- Co-dimension-one events in dissipative systems.
- Safe, explosive and dangerous forms.
- Hysteresis and basin boundary structure
- Slowing of transients prior to an instability.

Our recent publications

All can be found in Jan Sieber's Homepage

<http://userweb.port.ac.uk/~sieberj>

- J.M.T. Thompson & J. Sieber, Predicting climate tipping points, in *Geo-Engineering Climate Change* (eds. Launder & Thompson) CUP 2010.
- J.M.T. Thompson & Jan Sieber, Climate tipping as a noisy bifurcation: a predictive technique, to appear in *IMA J. Appl. Maths.* <http://arxiv.org/abs/1007.1376>
- J.M.T. Thompson & Jan Sieber, Predicting climate tipping as a noisy bifurcation: a review, to appear in *Int. J. Bifurcation & Chaos* (this is an extended version of the top paper).