$L(q(x), h, H)$ шекаралық есептің меншікті мәндері және $\Delta(\lambda)$ характеристикалық функцияның нөлдері.

Сонымен

$$\Delta(\lambda) = V(\varphi) = -U(\psi) \qquad (5)$$

$\Delta(\lambda) = \langle \psi, \varphi \rangle$ - характеристикалық функция д/аталады

Т. 1.1. Характеристикалық $\Delta(\lambda)$ функцияның нөлдері L шекаралық есептің $\{\lambda_n\}$ нөлдерімен беттеседі. $\varphi(x, \lambda_n)$, $\psi(x, \lambda_n)$ - меншікті функциялар болып табылады және

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \ \beta_n \neq 0, \ \exists \{\beta_n\}_{n \geq 0} \qquad (6)$$

1) λ_0 – саны $\Delta(\lambda)$ функциясының нөлі, яғни $\Delta(\lambda_0)=0$ болсын. Сонда $\langle\psi,\varphi\rangle$ Вронскиан x-ке тәуелсіз болуынан

$$\Delta(\lambda_0)=0=\psi(x,\lambda_0)\cdot\varphi'(x,\lambda_0)-\psi'(x,\lambda_0)\varphi(x,\lambda_0)=0 \Leftrightarrow$$

$$\psi(x,\lambda_0)=\frac{\psi'(x,\lambda_0)}{\varphi'(x,\lambda_0)}\varphi(x,\lambda_0),$$ яғни

$$\beta=\frac{\psi'(x,\lambda_0)}{\varphi'(x,\lambda_0)},$$ ал (3) теңдік бойынша

$$U(\varphi):=\varphi'(0,\lambda_0)-h\varphi(0,\lambda_0)=\left|\begin{matrix}\varphi(0,\lambda_0)=1,\\ \varphi'(0,\lambda_0)=h,\end{matrix}\right|=0,$$

$$V(\psi):=\psi'(\pi,\lambda_0)+H\psi(\pi,\lambda_0)=\left|\begin{matrix}\psi(\pi,\lambda_0)=1,\\ \psi'(\pi,\lambda_0)=-H,\end{matrix}\right|=0.$$

$\varphi(x,\lambda_0),\psi(x,\lambda_0)$ функциялары (2) шартты қанағаттандырады $\Rightarrow$ λ_0 – меншікті мән, $\varphi(x,\lambda_0),\psi(x,\lambda_0)$ – меншікті функциялар.

2) Кері, λ_0 - меншікті мән (Шек. L есеп)

$\wedge\ y_0(x, \lambda_0)$ - сәйкес меншікті функция болсын. Сонда (2) $\Rightarrow$

$$U(y_0) := y_0'(0) - h y_0(0) = 0, \qquad y_0 = y_0(x, \lambda_0).$$

$$V(y_0) := y_0'(\pi) + H y_0(\pi) = 0,$$

Әрине, $y_0(0) \neq 0$, себебі, егер $y_0(0) = 0$, онда $U(y_0) := y_0'(0) - h y_0(0) = 0 \Rightarrow y_0'(0) = 0$, ал Коши есебінің шешімінің жалғыздығы туралы теорема (!) бойынша $y_0(x, \lambda_0) \equiv 0$

$$\begin{cases} y_0'(0, \lambda_0) = 0 \Rightarrow y_0(x) = C; \ y_0(\lambda_0) = 0. \end{cases}$$

Сондықтан $y_0(0, \lambda_0) \neq 0$. Жалпылықты шектемей $y_0(0, \lambda_0) = 1$ деп есептеуге болады.

Тогда из равенства $y_0'(0,\lambda_0) - h\,y_0(0,\lambda_0) = 0$, получим $y_0'(0,\lambda_0) = h$. Таким образом

$$\begin{cases} y_0(0,\lambda_0)=1, \\ y_0'(0,\lambda_0)=h, \end{cases}$$ а по условию $$\begin{cases} \varphi(0,\lambda)=1, \\ \varphi'(0,\lambda)=h, \end{cases}$$

$\forall \lambda$. Отсюда $y_0(x,\lambda_0) \equiv \varphi(x,\lambda_0)$. Тогда

по (5): $\Delta(\lambda) = V(\varphi(x,\lambda)) \Rightarrow$

$$\Delta(\lambda_0) = V(\varphi(x,\lambda_0)) = V(y_0(x,\lambda_0)) = 0 \Rightarrow$$

λ_0 - нуль характеристической функции $\Delta(\lambda)$. $\blacksquare$

Ескерту. Егер $y_0(0,\lambda_0) = c \neq 1$, онда

$$y_0'(0,\lambda_0) - h\,y_0(0,\lambda_0) = 0 \Rightarrow y_0'(0,\lambda_0) = c\,h.$$

Сонда $y_0(x,\lambda_0) = c\,\varphi(x,\lambda_0)$, өйткені

$\forall \lambda \to\ !$ меншікті функция $\varphi(x,\lambda)$

тұрақты көбейткіш ~~арқ~~ дәлдігімен жалғыз.

$$\alpha_n := \int_0^{\pi} \varphi^2(x,\lambda_n)\,dx. \qquad (7).$$

def $\{\alpha_n\}_{n\geq 0}$ сандары салмақтық сандар, ал $\{\lambda_n, \alpha_n\}_{n\geq 0}$ – жұптары спектральді деректер д/аталады.

Лемма 1.1. Справедливо равенство

$$\beta_n \cdot \alpha_n = -\dot{\Delta}(\lambda_n), \qquad (8)$$

где $\beta_n = \dfrac{\psi(x,\lambda_n)}{\varphi(x,\lambda_n)}$, а

$$\dot{\Delta}(\lambda_n) = \lim_{\lambda \to \lambda_n} \frac{\Delta(\lambda) - \Delta(\lambda_n)}{\lambda - \lambda_n}.$$

$\square$

$$\cancel{\frac{d}{dx}\langle \psi(x,\lambda_n), \varphi(x,\lambda_n)\rangle = (\lambda-\lambda_n)\psi(x,\lambda)\varphi(x,\lambda_n)}$$

$$\frac{d}{dx}\langle \psi(x,\lambda), \varphi(x,\lambda_n)\rangle = (\lambda-\lambda_n)\,\psi(x,\lambda)\cdot\varphi(x,\lambda_n).$$

Докажем данное равенство.

$$\frac{d}{dx}\langle \psi(x,\lambda), \varphi(x,\lambda_n)\rangle = \big(\psi(x,\lambda)\cdot\varphi'(x,\lambda_n) -$$

$$- \psi'(x,\lambda)\,\varphi(x,\lambda_n)\big)'_x = \big[\psi'(x,\lambda)\cdot\varphi'(x,\lambda_n) +$$

$$+ \psi(x,\lambda)\cdot\varphi''(x,\lambda_n)\big] - \big[\psi''(x,\lambda)\cdot\varphi(x,\lambda_n) +$$

$$+ \psi'(x,\lambda)\,\varphi'(x,\lambda_n)\big] = \psi(x,\lambda)\,\varphi''(x,\lambda_n) -$$

$$- \psi''(x,\lambda)\cdot\varphi(x,\lambda_n) = \left\{\begin{array}{l} -\varphi''(x,\lambda_n) + q(x)\,\varphi(x,\lambda_n) = \lambda_n\varphi(x,\lambda_n) \\ -\psi''(x,\lambda) + q\,\psi(x,\lambda) = \lambda\,\psi(x,\lambda) \end{array}\right\}$$

$$= \psi(x,\lambda)\big[q(x)\,\varphi(x,\lambda_n) - \lambda_n\varphi(x,\lambda_n)\big] -$$

$$- \varphi(x,\lambda_n)\cdot\big[q(x)\cdot\psi(x,\lambda) - \lambda\,\psi(x,\lambda)\big] = \lambda\,\psi(x,\lambda)\,\varphi(x,\lambda_n) -$$

$$- \lambda_n\psi(x,\lambda)\cdot\varphi(x,\lambda_n) = (\lambda-\lambda_n)\,\psi(x,\lambda)\,\varphi(x,\lambda_n).$$

В доказанном равенстве, проинтегрируем обе части на $[0,\pi]$. Тогда

$$(\lambda-\lambda_n)\int_0^{\pi}\psi(x,\lambda)\varphi(x,\lambda_n)\,dx = \langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle\Big|_0^{\pi}.$$

$$\langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle\Big|_0^{\pi} = \big(\psi(\pi,\lambda)\varphi'(\pi,\lambda_n)-\psi'(\pi,\lambda)\varphi(\pi,\lambda_n)\big)-$$

$$-\big(\psi(0,\lambda)\varphi'(0,\lambda_n)-\psi'(0,\lambda)\varphi(0,\lambda_n)\big)=$$

$$=\left[\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\right]-\left[\psi(0,\lambda)h-\psi'(0,\lambda)\right]=$$

$$=-\left[\psi'(0,\lambda)-h\psi(0,\lambda)\right]+\left[\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\right]=$$

$$=-\big(\Delta(\lambda)-\Delta(\lambda_n)\big).$$

$$(\lambda-\lambda_n)\int_0^{\pi}\psi(x,\lambda)\varphi(x,\lambda_n)\,dx = -\big(\Delta(\lambda)-\Delta(\lambda_n)\big) \Rightarrow$$

$$\Rightarrow \int_0^{\pi}\psi(x,\lambda)\varphi(x,\lambda_n)\,dx = -\frac{\Delta(\lambda)-\Delta(\lambda_n)}{\lambda-\lambda_n} \Big/ \lim_{\lambda\to\lambda_n} :$$

$$\lim_{\lambda\to\lambda_n}\int_0^{\pi}\psi(x,\lambda)\varphi(x,\lambda_n)\,dx = -\lim_{\lambda\to\lambda_n}\frac{\Delta(\lambda)-\Delta(\lambda_n)}{\lambda\to\lambda_n}.$$

$$\int_0^{\pi}\psi(x,\lambda_n)\varphi(x,\lambda_n)\,dx = -\dot{\Delta}(\lambda_n).$$

Т.к. $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n) \Rightarrow$

$$\int_0^{\pi} \beta_n \cdot \varphi(x,\lambda_n) \cdot \varphi(x,\lambda_n)\, dx = -\dot{\Delta}(\lambda_n) \Rightarrow$$

$$\beta_n \int_0^{\pi} \varphi^2(x,\lambda_n)\, dx = -\dot{\Delta}(\lambda_n) \Rightarrow$$

$$\beta_n \cdot \alpha_n = -\dot{\Delta}(\lambda_n)$$

$\blacksquare$

T. 1.2. $\{\lambda_n\}$ - меншікті мәндері Λ

$\varphi(x,\lambda_n)$, $\psi(x,\lambda_n)$ - меншікті функциялары нақты мәнді.

$\Delta(\lambda)$ характеристикалық функцияның барлық нөлдері $\{\lambda_n\}$ - жай нөлдер, яғни $\dot{\Delta}(\lambda_n) \neq 0$.

Әртүрлі меншікті мәндерге $\lambda_n \neq \lambda_k$ сәйкес $\{\varphi(x,\lambda_n), \varphi(x,\lambda_k\}$ V ψ меншікті функциялар өзара $L^2(0;\pi)$ кеңістігінде ортогональ болады.

$\boxtimes$ $\exists\, \lambda_n, \lambda_k$ $(\lambda_n \neq \lambda_k)$ – собственные значения и $\lambda_n \leftrightarrow y_n(x)$, $\lambda_k \leftrightarrow y_k(x)$.

$$\int_0^{\pi} (l y_n(x))\, y_k(x)\, dx = \int_0^{\pi} y_n(x) (l y_k(x))\, dx.$$

$$l y_n(x) := -y_n''(x) + q(x) y_n(x) \Rightarrow$$

$$\int_0^{\pi} [-y_n''(x) + q(x) y_n(x)]\, y_k(x)\, dx = -\int_0^{\pi} y_n''(x) y_k(x)\, dx +$$

$$+ \int_0^{\pi} q(x) y_n(x) y_k(x)\, dx = S_1 + S_2.$$

$$S_1 = \int_0^{\pi} y_n''(x) y_k(x)\, dx = \left| \begin{array}{ll} u = y_k(x) & dv = y_n''(x) \\ du = y_k'(x) dx & v = y_n'(x) \end{array} \right| =$$

$$= y_k(x) \cdot y_n'(x) \Big|_0^{\pi} - \int_0^{\pi} y_k'(x) y_n'(x)\, dx = \left| \begin{array}{ll} u = y_k'(x) & dv = y_n'(x) \\ du = y_k''(x) & v = y_n(x) \end{array} \right| =$$

$$= \left[y_k(x) y_n'(x) \Big|_0^{\pi} - (y_k'(x) \cdot y_n(x)) \Big|_0^{\pi} \right] + \int_0^{\pi} y_k''(x) y_n(x)\, dx \Rightarrow$$

$$\left[y_k(x)\, y_n'(x)\right]\Big|_0^{\pi} - \left[y_k'(x)\, y_n(x)\right]\Big|_0^{\pi} =$$

$$= \left(y_k(\pi)\, y_n'(\pi) - y_k(0)\cdot y_n'(0)\right) - \left(y_k'(\pi)\, y_n(\pi) - y_k'(0)\, y_n(0)\right) = \left[y_k(\pi)\, \underline{y_n'(\pi)} - \underline{y_k'(\pi)}\, y_n(\pi)\right] -$$

$$- \left[y_k(0)\, \underline{y_n'(0)} - \underline{y_k'(0)}\, y_n(0)\right] = \left\{ \begin{array}{l} \cancel{y_s' + Hy} \\ y_s'(\pi) + H y_s(\pi) = 0 \\ y_s'(0) - h\, y_s(0) = 0 \end{array} \right| =$$

$$= \left(y_k(\pi)\left[-H y_n(\pi)\right] - \left[-H y_k(\pi)\right]\cdot y_n(\pi)\right) -$$

$$- \left(y_k(0)\left[h\, y_n(0)\right] - \left[h\, y_k(0)\right] y_n(0)\right) = 0$$

$$S_1 = -\int_0^{\pi} y_k''(x)\, y_n(x)\, dx \implies$$

$$\int_0^{\pi} l y_n(x)\, y_k(x)\, dx = \int_0^{\pi} -y_k''(x)\, y_n(x)\, dx + \int_0^{\pi} q(x)\, y_k(x)\, y_n(x)\, dx =$$

$$= \int_0^{\pi} y_n(x)\left[-y_k''(x) + q(x)\, y_k(x)\right] dx =$$

$$= \int_0^{\pi} y_n(x)\, l y_k(x)\, dx.$$

$$\int_0^{\pi} l y_n(x)\, y_k(x)\, dx = \int_0^{\pi} y_n(x)\, l y_k(x)\, dx$$

рав-во доказано

$$\ell y_n(x) := -y_n''(x) + q(x)\, y_n(x) \Rightarrow (1) \Rightarrow$$

$$\ell y_n(x) = \lambda_n y_n(x)\,, \quad \wedge\ \ell y_k(x) = \lambda_k y_k(x) \Rightarrow$$

$$\lambda_n \int_0^{\pi} y_n(x)\, y_k(x)\, dx = \lambda_k \int_0^{\pi} y_n(x)\, y_k(x)\, dx \Leftrightarrow$$

$$(\lambda_n - \lambda_k) \int_0^{\pi} y_n(x)\, y_k(x)\, dx = 0$$

$$\lambda_n \neq \lambda_k \Leftrightarrow \lambda_n - \lambda_k \neq 0 \Leftrightarrow$$

$$\int_0^{\pi} y_n(x)\, y_k(x)\, dx = 0.$$

def. $y_n(x) \perp y_k(x)$ в $L^2(0;\pi) \Longleftrightarrow$

$$(y_n(x), y_k(x))_{L^2(0;\pi)} = 0,$$

$$(y_n(x), y_k(x))_{L^2(0;\pi)} \stackrel{def}{=\!=} \int_0^\pi y_n(x)\, y_k(x)\, dx \Longleftrightarrow$$

$y_n(x) \perp y_k(x)$ в $L^2(0;\pi)$. 3) - док-во.

Пусть $\lambda^0 = u + i v$, $v \neq 0$; $\lambda^0 \in \mathbb{C}$

$\lambda^0 \Leftrightarrow y^0(x) \not\equiv 0$.

$q(x), h, H \in R \Longrightarrow \overline{\lambda^0} = u - i v$, $\overline{\lambda^0}$ – с.з.

$\overline{\lambda^0} \Leftrightarrow \overline{y^0}(x)$. $\lambda^0 \neq \overline{\lambda^0} \Longleftrightarrow$

$$\|y^0\|_{L^2(0;\pi)} \stackrel{def}{=\!=} \int_0^\pi y^0(x)\, \overline{y^0}(x)\, dx = 0 \Longleftrightarrow$$

$y^0(x) \equiv 0$. $\Longleftrightarrow \lambda^0 = \overline{\lambda^0} \Longleftrightarrow v = 0$!

Отсюда также следует, что собственные функции $\varphi(x,\lambda_n)$, $\psi(x,\lambda_n)$ – вещественные.

Пусть $\varphi(x,\lambda_n) = u(x,\lambda_n) + i\,v(x,\lambda_n)$ комплексно-значная функция. Тогда

$\varphi''(x,\lambda_n) = u''(x,\lambda_n) + i\,v''(x,\lambda_n)$ и

$-\varphi''(x,\lambda_n) + q(x)\,\varphi(x,\lambda_n) = \lambda_n\,\varphi(x,\lambda_n)$, след-но

$\overline{\varphi}(x,\lambda_n) = u(x,\lambda_n) - i\,v(x,\lambda_n)$ также собственная функция. Докажем.

$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n\varphi(x,\lambda_n) \Longrightarrow$$

$$\begin{cases} -u''(x,\lambda_n)+q(x)u(x,\lambda_n)=\lambda_n u(x,\lambda_n) \\ -v''(x,\lambda_n)+q(x)v(x,\lambda_n)=\lambda_n v(x,\lambda_n), \end{cases}$$

отсюда, умножая второе равенство на $(-i)$, т.е.

$$\begin{cases} -u''(x,\lambda_n)+q(x)u(x,\lambda_n)=\lambda_n u(x,\lambda_n), \\ i v''(x,\lambda_n)-iq(x)v(x,\lambda_n)=-i\lambda_n v(x,\lambda_n), \end{cases}$$

затем сложив их, получим.

$$-u''(x,\lambda_n)+iv''(x,\lambda_n)+q(x)\big(u(x,\lambda_n)-iv(x,\lambda_n)\big)=$$
$$=\lambda_n\big(u(x,\lambda_n)-iv(x,\lambda_n)\big).$$

$$-\left[u''(x,\lambda_n)-iv''(x,\lambda_n)\right]+q(x)\overline{\varphi}(x,\lambda_n)=\lambda_n\overline{\varphi}(x,\lambda_n)$$

$$-\overline{\varphi}''(x,\lambda_n)+q(x)\overline{\varphi}(x,\lambda_n)=\lambda_n\overline{\varphi}(x,\lambda_n).$$

$\overline{\varphi}(x,\lambda_n) \Leftrightarrow \lambda_n^*$ - собственное знач.

По доказанному равенству.

$$(\lambda_n-\lambda_n^*)\int_0^\pi \varphi(x,\lambda_n)\overline{\varphi}(x,\lambda_n)dx = 0,$$ а

$$\int_0^\pi \varphi(x,\lambda_n)\overline{\varphi}(x,\lambda_n)dx = \|\varphi(x,\lambda_n)\|_{L^2(0;\pi)} = 0 \Rightarrow$$

$\varphi(x,\lambda_n) \equiv 0$, что не возможно.

Тогда $\lambda_n - \lambda_n^* = 0 \Leftrightarrow \lambda_n = \lambda_n^*$ также противоречие, с тем, что $\Delta(\lambda)$ имеет простые нули, т.е. $\lambda_n = \lambda_n^*$ не м/б.

Так как $\alpha_n = \int_0^{\pi} \varphi_1^2(x, \lambda_n)\,dx \neq 0$ и

$\beta_n \neq 0$, то

$$\dot{\Delta}(\lambda_n) \neq 0.$$

Пример 1.1. Пусть $q(x) \equiv 0$, $h = H = 0$, $\lambda = \rho^2$. Тогда

$C(x, \lambda) = \varphi(x, \lambda) = \cos \rho x$,

$S(x, \lambda) = \dfrac{\sin \rho x}{\rho}$

$\psi(x, \lambda) = \cos \rho(\pi - x)$

$\Delta(\lambda) = -\rho \sin \rho\pi$;

$\lambda_n = n^2 \ (n \geq 0)$.

$\varphi(x, \lambda_n) = \cos nx$;

$\beta_n = (-1)^n$.

$\alpha_n = ?$

Решение.

$-y''(x) + q(x)y(x) = \lambda y(x)$, $q(x) = 0 \Leftrightarrow$

$-y''(x) = \rho^2 y(x) \Leftrightarrow$

$-y''(x) - \rho^2 y(x) = 0$. Характеристическое уравнение диф. ур.

$-(y''(x) + \rho^2 y(x)) = 0 \Leftrightarrow k^2 + \rho^2 = 0 \Leftrightarrow$

$k^2 - (i\rho)^2 = 0 \Leftrightarrow k = \pm i\rho \Leftrightarrow |k = \alpha \pm i\rho \Leftrightarrow$

$y(x) = \cancel{C_1}\, e^{\alpha x}(C_1 \cos\rho x + C_2 \sin\rho x)$, а т.к. $\alpha = 0$

$y(x) = C_1 \cos\rho x + C_2 \sin\rho x$.

Т.к. $h = H = 0 \Leftrightarrow U(y) = y'(0) = 0,\ V(y) =$

$= y'(\pi) = 0.$

$y'(x) = -\rho C_1 \sin \rho x + \rho C_2 \cos \rho x$, отсюда

$x = 0 \Rightarrow y'(0) = -\rho C_1 \sin 0 + \rho C_2 \cos 0 = \rho C_2' = 0.\ C_2 = 0.$

* Тогда $y(x) = C_1 \cos \rho x$, без ограничения общности $y(x) = \cos \rho x$, а т.к.

$y(0) = \cos 0 = 1$, а

$y'(0) = -\rho \sin \rho \cdot 0 = 0$, то

$C(x,\lambda) = \cos \rho x.$

$y(x) = C(x,\lambda)$, т.е.

~~$\varphi(x,\lambda)$~~ $\varphi(x,\lambda)$:

$$\begin{cases} \varphi(0,\lambda) = 1 \\ \varphi'(0,\lambda) = h = 0. \end{cases}$$ то

С другой стороны

$\varphi(x,\lambda) = C(x,\lambda) = \cos \rho x.$

$$y(x) = C_1 \cos \rho x + C_2 \sin \rho x.$$

$$S(x,\lambda): \quad \begin{cases} S(0,\lambda) = 0; \\ S'(0,\lambda) = 1. \end{cases}$$

$$y(0) = C_1 \cos 0 + C_2 \sin 0 = 0 \Rightarrow C_1 = 0.$$

$$y'(x) = -C_1 \rho \sin \rho x + \rho C_2 \cos \rho x \quad \Rightarrow$$

$$y'(0) = -C_1 \rho \sin 0 + \rho C_2 \cos 0. \overset{!}{=} 1 \quad \Rightarrow$$

$$\Rightarrow \rho C_2 = 1 \Rightarrow C_2 = \frac{1}{\rho}.$$

Т. О. $C_1 = 0; \; C_2 = \frac{1}{\rho} \quad \Rightarrow$

$$y(x) = \frac{1}{\rho} \sin \rho x = S(x,\lambda).$$

$$y(x) = C_1 \cos \rho x + C_2 \sin \rho x.$$

$$\psi(x,\lambda): \begin{cases} \psi(\pi,\lambda) = 1; \\ \psi'(\pi,\lambda) = -H = 0. \end{cases}$$

$$y(\pi) = C_1 \cos \rho \pi + C_2 \sin \rho \pi = 1.$$

$$y'(x) = -\rho C_1 \sin \rho x + \rho C_2 \cos \rho x \Rightarrow$$

$$y'(\pi) = -\rho C_1 \sin \rho \pi + \rho C_2 \cos \rho \pi = 0.$$

$$\begin{cases} C_1 \cos \rho \pi + C_2 \sin \rho \pi = 1. \\ -C_1 \sin \rho \pi + C_2 \cos \rho \pi = 0. \end{cases}$$

$$C_2 = \frac{C_1 \sin p\pi}{\cos p\pi} \Rightarrow$$

$$C_1 \cos p\pi + \frac{C_1 \sin^2 p\pi}{\cos p\pi} = 1 \Rightarrow$$

$$\frac{C_1 \cos^2 p\pi + C_1 \sin^2 p\pi}{\cos p\pi} = 1 \Leftrightarrow C_1 = \cos p\pi \Rightarrow$$

$$C_2 = \sin p\pi \Rightarrow$$

$$\psi(x,\lambda) = \cos p\pi \cdot \cos px + \sin p\pi \sin px =$$

$$= \cos p(\pi - x).$$

$$\Delta(\lambda) = \langle \psi, \varphi \rangle = \psi(x,\lambda)\,\varphi'(x,\lambda) - \psi'(x,\lambda)\cdot\varphi(x,\lambda) =$$

$$= \left| \begin{array}{l} \psi(x,\lambda) = \cos\rho(\pi - x) \;\Rightarrow\; \psi'(x,\lambda) = \rho\sin\rho(\pi - x) \\ \varphi(x,\lambda) = \cos\rho x \;\Rightarrow\; \varphi'(x,\lambda) = -\rho\sin\rho x \end{array} \right| =$$

$$= \cos\rho(\pi - x)\cdot(-\rho\sin\rho x) \overset{+}{-} \rho\sin\rho(\pi - x)\cdot\cos\rho x =$$

$$= -\rho\Big[(\cos\rho\pi\cdot\cos\rho x + \sin\rho\pi\sin\rho x)\sin\rho x +$$

$$+ (\sin\rho\pi\cos\rho x - \cos\rho\pi\cdot\sin\rho x)\cos\rho x\Big] = -\rho\cdot$$

$$\cdot\big(\underline{\cos\rho\pi\cos\rho x\sin\rho x} + \sin\rho\pi\sin^2\rho x + \sin\rho\pi\cos^2\rho x -$$

$$- \underline{\cos\rho\pi\cdot\sin\rho x\cdot\cos\rho x}\big) = -\rho\cdot(\sin\rho\pi\sin^2\rho x + \sin\rho\pi\cos^2\rho x) =$$

$$= -\rho\sin\rho\pi.$$

$$\beta_n = \frac{\psi(x,\lambda_n)}{\varphi(x,\lambda_n)} = \left| \lambda_n = n^2 \right| =$$

$$= \frac{\cos n(\pi - x)}{\cos nx} = \frac{\cos n\pi \cdot \cos nx + \overbrace{\sin n\pi}^{=0} \cdot \sin nx}{\cos nx} =$$

$$= \left| \forall n \in \mathbb{N} \Rightarrow \sin n\pi = 0 \right| = \frac{\cos n\pi \cos nx}{\cos nx} =$$

$$= \cos n\pi = (-1)^n.$$

$$\beta_n = (-1)^n.$$

$$\alpha_n = \int_0^{\pi} \varphi^2(x, \lambda_n)\, dx = \int_0^{\pi} \cos^2 nx\, dx =$$

$$= \int_0^{\pi} \frac{1}{2}\left[1 + \cos 2nx\right] dx = \frac{1}{2}\int_0^{\pi} dx +$$

$$+ \frac{1}{2}\int_0^{\pi} \cos 2nx\, dx = \frac{1}{2} x \Big|_0^{\pi} +$$

$$+ \frac{1}{2} \cdot \frac{1}{2n} \cdot \sin 2nx \Big|_0^{\pi} = \frac{1}{2}\pi + \frac{1}{4n}\left(\sin 2n\pi - 0\right) =$$

$$= \frac{1}{2}\pi.$$

$$\lambda = \rho^2$$

$C(x,\lambda) = \varphi(x,\lambda) = \cos\rho x;$ | $\Delta(\lambda) = -\rho \sin \rho\pi$

$S(x,\lambda) = \frac{1}{\rho}\sin\rho x;$

$\psi(x,\lambda) = \cos\rho(\pi - x)$

$\lambda_n = n^2 \ (n \geq 0)$

$C(x,\lambda_n) = \cos nx; \quad S(x,\lambda_n) = \frac{1}{n}\sin nx;$

$\psi(x,\lambda) = \cos n(\pi - x) = (-1)^n \cdot \cos nx;$

$\Delta(\lambda_n) = -n \cdot \sin n\pi = 0.$ λ_n – нули.

$\beta_n = (-1)^n; \qquad \alpha_n = \frac{1}{2}\pi = \text{const}.$

Задание 2.

Определить Вронскиан для решении $y_1(x), y_2(x)$ диар. ур.

1) $y'' + (3x-1)y' + 2y = 0$

2) $y'' + (5x+1)y' - 2y = 0$

3) $y'' + (x^2+2)y' + y = 0$

4) $y'' - (2x+5)y' - y = 0.$

применить формулу Остроградского – Лиувилля