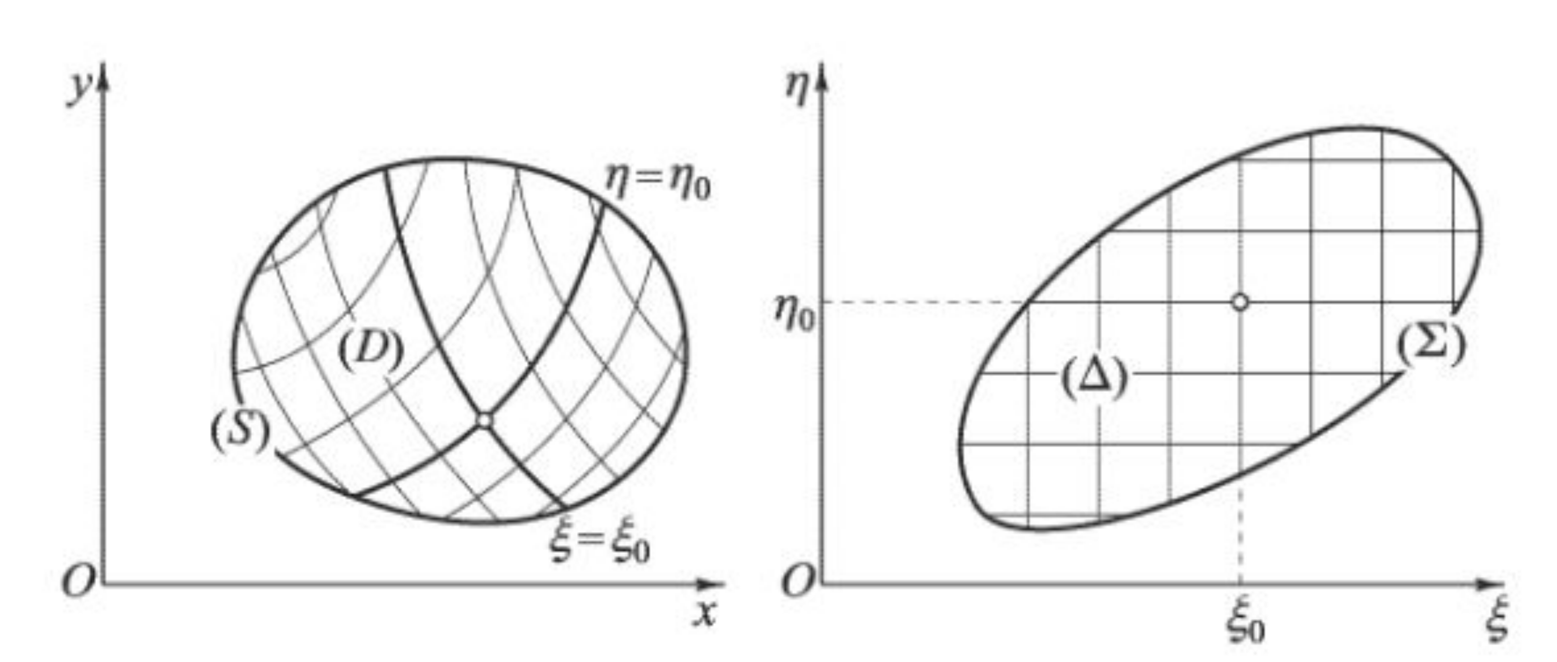
Lecture 5.

Change of Variables for a Double Integral. Triple Integrals.

Let's consider the double integral $\iint_{(D)} f(x,y)dxdy$ (1.1) where the domain or region (D) is bounded by continuous curve (L) and function f is continuous at this domain. Suppose now that region (D) is connected with another region (Δ) by formulas
$x = x(\xi,\eta), y = y(\xi,\eta)$ (1.2) so that between points belonged to (D) and (Δ) one-to-one transformation exists. It's necessary to express the integral (1.1) in domain (Δ) by changing variables. Let's divide the region(Δ) into parts (Δ_i) (i=1,2,…,n); at the same time the region (D) also will be divided into parts (D_i). At each part (D_i) let's choose an arbitrary point (x_i, y_i); and finally let's compose the integral sum for integral (1.1) $\sigma = \sum_{i=1}^{n} F(x_i, y_i) D_i$.



It's known that domains D_i and Δ_i are connected by formula

$D_i = \left|J(\xi_i^*, \eta_i^*)\right| \cdot \Delta_i$ where J is called the jacobian - $J = \dfrac{D(x,y)}{D(\xi,\eta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$ and

(ξ_i^*, η_i^*) is some point belonged the region (Δ). This point is defined by mean value theorem and we can't choose it arbitrarily, but point (x_i, y_i) in region (D_i) we can take arbitrarily. Using this fact we suppose that $x_i = x(\xi_i^*, \eta_i^*), y_i = y(\xi_i^*, \eta_i^*)$. Then the integral sum has form $\sigma = \sum_i f(x(\xi_i^*, \eta_i^*), y(\xi_i^*, \eta_i^*)\left|J(\xi_i^*, \eta_i^*)\right|\Delta_i$. In this form it represents the integral sum for the integral $\iint\limits_{(\Delta)} f(x(\xi,\eta), y(\xi,\eta))\left|J(\xi,\eta)\right| d\xi d\eta$ (1.3). Existence of the last integral follows from continuity of function f.

Finally, we can write $\iint\limits_{(D)} f(x,y)dxdy = \iint\limits_{(\Delta)} f(x(\xi,\eta), y(\xi,\eta))\left|J(\xi,\eta)\right| d\xi d\eta$ (1.4).

So, the rule for changing variables in double integral: it's necessary to change variables x and y by formulas (1.2) in integrating function and multiply integrand by the jacobian.

Example 1.

Calculate the double integral

$$\iint\limits_{R} (y - x)\,dx\,dy,$$

where the region R is bounded by

$$y = x + 1, y = x - 3, y = -\frac{x}{3} + 2, y = -\frac{x}{3} + 4.$$

Solution.

The region R is sketched in Figure 1.

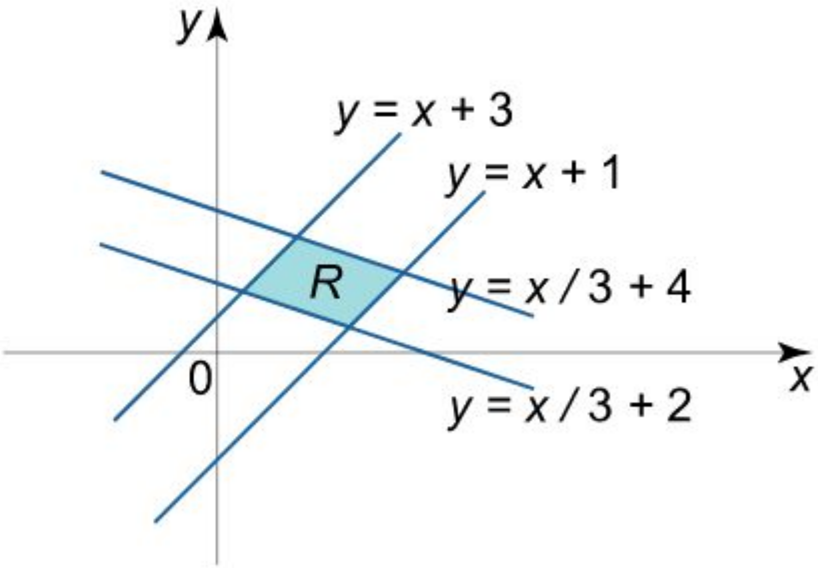


Figure 1.

We use change of variables to simplify the integral. By letting $u = y - x$, $v = y + \frac{x}{3}$, we have

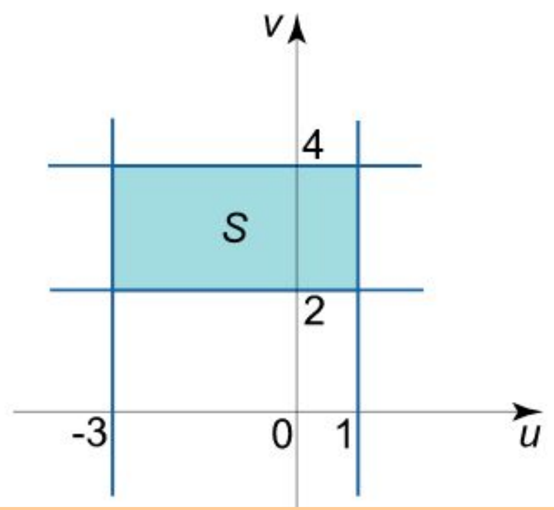
$$y = x + 1, \;\; \Rightarrow y - x = 1, \;\; \Rightarrow u = 1,$$

$$y = x - 3, \;\; \Rightarrow y - x = -3, \;\; \Rightarrow u = -3,$$

$$y = -\frac{x}{3} + 2, \;\; \Rightarrow y + \frac{x}{3} = 2, \;\; \Rightarrow v = 2,$$

$$y = -\frac{x}{3} + 4, \;\; \Rightarrow y + \frac{x}{3} = 4, \;\; \Rightarrow v = 4.$$

Hence, the pullback S of the region R is the rectangle shown in Figure 2.



Calculate the Jacobian of this transformation.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial(y-x)}{\partial x} & \frac{\partial(y-x)}{\partial y} \\ \frac{\partial\left(y+\frac{x}{3}\right)}{\partial x} & \frac{\partial\left(y+\frac{x}{3}\right)}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{vmatrix} = -1\cdot 1 - 1\cdot\frac{1}{3} = -\frac{4}{3}.$$

Then the absolute value of the Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}\right| = \left|\frac{1}{-\frac{4}{3}}\right| = \frac{3}{4}.$$

Hence, the differential is

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{3}{4}dudv.$$

As it can be seen, calculating the integral in the new variables (u,v) is much simpler:

$$\iint\limits_R (y-x)dxdy = \iint\limits_S \left(u\cdot\frac{3}{4}dudv\right) = \frac{3}{4}\int_{-3}^{1} udu\int_{2}^{4} dv = \frac{3}{4}\left(\frac{u^2}{2}\right)\Big|_{-3}^{1}\cdot v|_2^4 = \frac{3}{4}\left(\frac{1}{2}-\frac{9}{2}\right)$$

Example 3.

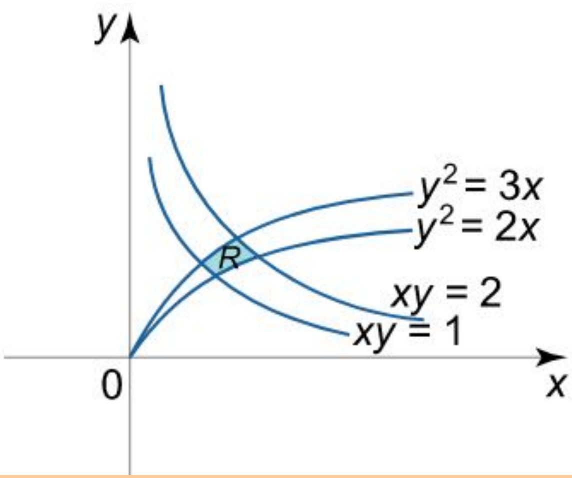
Calculate the double integral

$$\iint\limits_R dxdy,$$

where the region R is bounded by the parabolas $y^2 = 2x$, $y^2 = 3x$ and hyperbolas $xy = 1$, $xy = 2$.

Solution.

The region R is sketched in Figure 5.



We apply the following substitution of variables to simplify the region R :

$$\begin{cases} u = \frac{y^2}{x} \\ v = xy \end{cases}.$$

The pullback S of the region R is defined as follows:

$$y^2 = 2x, \;\Rightarrow \frac{y^2}{x} = 2, \;\Rightarrow u = 2,$$

$$y^2 = 3x, \;\Rightarrow \frac{y^2}{x} = 3, \;\Rightarrow u = 3,$$

$$xy = 1, \;\Rightarrow v = 1,$$

$$xy = 2, \;\Rightarrow v = 2.$$

As it can be seen, the region S is the rectangle. To find the Jacobian of the transformation, we express the variables x, y in terms of u, v.

$$u = \frac{y^2}{x}, \;\; \Rightarrow x = \frac{y^2}{u},$$

$$v = xy, \;\; \Rightarrow v = \frac{y^2}{u} \cdot y, \;\; \Rightarrow y^3 = uv.$$

Then

$$y = \sqrt[3]{uv} = u^{\frac{1}{3}} v^{\frac{1}{3}},$$

$$x = \frac{y^2}{u} = \frac{\sqrt[3]{u^2 v^2}}{u} = \sqrt[3]{\frac{v^2}{u}} = u^{-\frac{1}{3}} v^{\frac{2}{3}}.$$

Find the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial\left(u^{-\frac{1}{3}}v^{\frac{2}{3}}\right)}{\partial u} & \frac{\partial\left(u^{-\frac{1}{3}}v^{\frac{2}{3}}\right)}{\partial v} \\ \frac{\partial\left(u^{\frac{1}{3}}v^{\frac{1}{3}}\right)}{\partial u} & \frac{\partial\left(u^{\frac{1}{3}}v^{\frac{1}{3}}\right)}{\partial v} \end{vmatrix} = \begin{vmatrix} v^{\frac{2}{3}}\left(-\frac{1}{3}u^{-\frac{4}{3}}\right) & u^{-\frac{1}{3}}\cdot\frac{2}{3}v^{-\frac{1}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}v^{-\frac{2}{3}}u^{\frac{1}{3}} \end{vmatrix} = -\frac{1}{3}u^{-\frac{4}{3}}v^{\frac{2}{3}}$$

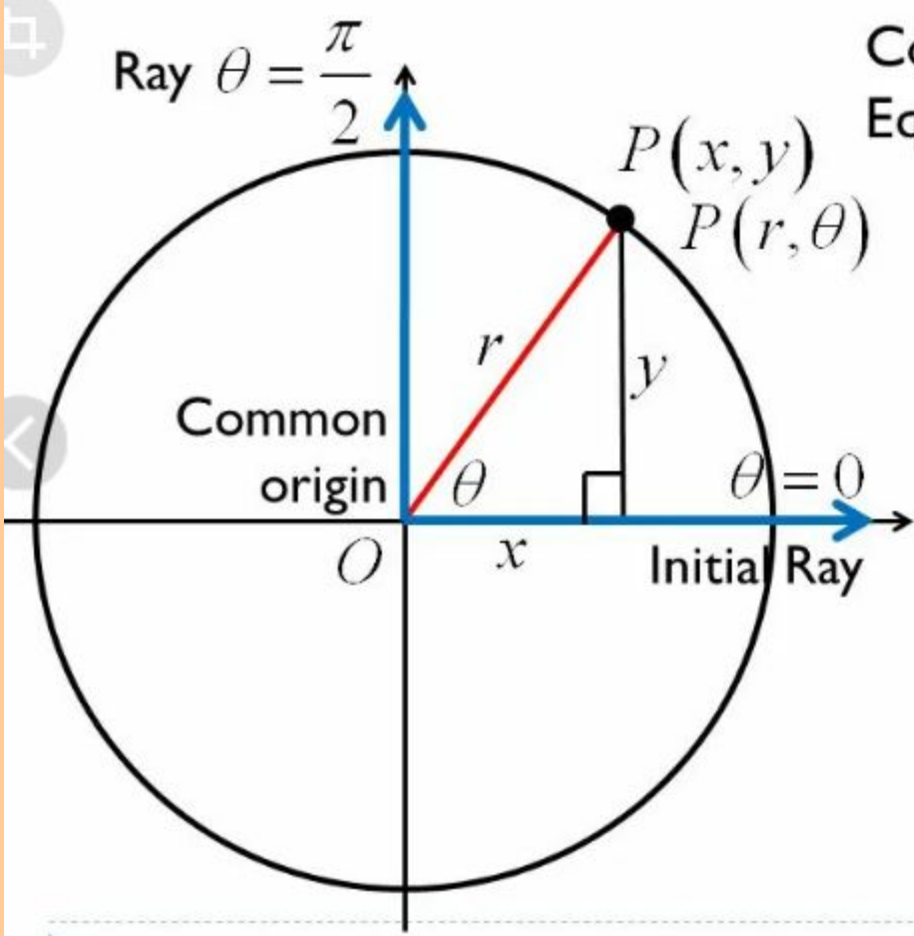
The relationship between the differentials is

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \left|-\frac{1}{3u}\right| dudv = \frac{dudv}{3u}.$$

Then we can write the integral as

$$\iint\limits_R dxdy = \iint\limits_S \frac{dudv}{3u} = \int\limits_2^3 \frac{du}{3u}\int\limits_1^2 dv = \frac{1}{3}(\ln u)|_2^3 \cdot v|_2^3 = \frac{1}{3}(\ln 3 - \ln 2)\cdot(2-1) = \frac{1}{3}\ln\frac{3}{2}.$$

Polar coordinates



Coordinate Conversion Equations:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$\frac{y}{x} = \tan\theta$$

The most usable is polar coordinates system: $x = \rho\cos\varphi, y = \rho\sin\varphi$.

Jacobian for this system is

$J = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho(\cos^2\varphi + \sin^2\varphi) = \rho$. Then formula (1.4) takes form $\iint\limits_{(D)} f(x,y)dxdy = \iint\limits_{(\Delta)} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho d\varphi$ (1.5)

Mechanical applications of double integral.

We know, that double integral $\iint\limits_{(D)} f(x,y)dxdy$ represents itself the volume of some cylindrical body bounded by plane area at bottom and by surface f(x,y) at top. Suppose that f(x,y)=1, then, obviously, the volume will coincide numerically with **area** (D):

$$S = \iint\limits_{(D)} dxdy \quad (2.1)$$

Next, suppose that f(x,y)=p(x,y) represents itself some density function at region (D), then the **mass** of this region may be evaluated by formula

$m = \iint\limits_{(D)} p(x,y)dxdy$ (2.2) – we must take the density at each point and multiply it by elementary area dxdy and sum up all products passing to limit; and by definition we have got double integral (2.2).

Other applications are based on this result. For example, static moments and moments of inertia with relation to axes x and y may be calculated by formulas

$$\begin{cases} M_x = \iint\limits_{(D)} yp(x,y)dxdy, \ \mathrm{M_y} = \iint\limits_{(D)} xp(x,y)dxdy \\ I_x = \iint\limits_{(D)} y^2 p(x,y)dxdy, \ \mathrm{I_y} = \iint\limits_{(D)} x^2 p(x,y)dxdy \end{cases} \quad (2.3)$$

Next, coordinates of center of mass may be calculated by formulas

$$x_0 = \frac{M_x}{m} = \frac{\iint\limits_{(D)} xpdxdy}{\iint\limits_{(D)} pdxdy}, y_0 = \frac{M_y}{m} = \frac{\iint\limits_{(D)} ypdxdy}{\iint\limits_{(D)} pdxdy} \quad (2.4)$$

Triple integral. Task about mass calculation. Definition and conditions of existence.

Let's consider the next problem. Let some body (V) is given with density $\rho = \rho(M) = \rho(x, y, z)$ at each point. It is necessary to find the mass m of this body. As usually, we divide body (V) into parts $(V_1),(V_2),\ldots,(V_n)$. Then we choose an arbitrary point (ξ_i, η_i, ζ_i) at each part. Approximately we can suppose that the density is constant and equals to the density of chosen point $\rho(\xi_i, \eta_i, \zeta_i)$ throughout of each part (Vi). Then the mass of this part will be

$$m_i \approx \rho(\xi_i, \eta_i, \zeta_i) \cdot V_i .$$

And the mass of whole body will be

$$m \approx \sum_{i=1}^{n} \rho(\xi_i, \eta_i, \zeta_i) V_i .$$

If diameters of all parts tend to zero, then if we pass to the limit this equality will become exact

$$m = \lim \sum_{i=1}^{n} \rho(\xi_i, \eta_i, \zeta_i) V_i \quad (1)$$

and problem is solved.

Limits of this kind are called triple integrals. The last result may be rewritten as

$$m = \iiint\limits_{(V)} p(x, y, z)dV \text{ .(2)}$$

Now let's define triple integral in general sense. Let function f(x,y,z) is defined in some space domain (V). Let's divide this space into parts (V_1),(V_2),…,(V_n) with the help of the surface network. After this we take an arbitrary point (ξ_i, η_i, ζ_i) at each part, multiply function value $f(\xi_i, \eta_i, \zeta_i)$ in this point by the volume V_i of this part and, finally, compose the integral sum

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)V_i \text{ .}$$

Def. Finite limit I of integral sum σ when the most of diameters Vi tends to zero is called the triple integral of function f(x,y,z) in domain (V). It denotes by symbol $I = \iiint\limits_{(V)} f(x, y, z)dV = \iiint\limits_{(V)} f(x, y, z)dxdydz$

By usual way it may be got that for triple integral existence condition $\lim(S - s) = 0$ or $\lim \sum_{i=1}^{n} \omega_i V_i = 0$ is necessary and sufficient, where

$\omega_i = M_i - m_i$ - oscillation of function at domain (V_i). So the next statement follows: every continuous function is integrated.

Properties of integrated function and triple integrals.

Most of properties is analogical to properties of double integral and we consider its without proof.

1. If (V)=(V')+(V'') then $\iiint\limits_{(V)} f dV = \iiint\limits_{(V')} f dV + \iiint\limits_{(V'')} f dV$ and from existence of the left integral existence of the right integrals follows and vice versa.
2. If k=const then $\iiint\limits_{(V)} kf dV = k\iiint\limits_{(V)} f dV$ and from existence of the right integral existence of the left integral follows.
3. If function f and g are integrated in area (V) then function $f \pm g$ is integrated too and $\iiint\limits_{(V)} (f \pm g) dV = \iiint\limits_{(V)} f dV \pm \iiint\limits_{(V)} g dV$
4. If for integrated in area (V) functions f and g inequality $f \le g$ takes place then $\iiint\limits_{(V)} f dV \le \iiint\limits_{(V)} g dV$.
5. If function f is integrated then function $|f|$ is integrated too and inequality $\iiint\limits_{(V)} f dV \le \iiint\limits_{(V)} |f| dV$ takes place.
6. If integrated in area (V) function f satisfies to inequality $m \le f \le M$ then $mV \le \iiint\limits_{(V)} f dV \le MV$. By other words, we have mean value theorem $\iiint\limits_{(V)} f dV = \mu V$ $(\mathrm{m} \le \mu \le \mathrm{M})$. In a case of continuous function this formula may be rewrote as $\iiint f dV = f(\bar{x}, \bar{y}, \bar{z}) \cdot V$ (3).