

Lecture 5.
**Change of Variables for a
Double Integral.
Triple Integrals.**

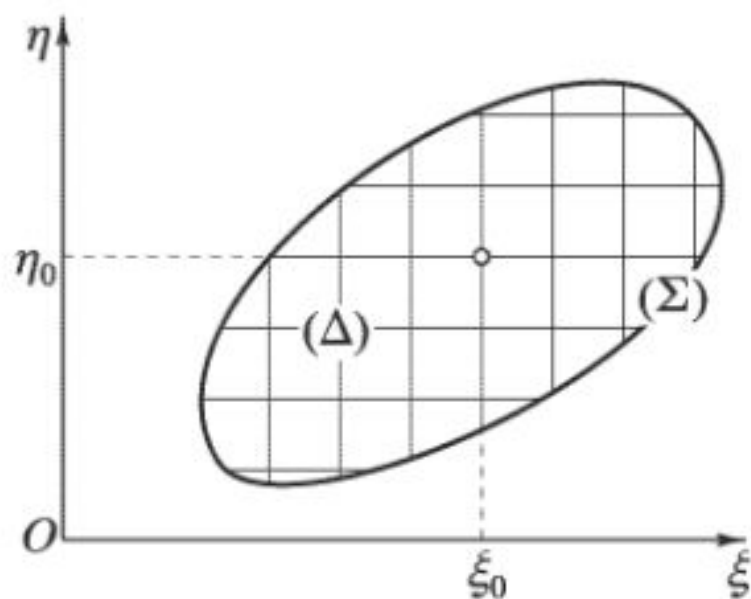
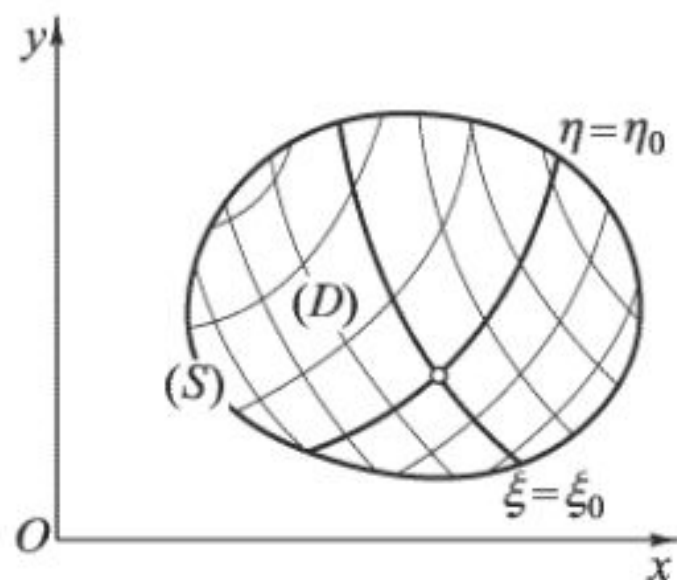
Let's consider the double integral $\iint_{(D)} f(x,y)dx dy$ (1.1) where the domain or region (D) is

bounded by continuous curve (L) and function f is continuous at this domain. Suppose now that region (D) is connected with another region (Δ) by formulas

$x = x(\xi, \eta), y = y(\xi, \eta)$ (1.2) so that between points belonged to (D) and (Δ) one-to-one

transformation exists. It's necessary to express the integral (1.1) in domain (Δ) by changing variables. Let's divide the region (Δ) into parts (Δ_i) ($i=1,2,\dots,n$); at the same time the region (D) also will be divided into parts (D_i). At each part (D_i) let's choose an arbitrary point

(x_i, y_i); and finally let's compose the integral sum for integral (1.1) $\sigma = \sum_{i=1}^n F(x_i, y_i) D_i$.



It's known that domains D_i and Δ_i are connected by formula

$$D_i = |J(\xi_i^*, \eta_i^*)| \cdot \Delta_i \text{ where } J \text{ is called the jacobian - } J = \frac{D(x, y)}{D(\xi, \eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} \text{ and}$$

(ξ_i^*, η_i^*) is some point belonged the region (Δ) . This point is defined by mean value theorem and we can't choose it arbitrarily, but point (x_i, y_i) in region (D_i) we can take arbitrarily. Using this fact we suppose that $x_i = x(\xi_i^*, \eta_i^*), y_i = y(\xi_i^*, \eta_i^*)$. Then the integral sum has form $\sigma = \sum_i f(x(\xi_i^*, \eta_i^*), y(\xi_i^*, \eta_i^*)) |J(\xi_i^*, \eta_i^*)| \Delta_i$. In this form it represents the integral sum for the integral $\iint_{(\Delta)} f(x(\xi, \eta), y(\xi, \eta)) |J(\xi, \eta)| d\xi d\eta$ (1.3). Existence of the

last integral follows from continuity of function f .

Finally, we can write $\iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(x(\xi, \eta), y(\xi, \eta)) |J(\xi, \eta)| d\xi d\eta$ (1.4).

So, the rule for changing variables in double integral: it's necessary to change variables x and y by formulas (1.2) in integrating function and multiply integrand by the jacobian.

Example 1.

Calculate the double integral

$$\iint_R (y - x) dx dy,$$

where the region R is bounded by

$$y = x + 1, y = x - 3, y = -\frac{x}{3} + 2, y = -\frac{x}{3} + 4.$$

Solution.

The region R is sketched in Figure 1.

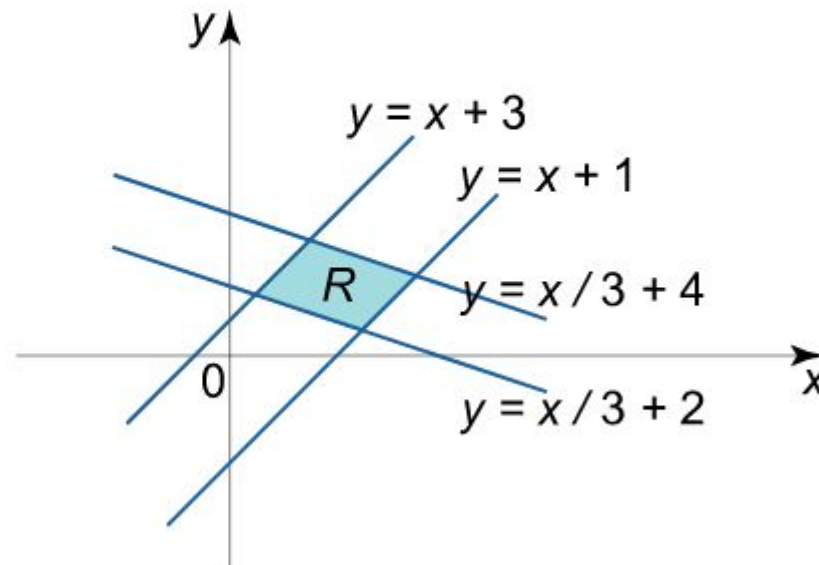


Figure 1.

We use change of variables to simplify the integral. By letting $u = y - x$, $v = y + \frac{x}{3}$, we have

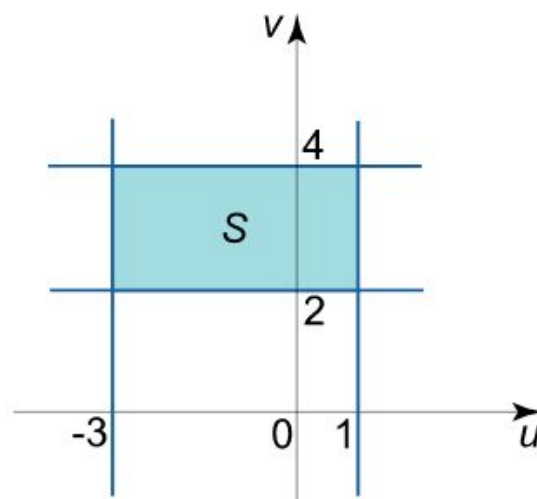
$$y = x + 1, \Rightarrow y - x = 1, \Rightarrow u = 1,$$

$$y = x - 3, \Rightarrow y - x = -3, \Rightarrow u = -3,$$

$$y = -\frac{x}{3} + 2, \Rightarrow y + \frac{x}{3} = 2, \Rightarrow v = 2,$$

$$y = -\frac{x}{3} + 4, \Rightarrow y + \frac{x}{3} = 4, \Rightarrow v = 4.$$

Hence, the pullback S of the region R is the rectangle shown in Figure 2.



Calculate the Jacobian of this transformation.

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial(y-x)}{\partial x} & \frac{\partial(y-x)}{\partial y} \\ \frac{\partial(y+\frac{x}{3})}{\partial x} & \frac{\partial(y+\frac{x}{3})}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{vmatrix} = -1 \cdot 1 - 1 \cdot \frac{1}{3} = -\frac{4}{3}.$$

Then the absolute value of the Jacobian is

$$\left| \frac{\partial (x, y)}{\partial (u, v)} \right| = \left| \left(\frac{\partial (u, v)}{\partial (x, y)} \right)^{-1} \right| = \left| \frac{1}{-\frac{4}{3}} \right| = \frac{3}{4}.$$

Hence, the differential is

$$dxdy = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| du dv = \frac{3}{4} du dv.$$

As it can be seen, calculating the integral in the new variables (u, v) is much simpler:

$$\iint_R (y-x) dxdy = \iint_S \left(u \cdot \frac{3}{4} du dv \right) = \frac{3}{4} \int_{-3}^1 u du \int_2^4 dv = \frac{3}{4} \left(\frac{u^2}{2} \right) \Big|_{-3}^1 \cdot v \Big|_2^4 = \frac{3}{4} \left(\frac{1}{2} - \frac{9}{2} \right)$$

Example 3.

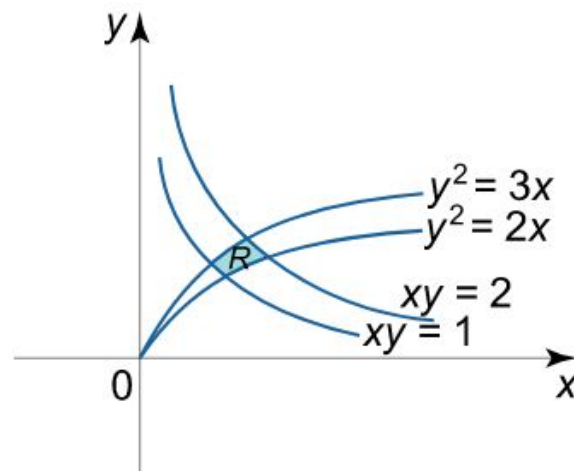
Calculate the double integral

$$\iint_R dx dy,$$

where the region R is bounded by the parabolas $y^2 = 2x$, $y^2 = 3x$ and hyperbolas $xy = 1$, $xy = 2$.

Solution.

The region R is sketched in Figure 5.



We apply the following substitution of variables to simplify the region R :

$$\begin{cases} u = \frac{y^2}{x} \\ v = xy \end{cases}$$

The pullback S of the region R is defined as follows:

$$y^2 = 2x, \Rightarrow \frac{y^2}{x} = 2, \Rightarrow u = 2,$$

$$y^2 = 3x, \Rightarrow \frac{y^2}{x} = 3, \Rightarrow u = 3,$$

$$xy = 1, \Rightarrow v = 1,$$

$$xy = 2, \Rightarrow v = 2.$$

As it can be seen, the region S is the rectangle. To find the Jacobian of the transformation, we express the variables x, y in terms of u, v .

$$u = \frac{y^2}{x}, \quad \Rightarrow x = \frac{y^2}{u},$$

$$v = xy, \quad \Rightarrow v = \frac{y^2}{u} \cdot y, \quad \Rightarrow y^3 = uv.$$

Then

$$y = \sqrt[3]{uv} = u^{\frac{1}{3}}v^{\frac{1}{3}},$$

$$x = \frac{y^2}{u} = \frac{\sqrt[3]{u^2v^2}}{u} = \sqrt[3]{\frac{v^2}{u}} = u^{-\frac{1}{3}}v^{\frac{2}{3}}.$$

Find the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial(u^{-\frac{1}{3}}v^{\frac{2}{3}})}{\partial u} & \frac{\partial(u^{-\frac{1}{3}}v^{\frac{2}{3}})}{\partial v} \\ \frac{\partial(u^{\frac{1}{3}}v^{\frac{1}{3}})}{\partial u} & \frac{\partial(u^{\frac{1}{3}}v^{\frac{1}{3}})}{\partial v} \end{vmatrix} = \begin{vmatrix} v^{\frac{2}{3}} \left(-\frac{1}{3}u^{-\frac{4}{3}}\right) & u^{-\frac{1}{3}} \cdot \frac{2}{3}v^{-\frac{1}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}v^{-\frac{2}{3}}u^{\frac{1}{3}} \end{vmatrix} = -\frac{1}{3}u^{-\frac{4}{3}}v^{\frac{2}{3}}.$$

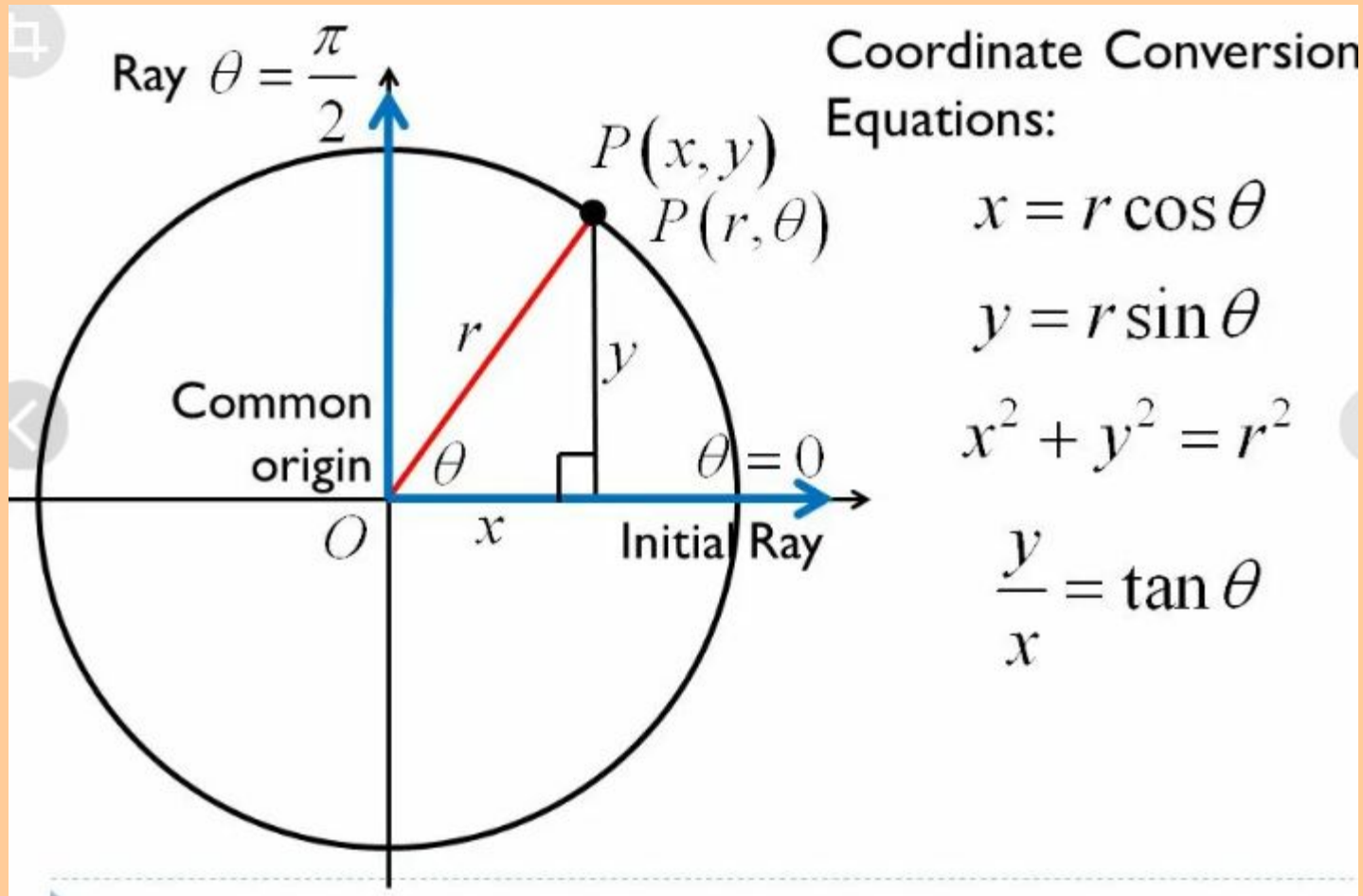
The relationship between the differentials is

$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \left| -\frac{1}{3u} \right| du dv = \frac{du dv}{3u}.$$

Then we can write the integral as

$$\iint_R dx dy = \iint_S \frac{du dv}{3u} = \int_2^3 \frac{du}{3u} \int_1^2 dv = \frac{1}{3} (\ln u) \Big|_2^3 \cdot v \Big|_1^2 = \frac{1}{3} (\ln 3 - \ln 2) \cdot (2 - 1) = \frac{1}{3} \ln \frac{3}{2}.$$

Polar coordinates



The most usable is polar coordinates system: $x = \rho \cos \varphi, y = \rho \sin \varphi$.

Jacobian for this system is

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos \varphi & -\rho \sin \varphi \\ \sin \varphi & \rho \cos \varphi \end{vmatrix} = \rho(\cos^2 \varphi + \sin^2 \varphi) = \rho. \text{ Then formula (1.4) takes}$$

$$\text{form } \iint_{(D)} f(x, y) dx dy = \iint_{(\Delta)} f(\rho \cos \varphi, \rho \sin \varphi) \rho d\rho d\varphi \quad (1.5)$$

Mechanical applications of double integral.

We know, that double integral $\iint_{(D)} f(x,y) dx dy$ represents itself the volume of some cylindrical body bounded by plane area at bottom and by surface $f(x,y)$ at top. Suppose that $f(x,y)=1$, then, obviously, the volume will coincide numerically with **area** (D):

$$S = \iint_{(D)} dx dy \quad (2.1)$$

Next, suppose that $f(x,y)=p(x,y)$ represents itself some density function at region (D), then the **mass** of this region may be evaluated by formula

$$m = \iint_{(D)} p(x,y) dx dy \quad (2.2) - \text{we must take the density at each point and}$$

multiply it by elementary area $dx dy$ and sum up all products passing to limit; and by definition we have got double integral (2.2).

Other applications are based on this result. For example, static moments and moments of inertia with relation to axes x and y may be

calculated by formulas

$$\begin{cases} M_x = \iint_{(D)} yp(x,y)dxdy, & M_y = \iint_{(D)} xp(x,y)dxdy \\ I_x = \iint_{(D)} y^2 p(x,y)dxdy, & I_y = \iint_{(D)} x^2 p(x,y)dxdy \end{cases} \quad (2.3)$$

Next, coordinates of center of mass may be calculated by formulas

$$x_0 = \frac{M_y}{m} = \frac{\iint_{(D)} xp(x,y)dxdy}{\iint_{(D)} p(x,y)dxdy}, \quad y_0 = \frac{M_x}{m} = \frac{\iint_{(D)} yp(x,y)dxdy}{\iint_{(D)} p(x,y)dxdy} \quad (2.4)$$

Triple integral. Task about mass calculation. Definition and conditions of existence.

Let's consider the next problem. Let some body (V) is given with density $\rho = \rho(M) = \rho(x, y, z)$ at each point. It is necessary to find the mass m of this body. As usually, we divide body (V) into parts $(V_1), (V_2), \dots, (V_n)$. Then we choose an arbitrary point (ξ_i, η_i, ζ_i) at each part. Approximately we can suppose that the density is constant and equals to the density of chosen point $\rho(\xi_i, \eta_i, \zeta_i)$ throughout of each part (V_i) . Then the mass of this part will be

$$m_i \approx \rho(\xi_i, \eta_i, \zeta_i) \cdot V_i.$$

And the mass of whole body will be

$$m \approx \sum_{i=1}^n \rho(\xi_i, \eta_i, \zeta_i) V_i.$$

If diameters of all parts tend to zero, then if we pass to the limit this equality will become exact

$$m = \lim \sum_{i=1}^n \rho(\xi_i, \eta_i, \zeta_i) V_i \quad (1)$$

and problem is solved.

Limits of this kind are called triple integrals. The last result may be rewritten as

$$m = \iiint_{(V)} p(x, y, z) dV \quad (2)$$

Now let's define triple integral in general sense. Let function $f(x, y, z)$ is defined in some space domain (V) . Let's divide this space into parts $(V_1), (V_2), \dots, (V_n)$ with the help of the surface network. After this we take an arbitrary point (ξ_i, η_i, ζ_i) at each part, multiply function value $f(\xi_i, \eta_i, \zeta_i)$ in this point by the volume V_i of this part and, finally, compose the integral sum

$$\sigma = \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) V_i.$$

Def. Finite limit I of integral sum σ when the most of diameters V_i tends to zero is called the triple integral of function $f(x, y, z)$ in domain (V) . It denotes by symbol $I = \iiint_{(V)} f(x, y, z) dV = \iiint_{(V)} f(x, y, z) dx dy dz$

By usual way it may be got that for triple integral existence condition $\lim(S - s) = 0$ or $\lim \sum_{i=1}^n \omega_i V_i = 0$ is necessary and sufficient, where

$\omega_i = M_i - m_i$ - oscillation of function at domain (V_i) . So the next statement follows: every continuous function is integrated.

Properties of integrated function and triple integrals.

Most of properties is analogical to properties of double integral and we consider its without proof.

1. If $(V) = (V') + (V'')$ then $\iiint_{(V)} f dV = \iiint_{(V')} f dV + \iiint_{(V'')} f dV$ and from existence of the left integral existence of the right integrals follows and vice versa.
2. If $k = \text{const}$ then $\iiint_{(V)} k f dV = k \iiint_{(V)} f dV$ and from existence of the right integral existence of the left integral follows.
3. If function f and g are integrated in area (V) then function $f \pm g$ is integrated too and $\iiint_{(V)} (f \pm g) dV = \iiint_{(V)} f dV \pm \iiint_{(V)} g dV$
4. If for integrated in area (V) functions f and g inequality $f \leq g$ takes place then $\iiint_{(V)} f dV \leq \iiint_{(V)} g dV$.
5. If function f is integrated then function $|f|$ is integrated too and inequality $\iiint_{(V)} f dV \leq \iiint_{(V)} |f| dV$ takes place.
6. If integrated in area (V) function f satisfies to inequality $m \leq f \leq M$ then $mV \leq \iiint_{(V)} f dV \leq MV$. By other words, we have mean value theorem $\iiint_{(V)} f dV = \mu V$ ($m \leq \mu \leq M$). In a case of continuous function this formula may be rewrote as $\iiint_{(V)} f dV = f(\bar{x}, \bar{y}, \bar{z}) \cdot V$ (3).