

# Математик

а

## Приемы доказательства неравенств, содержащих переменные

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Учебное заведение: МОУ Лицей №1 г. Комсомольск-на-Амуре

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Руководитель: Будлянская Наталья Леонидовна

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

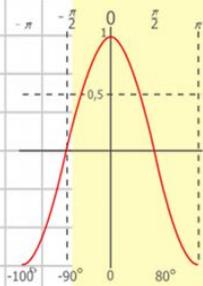
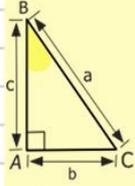
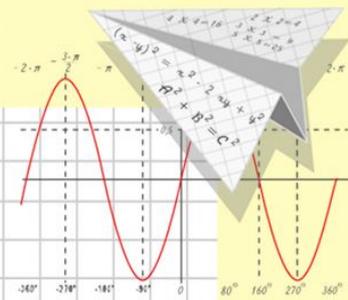
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

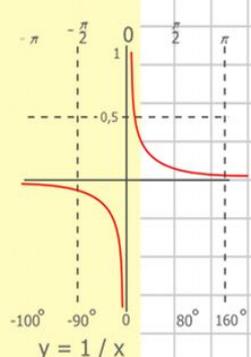
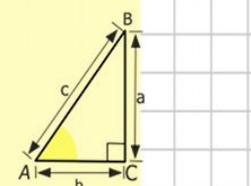
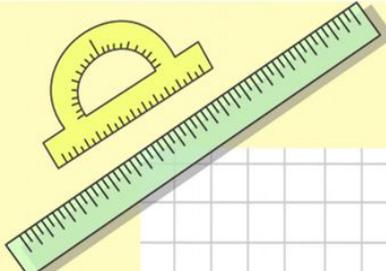
$$\begin{cases} y = 1 \\ x = 25 + 45 \\ x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$

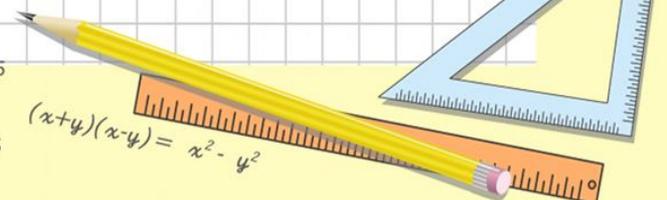
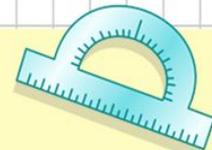


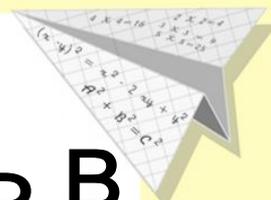
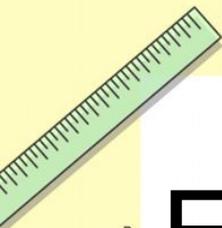
$$y = \cos x$$

- 2 x 2 = 4
- 3 x 3 = 9
- 4 x 4 = 16
- 5 x 5 = 25
- 6 x 6 = 36
- 7 x 7 = 49
- 8 x 8 = 64

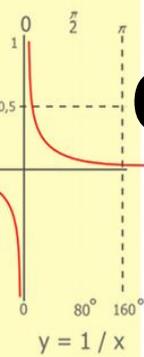
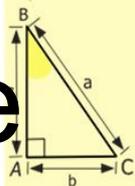
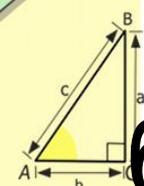


$$\begin{array}{r} 1 \\ 2500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 10500 \end{array}$$



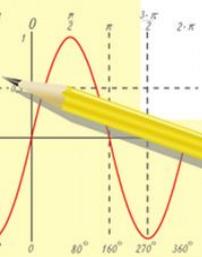


Если вы хотите участвовать в большой жизни, то наполняйте свою голову математикой, пока есть к тому возможность. Она окажет вам потом огромную помощь во всей вашей работе.  
(М.И. Калинин)



$$\begin{array}{r} 1 \\ \times 2500 \\ \hline 2500 \\ + 42 \\ \hline 210 \\ \hline 10500 \end{array}$$

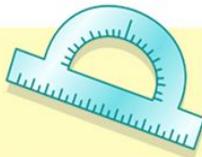
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$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

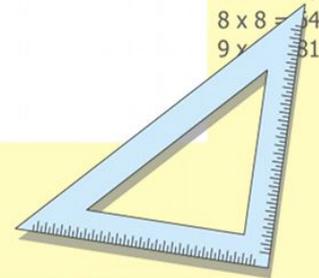
$$\sin 90^\circ = 1$$



$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \\ \hline x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$



1. Представление левой части неравенства в виде суммы неотрицательных слагаемых (правая часть равна 0) с использованием тождеств.

**Пример 1. Доказать что для любого  $x \in \mathbb{R}$**

$$x^4 - 7x^2 - 2x + 20 > 0$$

**Доказательство. 1 способ.**

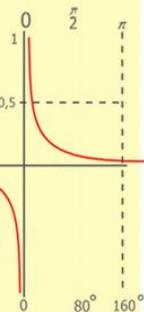
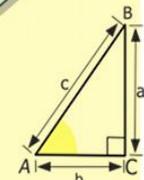
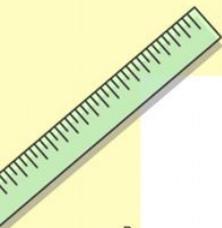
$$\begin{aligned} x^4 - 7x^2 - 2x + 20 &= (x^4 - 8x^2 + 16) + (x^2 - 2x + 1) + 3 = \\ &= (x^2 - 4)^2 + (x - 1)^2 + 3 > 0 \end{aligned} \quad \text{для } x \in \mathbb{R}$$

**2 способ.**

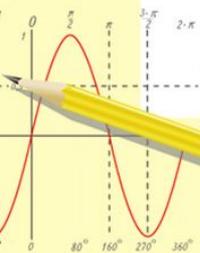
$$\begin{aligned} x^4 - 7x^2 - 2x + 20 &= (x^2 - 4)^2 + x^2 - 2x + 4 \\ (x^2 - 4)^2 &\geq 0 \end{aligned} \quad \text{для } x \in \mathbb{R}$$

$$x^2 - 2x + 4 > 0 \quad \text{для } x \in \mathbb{R} \quad \text{т. к.}$$

$a = 1$   $\frac{D}{4} = 1 - 4 = -3 < 0$  — отрицательной функции  $y = x^2 - 2x + 4$ , что означает ее положительность при любом действительном  $x$ .



$$\begin{array}{r} 1\ 2\ 5\ 00 \\ \times 4\ 2 \\ \hline 2\ 1\ 0 \\ + 8\ 4 \\ \hline 105\ 0\ 00 \end{array}$$



$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

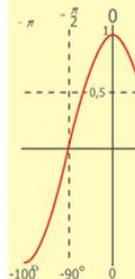
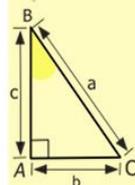
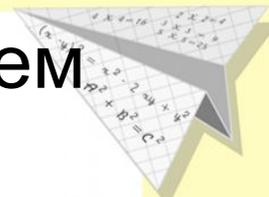


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

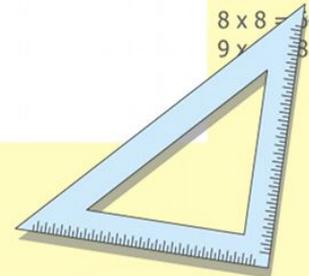
$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



$$y = \cos$$

$$\begin{aligned} 2 \times 2 &= 4 \\ 3 \times 3 &= 9 \\ 4 \times 4 &= 16 \\ 5 \times 5 &= 25 \\ 6 \times 6 &= 36 \\ 7 \times 7 &= 49 \\ 8 \times 8 &= 64 \\ 9 \times 9 &= 81 \end{aligned}$$



**Пример 2. Доказать, что для любых  $x$  и  $y$**

$$x^2 + y^2 - 2xy + 2x - 2y + 1 \geq 0$$

**Доказательство.**

$$\begin{aligned} x^2 + y^2 - 2xy + 2x - 2y + 1 &= (x - y)^2 + 2(x - y) + 1 = \\ &= (x - y + 1)^2 \geq 0 \end{aligned}$$

и у

**Пример 3. Доказать, что  $x^2 + 4y^2 + 4y - 4x + 5 \geq 0$**

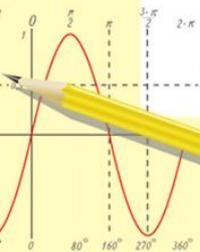
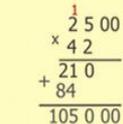
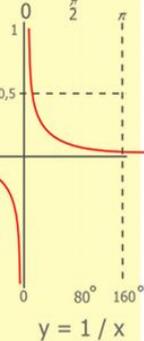
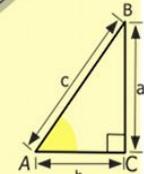
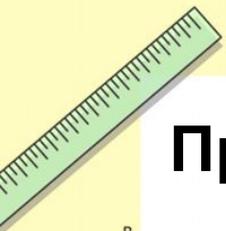
**Доказательство.**

$$\begin{aligned} x^2 + 4y^2 + 4y - 4x + 5 &= (x^2 - 4x + 4) + (4y^2 + 4y + 1) = \\ &= (x - 2)^2 + (2y + 1)^2 \geq 0 \end{aligned}$$

**Пример 4. Доказать, что для любых  $a$  и  $b$**

**Доказательство**  $a^2 + b^2 - 2ab(a + b) + 2a^2b^2 \geq 0$

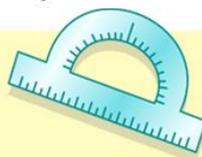
$$\begin{aligned} a^2 + b^2 - 2ab(a + b) + 2a^2b^2 &= \\ &= (a^2 - 2a^2b + a^2b^2) + (b^2 - 2ab^2 + a^2b^2) = \\ &= a^2(1 - b)^2 + b^2(1 - a)^2 \geq 0 \end{aligned}$$



$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

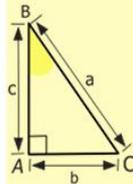
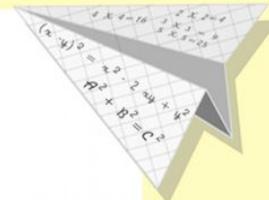
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

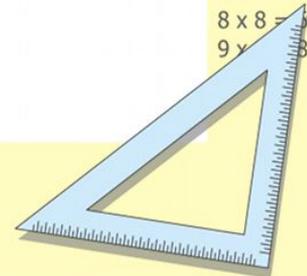


$$\begin{cases} x = 25y + 45 \\ y = 1 \\ x = 25 + 45 \\ x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$



|         |    |
|---------|----|
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# 2. Метод от противного

Вот хороший пример применения данного метода.

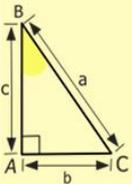
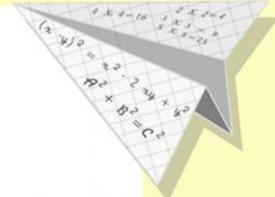
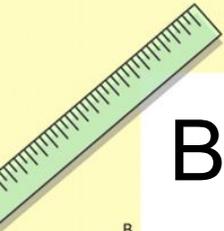
Доказать, что  $a^2 + ab + b^2 \geq 0$  для  $a, b \in R$ .

Доказательство.

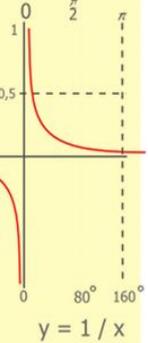
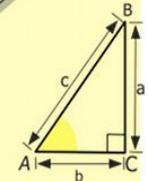
Предположим, что  $a^2 + ab + b^2 < 0$ .

Но  $a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3}{4}b^2$  и доказывает, что наше предположение неверно.

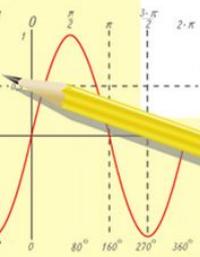
Ч.Т.Д.



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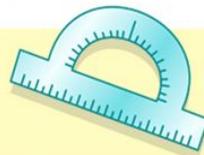
$$\begin{array}{r} 1 \\ 2500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 10500 \end{array}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

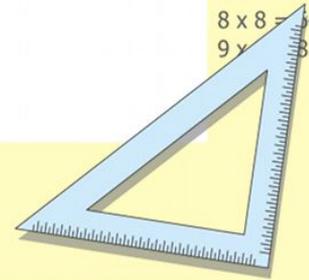


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



**Пример 5. Доказать, что для любых чисел  $A, B, C$  справедливо неравенство**

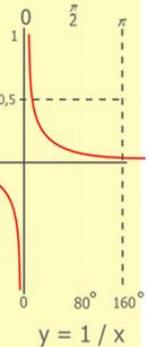
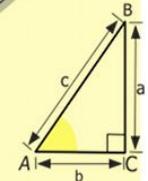
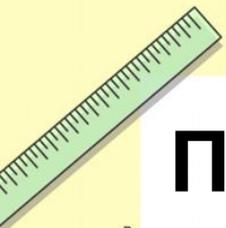
$$\frac{A + B + C}{3} \leq \sqrt{\frac{A^2 + B^2 + C^2}{3}}$$

**Доказательство.** Очевидно, что данное неравенство достаточно установить для неотрицательных  $A, B, C$

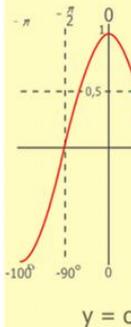
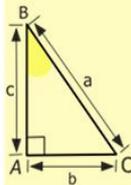
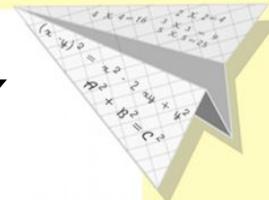
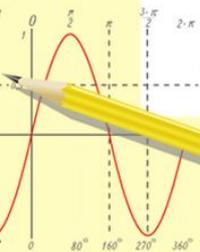
С помощью соотношения

$$\sqrt{\frac{A^2 + B^2 + C^2}{3}} = \sqrt{\frac{|A|^2 + |B|^2 + |C|^2}{3}} \geq \frac{|A| + |B| + |C|}{3} \geq \frac{A + B + C}{3}$$

, что является обоснованием исходного неравенства.



$$\begin{array}{r} 1 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 10500 \end{array}$$

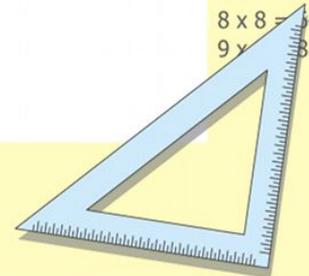


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$$\begin{array}{l} y = \sin 90 \\ x = 25 + 45 \\ y = 1 \\ x = 25 + 45 \\ \hline x = 70 \end{array}$$

$$(x+y)(x-y) = x^2 - y^2$$



Пусть теперь нашлись такие неотрицательные числа  $A, B$  и  $C$ , для которых выполняется

неравенство

$$\frac{A + B + C}{3} \leq \sqrt{\frac{A^2 + B^2 + C^2}{3}}$$

$$\left(\frac{A + B + C}{3}\right)^2 > \frac{A^2 + B^2 + C^2}{3}$$

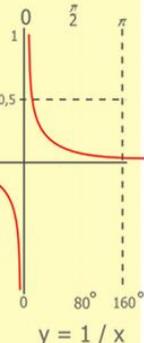
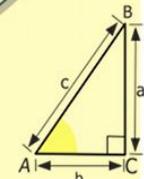
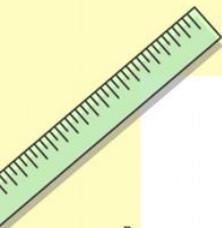
$$\frac{A^2 + B^2 + C^2 + 2AB + 2AC + 2BC}{9} > \frac{A^2 + B^2 + C^2}{3}$$

$$A^2 + B^2 + C^2 + 2AB + 2AC + 2BC - 3A^2 - 3B^2 - 3C^2 > 0$$

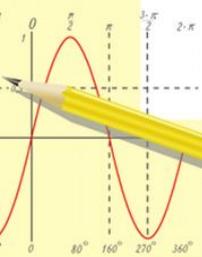
$$-(2A^2 + 2B^2 + 2C^2 - 2AB - 2AC - 2BC) > 0$$

$$-((A - B)^2 + (A - C)^2 + (B - C)^2) > 0$$

, что невозможно ни при каких действительных  $A, B$  и  $C$ . Сделанное выше предположение опровергнуто, что доказывает исследуемое исходное неравенство.



$$\begin{array}{r} 1\ 5\ 00 \\ \times 42 \\ \hline 21\ 0 \\ + 84 \\ \hline 105\ 0\ 00 \end{array}$$



$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

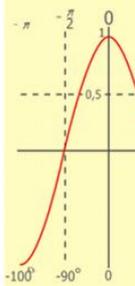
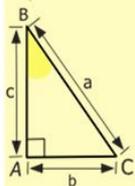
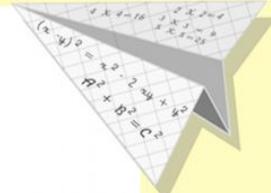
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$



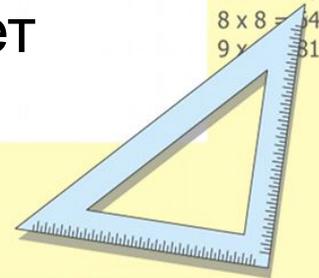
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$$(x+y)(x-y) = x^2 - y^2$$



$$y = \cos$$

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- 9 x 9 = 81



# Использование свойств квадратного трехчлена

Метод основан на свойстве

неотрицательности квадратного трехчлена  $P(x) = ax^2 + bx + c$  для  $a > 0$

$D < 0$ , если

и

$$2x^2 - 6x + 13 > 0 \text{ для } x \in \mathbb{R}$$

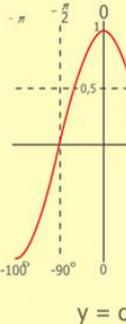
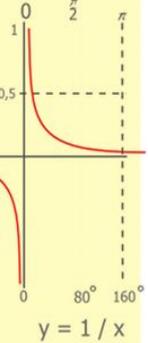
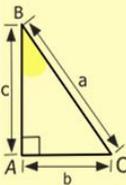
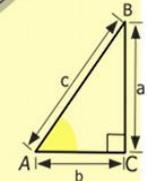
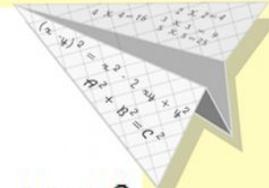
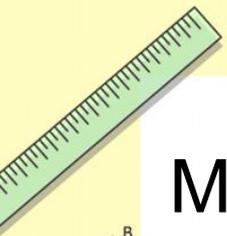
**Пример 6. Доказать, что**

**Доказательство:**

$$P(x) = 2x^2 - 6x + 13 > 0$$

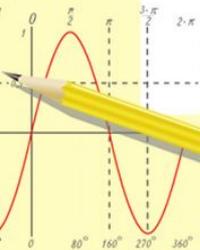
$$D = 9 - 13 \cdot 2 = -17, \frac{D}{4} < 0 \Rightarrow 2x^2 - 6x + 13 > 0 \text{ для } x \in \mathbb{R}$$

$\Rightarrow$



$$\begin{array}{r} 1 \\ \times 2500 \\ \hline 2500 \\ + 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$

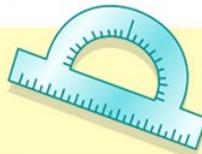
$$\begin{array}{l} 2 \times 2 = 4 \\ 3 \times 3 = 9 \\ 4 \times 4 = 16 \\ 5 \times 5 = 25 \\ 6 \times 6 = 36 \\ 7 \times 7 = 49 \\ 8 \times 8 = 64 \\ 9 \times 9 = 81 \end{array}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

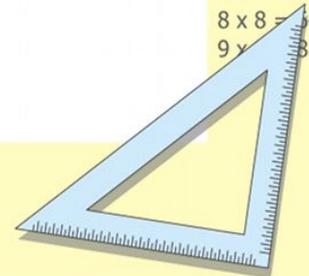


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



**Пример 7. Доказать, что для любых действительных  $x$  и  $y$  имеет место быть неравенство  $x^2 + y^2 + xy + x - y + 3 > 0$**

**Доказательство. Рассмотрим левую часть**

**неравенства как квадратный трехчлен относительно  $x$ :**

$$(y + 1)x^2 + (y + 1)x + y^2 - y + 3 \quad \text{для } x \in \mathbb{R}$$

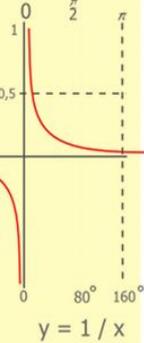
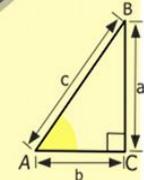
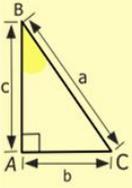
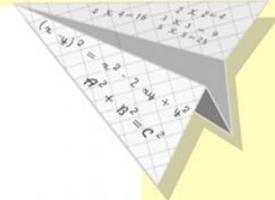
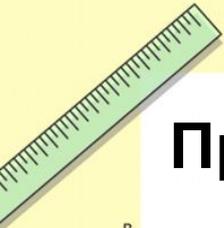
$$(y + 1)^2 - 4y^2 + 4y - 12 = -3(y - 1)^2$$

$$a > 0 \quad D < 0$$

$$x^2 + y^2 + xy + x - y + 3 > 0$$

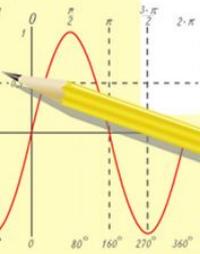
$$D = \quad \Rightarrow P(x) > 0 \quad \text{и}$$

**верно при любых действительных значениях  $x$  и  $y$ .**



$$\begin{array}{r} 1 \ 2 \ 5 \ 00 \\ \times 4 \ 2 \\ \hline 21 \ 0 \\ + 84 \\ \hline 105 \ 0 \ 00 \end{array}$$

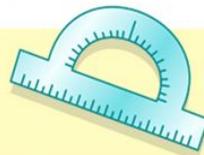
- 2 x 2 = 4
- 3 x 3 = 9
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$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

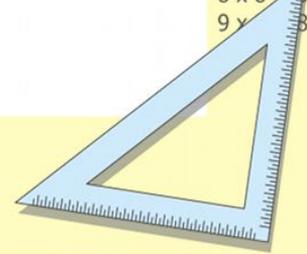


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



**Пример 8. Доказать, что  $x^2 + 2xy + 3y^2 + 2x + 6y + 3 \geq 0$**

**для любых действительных значениях  $x$  и  $y$ .**

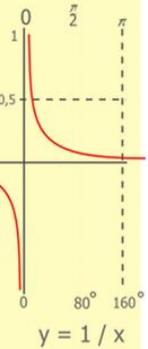
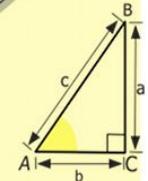
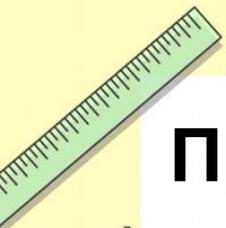
**Доказательство. Пусть  $P(y) = 3y^2 + 2(x + 3)y + x^2 + 2x + 3$**

$$\frac{D}{4} = (x + 3)^2 - 3(x^2 + 2x + 3) =$$

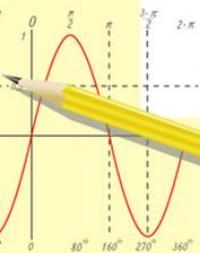
$$= x^2 + 6x + 9 - 3x^2 - 6x - 9 = -2x^2 \leq 0 \quad \text{для } x \in \mathbb{R}$$

**Это означает, что  $P(y) \geq 0$  для любых действительных  $y$  и неравенство**

**$x^2 + 2xy + 3y^2 + 2x + 6y + 3 \geq 0$  выполняется при любых действительных  $x$  и  $y$ .**



$$\begin{array}{r} 1 \\ \times 2500 \\ \hline 2500 \\ + 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$



$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

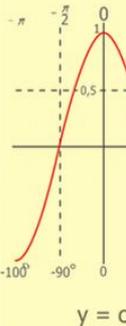
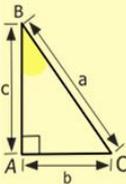
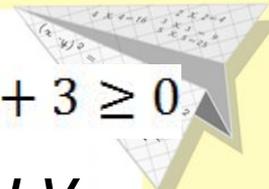


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

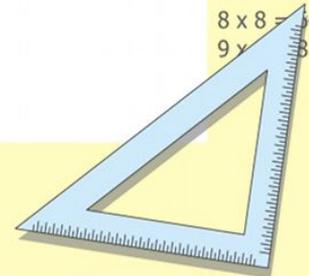
$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



$$\begin{array}{l} 2 \times 2 = 4 \\ 3 \times 3 = 9 \\ 4 \times 4 = 16 \\ 5 \times 5 = 25 \\ 6 \times 6 = 36 \\ 7 \times 7 = 49 \\ 8 \times 8 = 64 \\ 9 \times 9 = 81 \end{array}$$



# Метод введения новых переменных или метод подстановки

**Пример 9. Доказать, что для любых неотрицательных чисел  $x, y, z$**

$$x + y + z \geq \sqrt{xy} + \sqrt{yz} + \sqrt{xz}$$

**Доказательство.** Воспользуемся верным неравенством для  $a \geq 0, b \geq 0, c \geq 0$

$$a^2 + b^2 + c^2 = ab + bc + ac$$

$$a = \sqrt{x}$$

$$b = \sqrt{y}$$

$$c = \sqrt{z}$$

**Получаем исследуемое неравенство**

$$x + y + z \geq \sqrt{xy} + \sqrt{yz} + \sqrt{xz}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

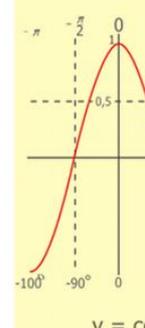
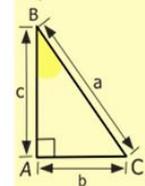
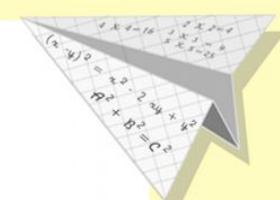
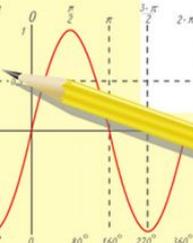
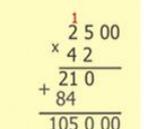
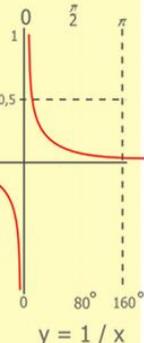
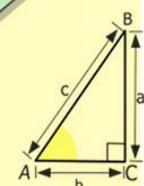
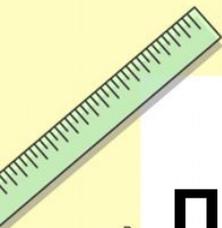
$$\sin 90^\circ = 1$$

$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

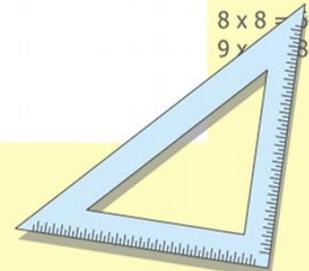
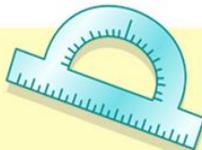
$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



- 2 x 2 = 4
- 3 x 3 = 9
- 4 x 4 = 16
- 5 x 5 = 25
- 6 x 6 = 36
- 7 x 7 = 49
- 8 x 8 = 64
- 9 x 9 = 81



# Использование свойств функций.

**Пример 10.** Докажем неравенство  $a^2 + ab + b^2 \geq 0$  для любых  $a$  и  $b$ .

Доказательство. Рассмотрим 2 случая:

1) Если  $a=b$ , то  $a^2 + ab + b^2 = 3a^2 \geq 0$  для  $a \in \mathbb{R}$  причем равенство достигается только при

$$a=b=\hat{a} \neq b$$

2) Если

$$a^2 + ab + b^2 = \frac{c^2 - b^2}{a - b}$$

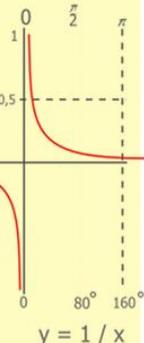
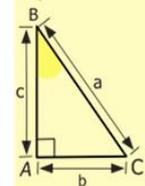
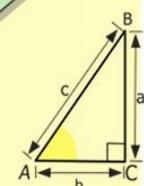
$$y = x^3 \downarrow$$

на  $\mathbb{R} \Rightarrow$

$$\left( \frac{a - b}{a - b} \right) * \left( \frac{a^3 - b^3}{a - b} \right) > 0,$$

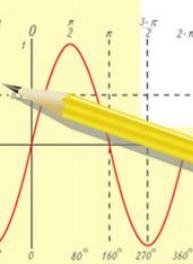
что доказывает неравенство

$$a^2 + ab + b^2 \geq 0$$



$\begin{array}{r} 2500 \\ \times 42 \\ \hline 2100 \\ + 840 \\ \hline 105000 \end{array}$

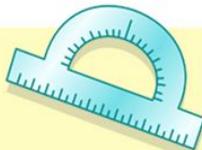
$\begin{array}{l} 2 \times 2 = 4 \\ 3 \times 3 = 9 \\ 4 \times 4 = 16 \\ 5 \times 5 = 25 \\ 6 \times 6 = 36 \\ 7 \times 7 = 49 \\ 8 \times 8 = 64 \\ 9 \times 9 = 81 \end{array}$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

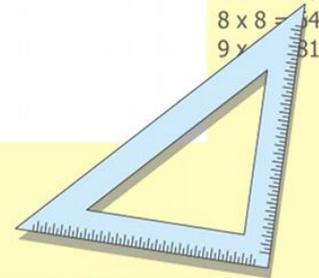


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$

$$\frac{x = 70}{x = 70}$$



**Пример 11.** Докажем, что для любых  $a \neq b$

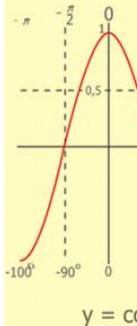
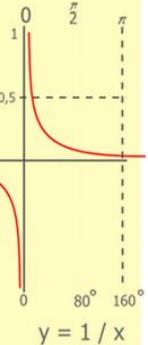
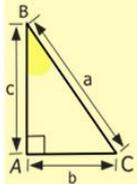
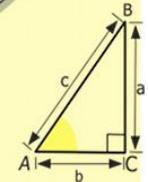
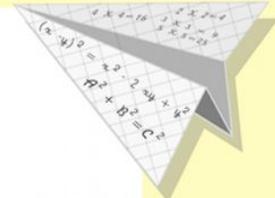
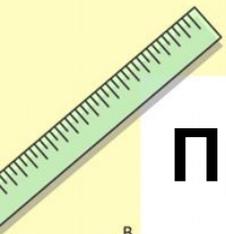
$$a^6 + b^6 > a^5b + ab^5$$

**Доказательство.**

$$\begin{aligned} a^6 + b^6 - a^5b - ab^5 &= a^6 - a^5b + b^6 - ab^5 = \\ &= a^5(a - b) - b^5(a - b) = (a - b)(a^5 - b^5) \end{aligned}$$

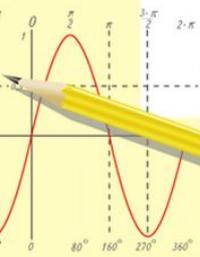
$y = x^5$  на  $\mathbb{R}$ .

Если  $a \neq b$ , то знаки чисел  $(a - b)$  и  $a^5 - b^5$  совпадают, что означает положительность исследуемой разности  $\Rightarrow a^6 + b^6 > a^5b + ab^5$



$$\begin{array}{r} 1 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 10500 \end{array}$$

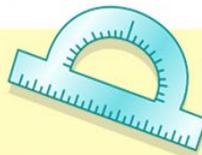
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- 6 x 6 = 36
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$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

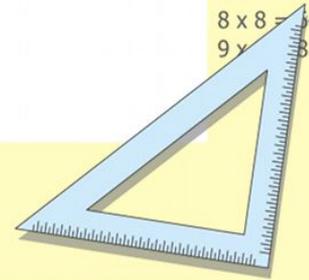


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



# Применение метода

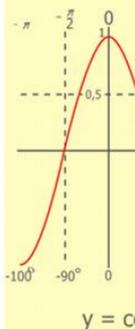
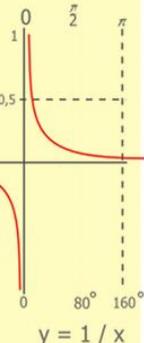
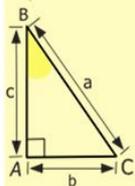
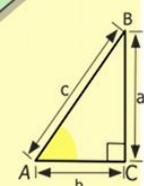
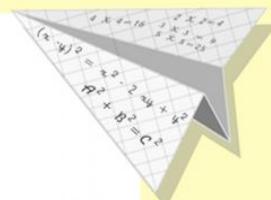
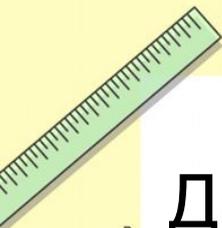
## математической индукции

Данный метод применяется для доказательства неравенств относительно натуральных чисел.

**Пример 12.** Докажем  $3^n > n + 1$  то для любого  $n \in \mathbb{N}$

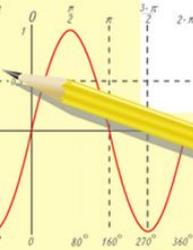
1) Проверим истинность утверждения при  $n = 1$   
 $3^1 > 1 + 1$   
- (верно)

2) Предположим верность утверждения при  $n = k$   
( $k > 1$ )  
 $3^k > k + 1$



$$\begin{array}{r} 1 \ 5 \ 00 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$

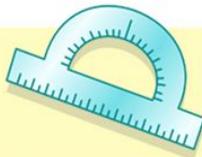
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$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

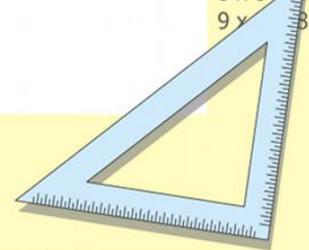
$$\sin 90^\circ = 1$$



$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \\ \hline x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$



### 3) Докажем истинность утверждения при

$$3^{k+1} > k + 2$$

$$3^k > k + 1 \quad | *3$$

$$3^{k+1} > 3k + 3$$

$$3k + 3 \quad k + 2 \quad 3k + 3 - k - 2 = 2k + 1 \quad 2k + 1 > 0$$

**Сравним:**  
 $3k + 3 > k + 2$

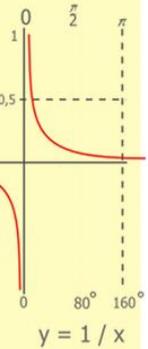
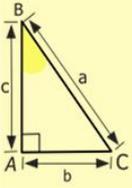
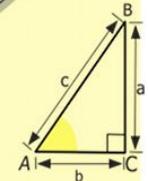
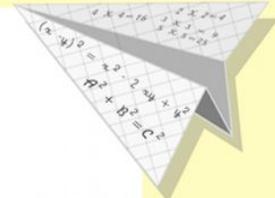
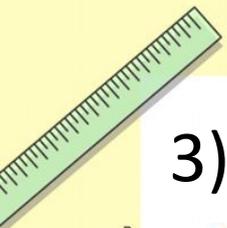
**И :**

$$3k + 3 > k + 3$$

**Имеем:**  $3k + 3 > k + 2$

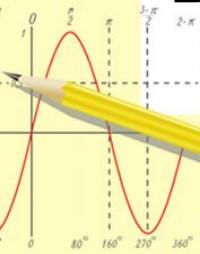
$$3^{k+1} > k + 2$$

**Вывод:** утверждение верно для любого  $n \in \mathbb{N}$ .



$$\begin{array}{r} 2500 \\ \times 42 \\ \hline 2100 \\ + 840 \\ \hline 105000 \end{array}$$

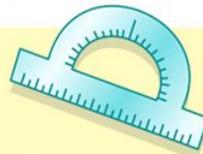
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$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

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$$\sin 90^\circ = 1$$

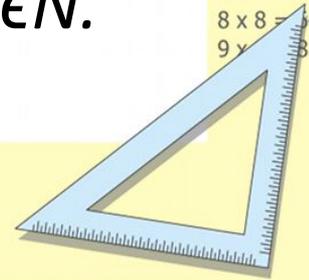


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

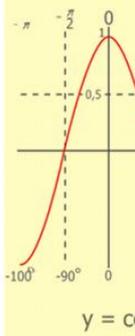
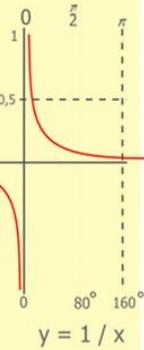
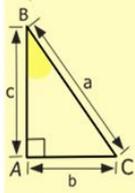
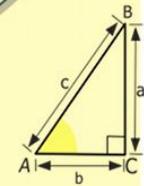
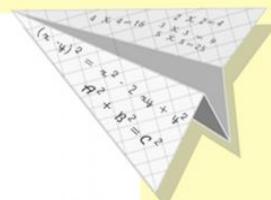
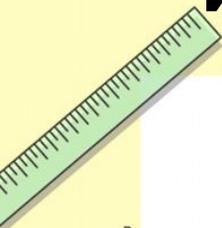
$$(x+y)(x-y) = x^2 - y^2$$



# Использование замечательных неравенств

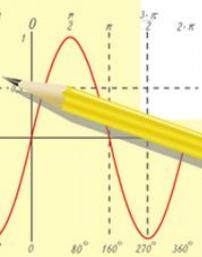
- Теорема о средних (неравенство Коши)
- Неравенство Коши – Буняковского
- Неравенство Бернулли

Рассмотрим каждое из перечисленных неравенств в отдельности.



$$\begin{array}{r} 2500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$

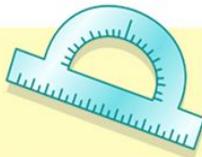
$$\begin{array}{l} 2 \times 2 = 4 \\ 3 \times 3 = 9 \\ 4 \times 4 = 16 \\ 5 \times 5 = 25 \\ 6 \times 6 = 36 \\ 7 \times 7 = 49 \\ 8 \times 8 = 64 \\ 9 \times 9 = 81 \end{array}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

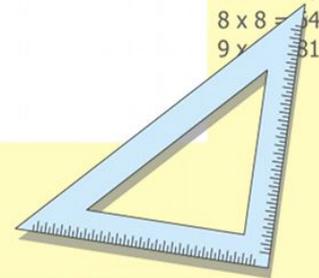
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$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

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$$(x+y)(x-y) = x^2 - y^2$$



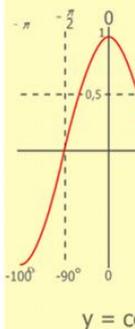
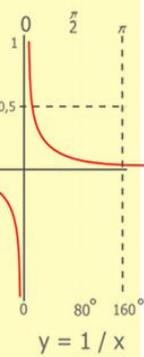
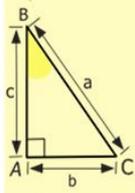
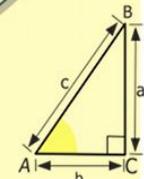
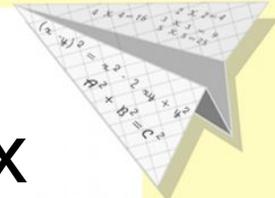
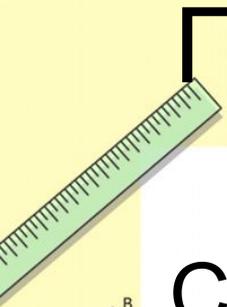
# Применение теоремы о средних (неравенства Коши)

Среднее арифметическое нескольких неотрицательных чисел больше или равно их среднего геометрического

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} \quad a \geq 0$$

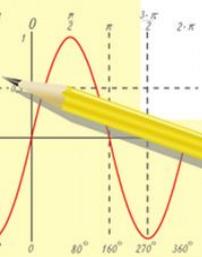
Знак равенства достигается тогда и только тогда,  $a_1 = a_2 = \dots = a_n$

Рассмотрим частные случаи этой теоремы:



$$\begin{array}{r} \frac{1}{2} 500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$

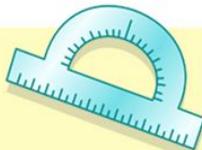
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$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

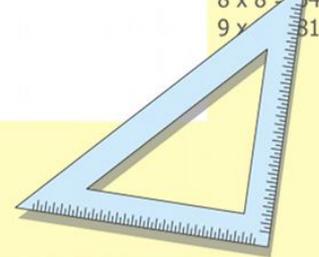
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$



$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases} \quad \begin{cases} y = 1 \\ x = 25 + 45 \\ \hline x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$



1. Пусть  $n=2, a \geq 0, b \geq 0$ , тогда  $\frac{a+b}{2} \geq \sqrt{ab}$

2. Пусть  $n=2, a > 0$ , тогда  $a + \frac{1}{a} \geq 2$

3. Пусть  $n=3, a \geq 0, b \geq 0, c \geq 0$ , тогда

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

Пример. Доказать, что для всех неотрицательных  $a, b, c$  выполняется неравенство  $a^2 + b^2 + c^2 \geq ab + bc + ac$

Доказательство.  $a^2 + b^2 \geq 2ab$

$$+ a^2 + c^2 \geq 2ac$$

$$+ c^2 + b^2 \geq 2cb$$

---


$$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ac$$

$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

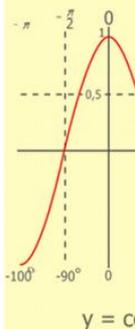
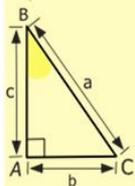
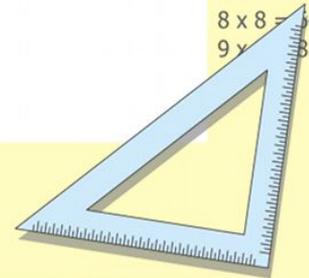
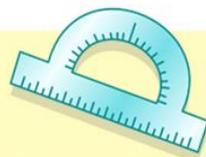
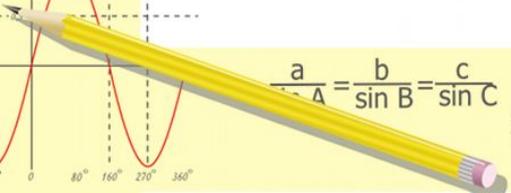
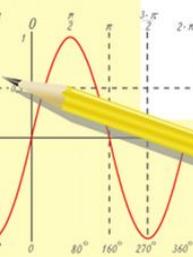
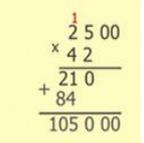
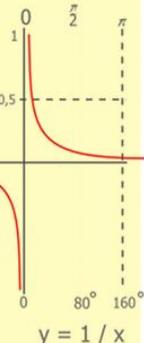
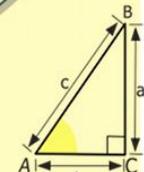
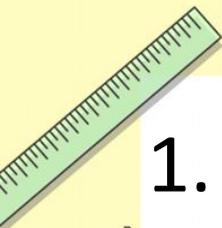
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$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



- 2 x 2 = 4
- 3 x 3 = 9
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# Неравенство Коши - Буняковского

Неравенство Коши - Буняковского

утверждает, что  $a_1, a_2, \dots, a_n \neq 0, b_1, b_2, \dots, b_n$  справедливо соотношение

$$a_1 b_1 + \dots + a_n b_n \leq \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}$$

Доказанное неравенство имеет геометрическую интерпретацию. Для  $n=2,3$  оно выражает известный факт, что скалярное произведение двух векторов на плоскости и в пространстве не превосходит произведение их длин. Для  $n=2$

нерез  $a_1 b_1 + a_2 b_2 \leq \sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}$  . Для  $n=3$

получим

$$a_1 b_1 + a_2 b_2 + a_3 b_3 \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

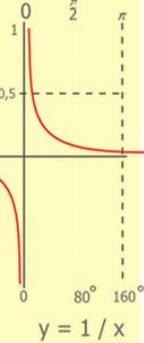
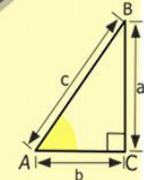
$$\sin 90^\circ = 1$$

$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

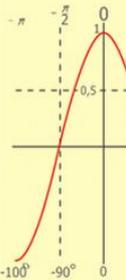
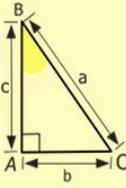
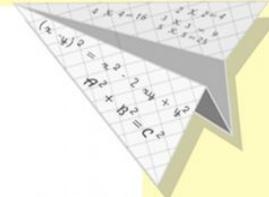
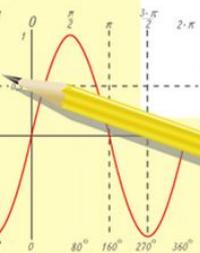
$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$

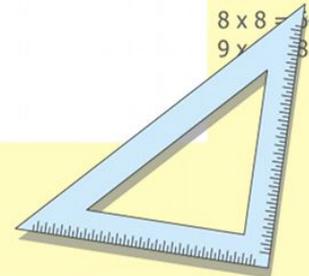


$$\begin{array}{r} 1 \ 2 \ 5 \ 00 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$



$$y = \cos$$

$$\begin{array}{l} 2 \times 2 = 4 \\ 3 \times 3 = 9 \\ 4 \times 4 = 16 \\ 5 \times 5 = 25 \\ 6 \times 6 = 36 \\ 7 \times 7 = 49 \\ 8 \times 8 = 64 \\ 9 \times 9 = 81 \end{array}$$



**Пример 14.** Доказать, что для любых  $a, b, c \in R$  справедливо неравенство  $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$

**Доказательство.** Запишем исследуемое неравенство в сл  $(1 \cdot a + 1 \cdot b + 1 \cdot c)^2 \leq (1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2)$

Это заведомо истинное неравенство, так как является частным случаем неравенства Коши – Буняковского.

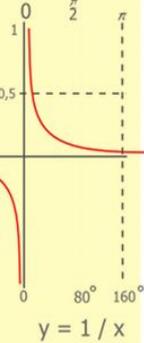
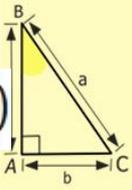
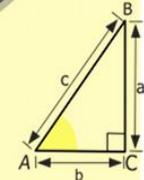
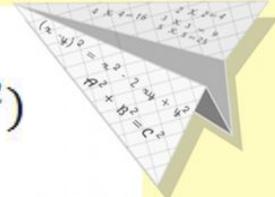
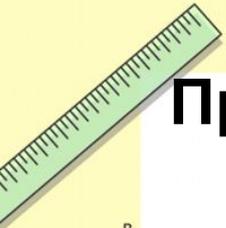
**Пример 15.** Доказать, что  $a^2b^2 + b^2c^2 + a^2c^2 \geq abc(a + b + c)$  справедливо неравенство

**Доказательство.** Достаточно записать данное

$$\sqrt{(ab)^2 + (bc)^2 + (ca)^2} \cdot \sqrt{(ac)^2 + (ab)^2 + (bc)^2} \geq abca + bcab + cabc$$

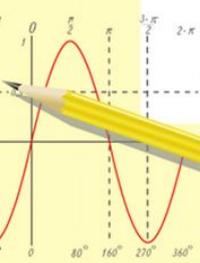
и сослаться на неравенство

Коши – Буняковского.



$$\begin{array}{r} 1 \\ 2500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 10500 \end{array}$$

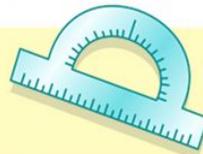
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$$\sin 90^\circ = 1$$

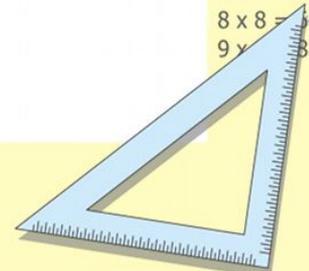


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$



# Неравенство Бернулли

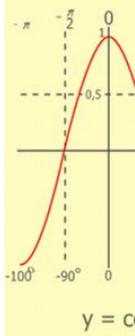
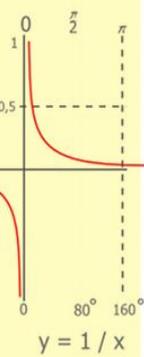
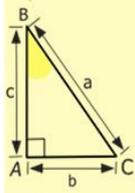
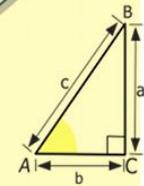
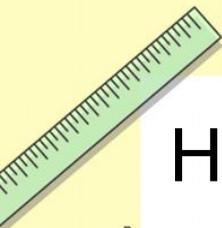
Неравенство Бернулли утверждает, что если  $x > -1$ , то для всех натуральных значений  $n$  выполняется неравенство  $(1 + x)^n \geq 1 + nx$

Неравенство может применяться для выражений  $1,005^{200}$  или  $0,992^{10}$

$$1,005^{200} = (1 + 0,005)^{200} \geq 1 + 200 \cdot 0,005 = 2$$

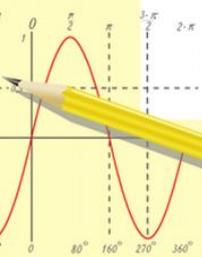
$$0,994^{10} = (1 - 0,006)^{10} \geq 1 - 10 \cdot 0,006 = 0,94$$

Кроме того, очень большая группа неравенств может быть легко доказана с помощью теоремы Бернулли.



$$\begin{array}{r} 12500 \\ \times 42 \\ \hline 210 \\ + 84 \\ \hline 105000 \end{array}$$

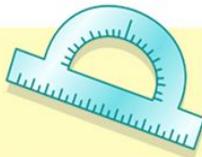
- 2 x 2 = 4
- 3 x 3 = 9
- 4 x 4 = 16
- 5 x 5 = 25
- 6 x 6 = 36
- 7 x 7 = 49
- 8 x 8 = 64
- 9 x 9 = 81



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

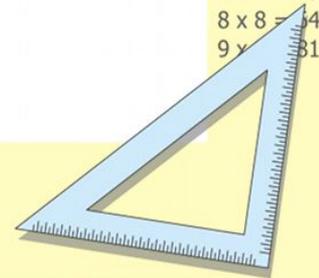
$$\sin 90^\circ = 1$$



$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 25 + 45 \\ \hline x = 70 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$



**Пример 16.** Доказать, что для любых  $n \in \mathbb{N}$

$$(1,5)^n \geq 1 + 0,5n$$

Доказательство  $(1,5)^n = (1 + 0,5)^n$ ожив  $x=0,5$  и применив теорему Бернулли для

$$(1 + 0,5)^n$$

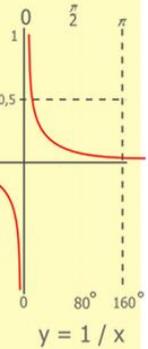
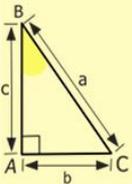
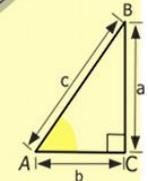
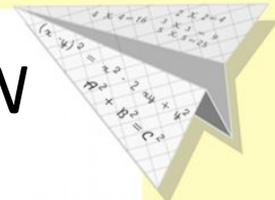
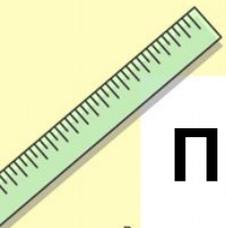
, получим требуемое неравенство.

**Пример 17.**  $(0,7)^n \geq 1 - 0,3n$ о для любых  $n \in \mathbb{N}$

Доказательство  $(0,7)^n = (1 - 0,3)^n \geq 1 - 0,3n$

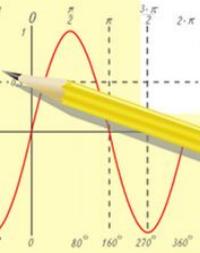
по теореме Бернулли, что и

требовалось.



$$\begin{array}{r} 2500 \\ \times 42 \\ \hline 2100 \\ + 840 \\ \hline 105000 \end{array}$$

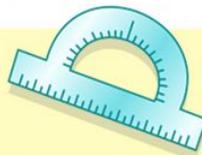
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$$\frac{a}{A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\sin 90^\circ = 1$$

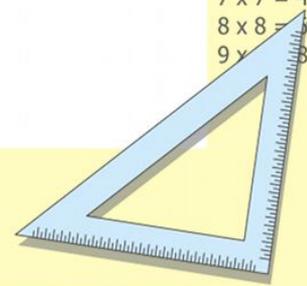


$$\begin{cases} y = \sin 90 \\ x = 25y + 45 \end{cases}$$

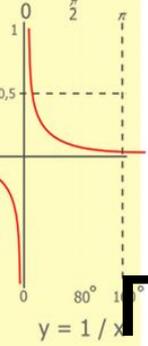
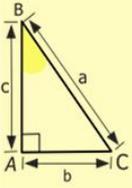
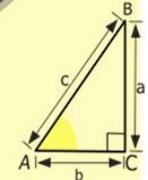
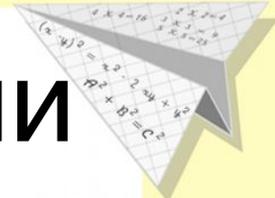
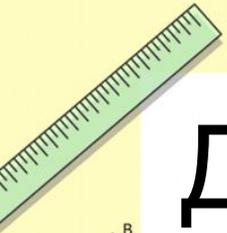
$$\begin{cases} y = 1 \\ x = 25 + 45 \end{cases}$$

$$x = 70$$

$$(x+y)(x-y) = x^2 - y^2$$

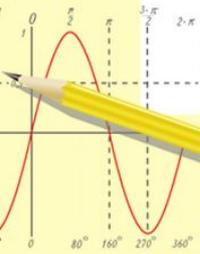


Давида Гильберта спросили  
 об одном из его бывших  
 учеников. "А, такой-то? -  
 вспомнил Гильберт. - Он стал  
 поэтом. Для математики у него  
 было слишком мало  
 воображения.



$$\begin{array}{r}
 2500 \\
 \times 42 \\
 \hline
 210 \\
 + 84 \\
 \hline
 105000
 \end{array}$$

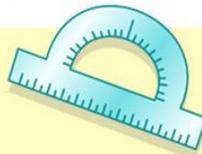
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$$\sin 90^\circ = 1$$



$$\begin{cases}
 y = \sin 90 \\
 x = 25y + 45
 \end{cases}$$

$$\begin{cases}
 y = 1 \\
 x = 25 + 45 \\
 \hline
 x = 70
 \end{cases}$$

$$(x+y)(x-y) = x^2 - y^2$$

