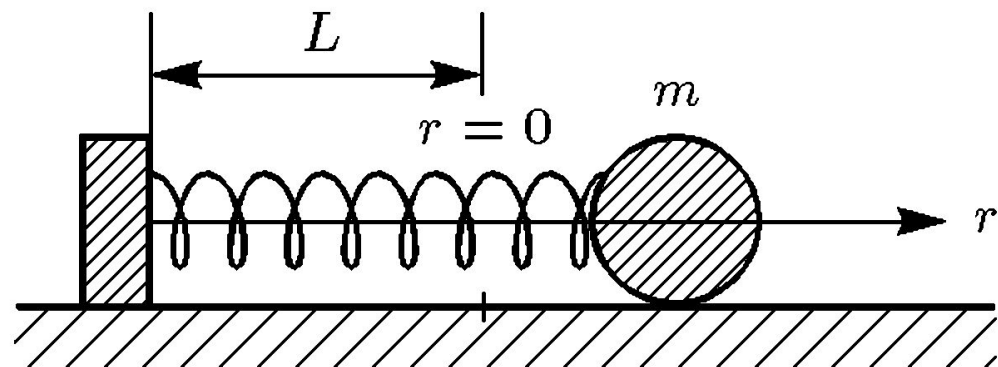


Принцип аналогій



$$E_{\text{K}} = \frac{mv^2}{2} = \frac{m (dr/dt)^2}{2}$$

$$E_{\text{II}} = - \int_0^r F dr' = \int_0^r kr' dr' = k \frac{r^2}{2}$$

$$E = E_{\text{K}} + E_{\text{II}}$$

$$E = \frac{m (dr/dt)^2}{2} + \frac{kr^2}{2}$$

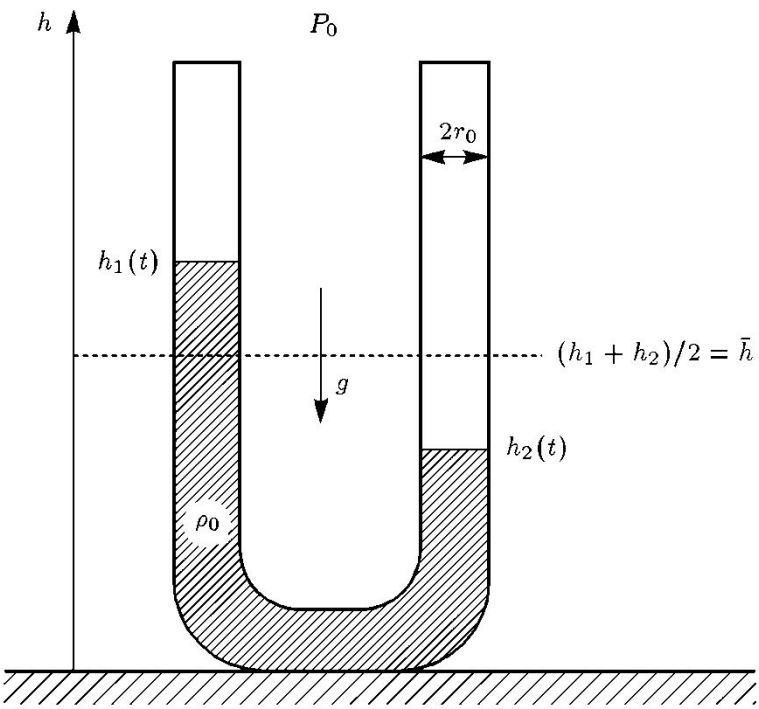
$$dE/dt \equiv 0 \quad \longrightarrow \quad m \frac{dr}{dt} \frac{d^2r}{dt^2} + k \frac{dr}{dt} r = \frac{dr}{dt} \left(m \frac{d^2r}{dt^2} + kr \right) = 0$$

$$F = ma = m \frac{d^2r}{dt^2}$$

$$F = -kr,$$

$$m \frac{d^2r}{dt^2} = -kr \quad \longrightarrow \quad r = A \sin \omega t + B \cos \omega t$$

$$\omega = \sqrt{k/m}$$



$$E_{\text{п}} = - \int_{\bar{h}}^{h_2} P dh_2 = - \int_{\bar{h}}^{h_2} \rho_0 s_0 (h_1 - h) g dh$$

$$\text{де } \bar{h} = \frac{h_1 + h_2}{2} \quad s_0 = \pi r_0^2$$

$$h_1(t) + h_2(t) = \text{const} \rightarrow E_{\text{п}} = -\rho_0 s_0 g (-h_2^2(t) + Ch_2(t) + C_1)$$

$$E_{\text{п}} = -\rho_0 s_0 g (-h_2^2(t) + Ch_2(t) + C_1)$$

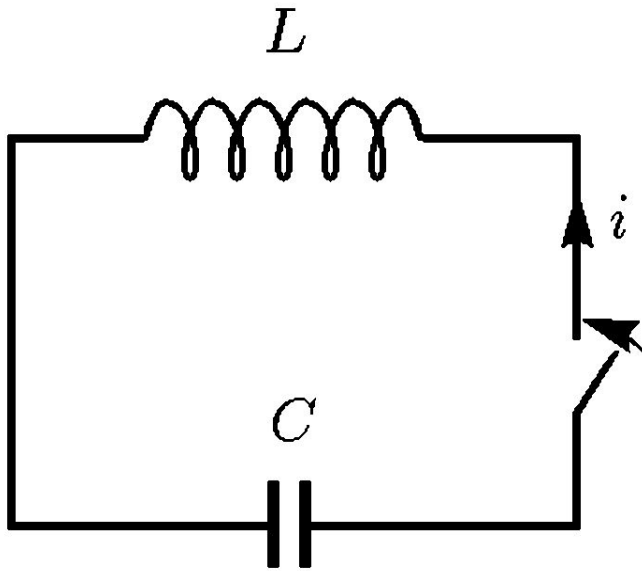
$$E_{\text{к}} = \frac{1}{2} M_0 \left(\frac{dh_2}{dt} \right)^2$$

$$E(t) = E_{\text{к}}(t) + E_{\text{п}}(t) = \frac{M_0}{2} \left(\frac{dh_2}{dt} \right)^2 - \rho_0 s_0 g (-h_2^2 + Ch_2 + C_1).$$

$$dE/dt = 0 \quad \rightarrow \quad M_0 \frac{d^2 h_2}{dt^2} = \rho_0 s_0 g (-2h_2 + C)$$

$$M_0 \frac{d^2 h}{dt^2} = -\rho_0 s_0 g h = -\pi \rho_0 r_0^2 g h \quad \text{де } h = (h_2 - h_1)/2$$

$$v(t) = -\varepsilon(t) \quad \text{де} \quad v(t) = q(t) C \quad \varepsilon = -L \ddot{i}/dt.$$



$$q(t) C = -\varepsilon(t) = L di/dt.$$

закон Ома

$$i = -dq/dt$$



$$L \frac{d^2 q}{dt^2} = -Cq$$

Динаміка системи “хижак-здобич”

$$\frac{dN}{dt} = (\alpha_1 - \beta_1 M) N. \quad \alpha_1 > 0, \beta_1 > 0$$

$$\frac{dM}{dt} = (-\alpha_2 + \beta_2 N) M. \quad \alpha_2 > 0, \beta_2 > 0$$

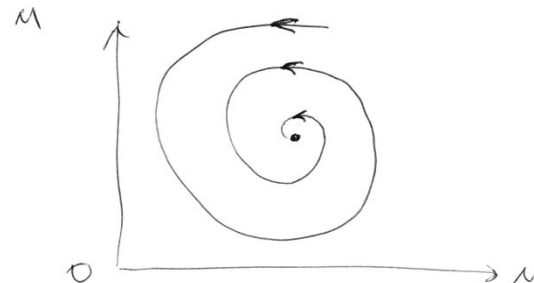
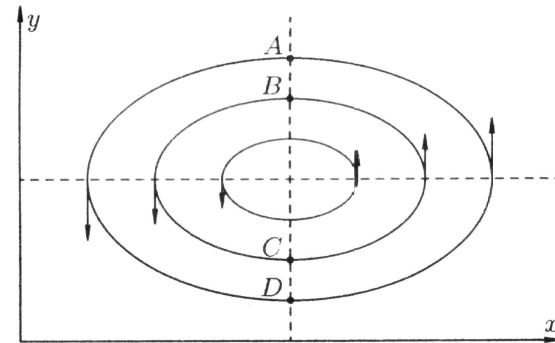
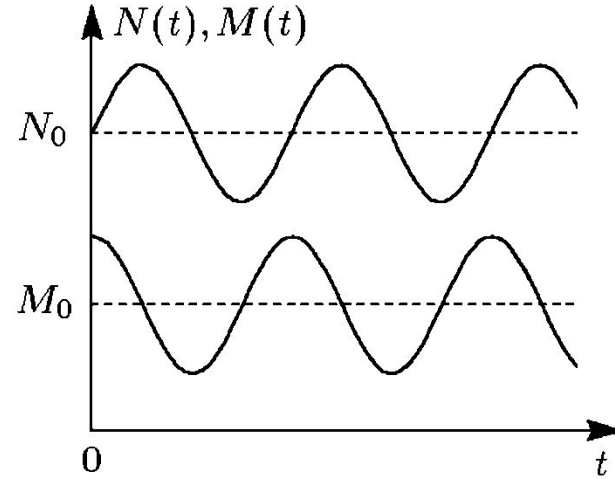
$$N_0 = \alpha_2 / \beta_2 \quad M_0 = \alpha_1 / \beta_1 \quad \text{р\iвновага}$$

$$N = N_0 + n, \quad M = M_0 + m, \quad n \ll N_0, \quad m \ll M_0$$

$$\frac{dn}{dt} = -\beta_1 N_0 m,$$

$$\frac{dm}{dt} = -\beta_2 M_0 n.$$

$$\frac{d^2 n}{dt^2} = -\alpha_1 \alpha_2 n. \quad \omega = \sqrt{\alpha_1 \alpha_2}$$



Динаміка зайнятості

$p(t)$ $p_0 > 0$ заробітна плата

$N(t)$ $N_0 > 0$ чисельність зайнятих

$$\left. \begin{aligned} \frac{dp}{dt} &= -\alpha_1 (N - N_0), & \alpha_1 > 0. \\ \frac{dN}{dt} &= \alpha_2 (p - p_0), & \alpha_2 > 0 \end{aligned} \right\} \begin{array}{l} \text{число робітників (зарплата) змінюється} \\ \text{пропорційно змінам зарплати (числу} \\ \text{робітників) відносно рівноважного} \\ \text{значення } N_0 \text{ (} p_0 \text{)} \end{array}$$

$$\frac{d^2(p - p_0)}{dt^2} = -\alpha_1 \alpha_2 (p - p_0)$$

$$\alpha_1 (N - N_0)^2 + \alpha_2 (p - p_0)^2 = \text{const} > 0$$

$$p = p_0 \longrightarrow N > N_0$$

$$N = N_0 \longrightarrow p > p_0$$

Принцип ієрархії моделей

Закон збереження імпульсу

$$m(t) v(t) = m(t + dt) v(t + dt) - dm [v(t + \xi dt) - u].$$

імпульс ракети

імпульс, що переданий газом

$$m(t + dt) = m(t) + (dm/dt) dt + O(dt^2) \longrightarrow m \frac{dv}{dt} = - \frac{dm}{dt} u.$$

$$v(t) = v_0 + u \ln \left(\frac{m_0}{m(t)} \right) \xrightarrow{\text{max}} v = u \ln \left(\frac{m_0}{m_p + m_s} \right)$$

m_0 початкова маса

m_p корисна маса

m_s структурна маса

$\lambda = \frac{m_s}{m_0 - m_p}$ характеризує відношення структурної і початкової мас ракети

формула
Ціолковського

навіть за умов $m_p = 0$ \longrightarrow $v = u \ln(1/\lambda) = 7 \text{ км/с}$
 $\lambda = 0,1, u = 3 \text{ км/с}$

Ієрархія моделей

λm_i | структурна маса ракети

$(1 - \lambda) \tilde{m}_i$ | маса палива

$m_0 = m_p + m_1 + m_2 + m_3$ початкова маса ракети при $n = 3$

(1)
$$v_1 = u \ln \left(\frac{m_0}{m_p + \lambda m_1 + m_2 + m_3} \right)$$

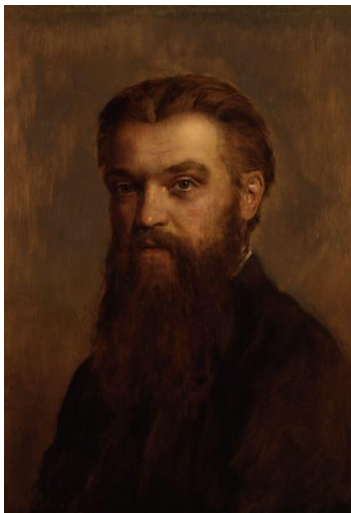
(2)
$$v_2 = v_1 + u \ln \left(\frac{m_p + m_2 + m_3}{m_p + \lambda m_2 + m_3} \right)$$

(3)
$$v_3 = v_2 + u \ln \left(\frac{m_p + m_3}{m_p + \lambda m_3} \right)$$

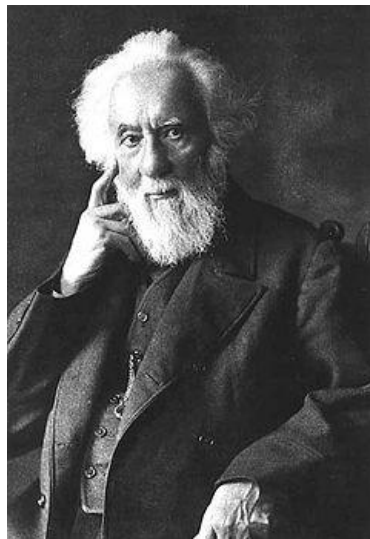
$$\frac{v_3}{u} = \ln \left\{ \left(\frac{m_0}{m_p + \lambda m_1 + m_2 + m_3} \right) \left(\frac{m_p + m_2 + m_3}{m_p + \lambda m_2 + m_3} \right) \left(\frac{m_p + m_3}{m_p + \lambda m_3} \right) \right\}$$

$\left. \begin{array}{l} v_n = 10,5 \text{ км/с} \\ \lambda = 0,1 \end{array} \right\} n = 2, 3, 4 \rightarrow m_0 = 149 m_p, m_0 = 77 m_p, m_0 = 65 m_p$

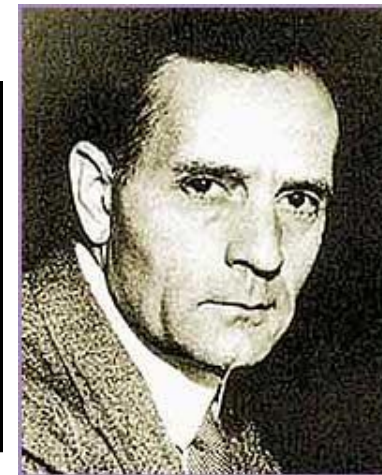
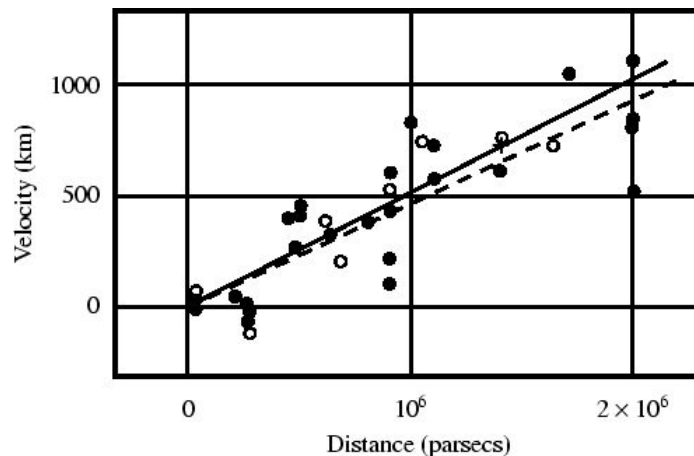
Ієрархія моделей (модель Всесвіту)



William Clifford
1845-1879



William Huggins
1824-1910



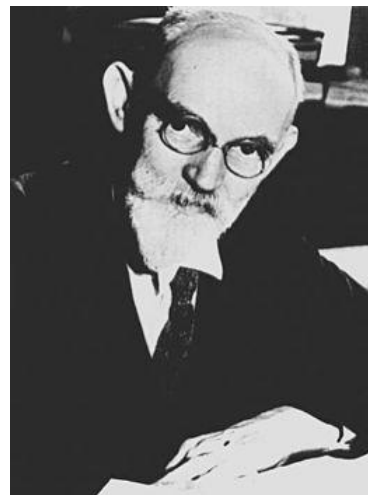
Edwin Hubble
1889-1953



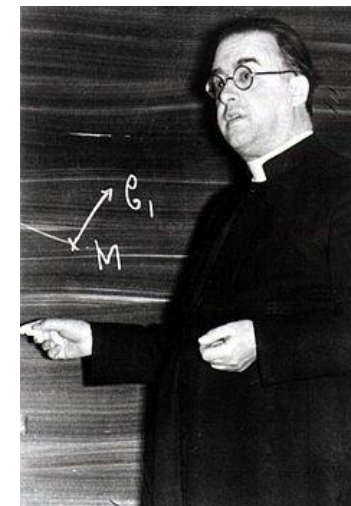
Vesto Slipher
1875-1969



Arthur Eddington
1882-1944



Willem de Sitter
1872-1934



Georges Lemaître
1894-1966

Ієрархія моделей (модель Всесвіту)



А. А. Фридман 1916

$$v = Hr \quad (a) \quad H \approx 50 \div 100 \text{ êì } / (\tilde{n} \cdot \dot{I} \tilde{n})$$

$$H = 72 \pm 8 \text{ êì } / (\tilde{n} \cdot \dot{I} \tilde{n})$$

$$\mathbf{M} \mathbf{R} \Delta m \longrightarrow v_R = HR$$

$$E_K = \frac{\Delta m v_R^2}{2} \quad E_P = -\gamma \frac{\mathbf{M} \Delta m}{2}$$

$$\Delta m \left(\frac{v_R^2}{2} - \gamma \frac{\mathbf{M}}{\mathbf{R}} \right) = const$$

$$\left. \begin{array}{l} const > 0 \\ const < 0 \end{array} \right\} \left. \begin{array}{l} v_R^2 = 2\gamma \frac{\mathbf{M}}{\mathbf{R}} + k \\ \mathbf{R} = -\frac{2\gamma \mathbf{M}}{k} \end{array} \right\} \left. \begin{array}{l} k = const / \Delta m \\ v_R^2 - 2\gamma \frac{\mathbf{M}}{\mathbf{R}} = 0 \end{array} \right\} (6)$$

$$\mathbf{M} = \frac{4}{3} \pi \mathbf{R}^3 \rho \longrightarrow \mathbf{R}^2 \left(H^2 - \frac{8\pi}{3} \gamma \rho \right) = 0 \quad \text{або} \quad \rho = \frac{3H^2}{8\pi\gamma}$$

$$\left. \begin{array}{l} v_R = \frac{dR}{dt} = \sqrt{2\gamma \frac{\mathbf{M}}{\mathbf{R}}} \\ \sqrt{\mathbf{R}} dR = \sqrt{2\gamma \mathbf{M}} dt \\ \frac{dR}{dt} = HR \end{array} \right\} \frac{2}{3} \mathbf{R}^{\frac{3}{2}} = \sqrt{2\gamma \mathbf{M}} t \longrightarrow t = \frac{2}{3H} \approx 10^{10}$$

Популяційна динаміка

$$u_1 = D_1 \Delta u_1 + f_1(u_1(t), u_2(t - \tau_1), \dots, u_m(t), u_m(t - \tau_m))$$

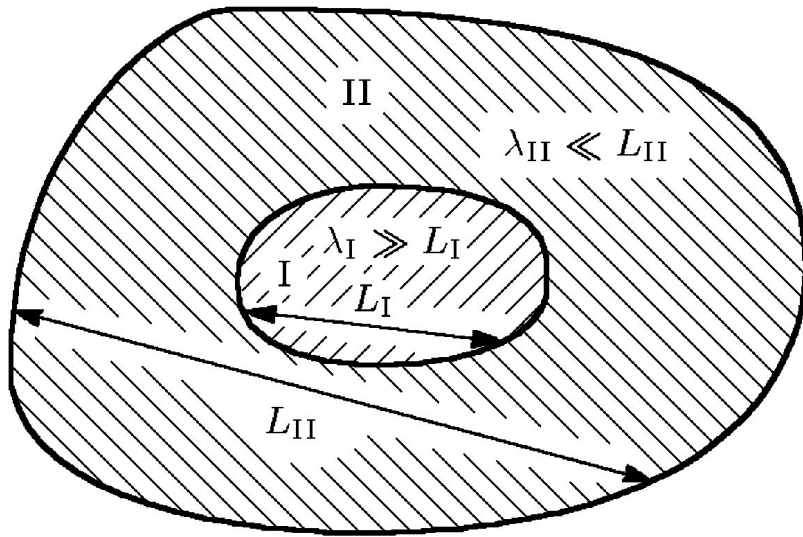
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$$u_m = D_m \Delta u_m + f_m(u_1(t), u_2(t - \tau_1), \dots, u_m(t), u_m(t - \tau_m))$$

$$u_k(\overset{\sqcup}{r}, 0) = h_k(\overset{\sqcup}{r})$$

$$u_k(\overset{\sqcup}{r}, t) = v_k(t) \quad k = 1, \dots, m$$

експонентне зростання



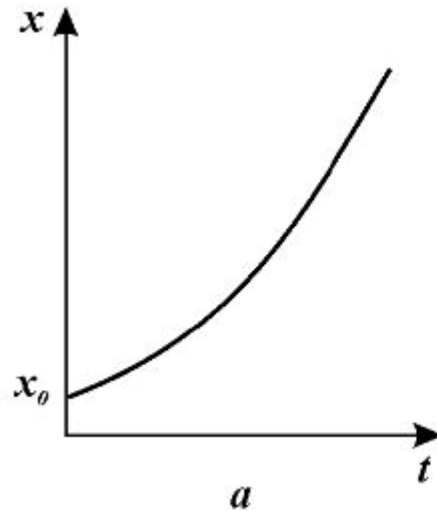
$$M_I(0) + M_{II}(0) = M_I(t) + M_{II}(t)$$

$$\frac{dN_I(t)}{dt} = -\alpha N_I(t) \longrightarrow \frac{dM_I(t)}{dt} = -\alpha M_I(t) \longrightarrow M_I(t) = M_I(0) e^{-\alpha t}$$

$$M_{II}(t) = M_{II}(0) + M_I(0) - M_I(0) e^{-\alpha t} = M_{II}(0) + M_I(0) (1 - e^{-\alpha t})$$

експонентне зростання

$$\frac{dx}{dt} = rx.$$

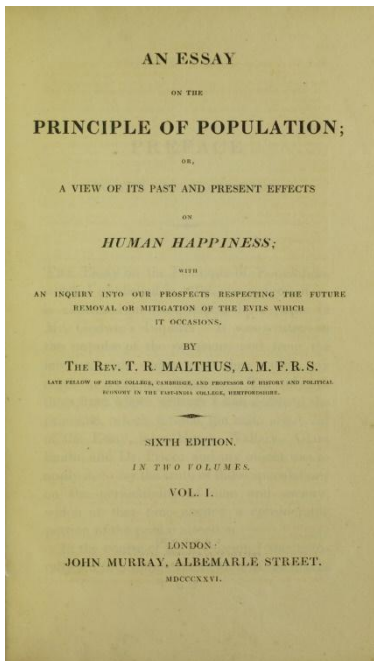


Augustin Louis Cauchy
1789-1857

$$y' = f(x, y)$$

$$P(x, y)dx + Q(x, y)dy = 0$$

$$x = \varphi(t) \quad y = \psi(t)$$



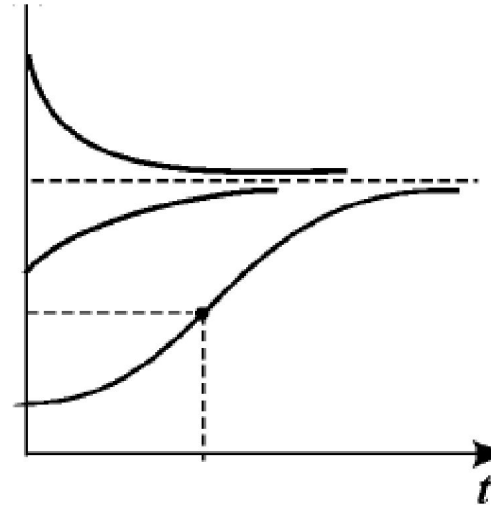
Томас Роберт Мальтус
1766-1834



Pierre Francois Verhulst

1804-1849

обмежене зростання

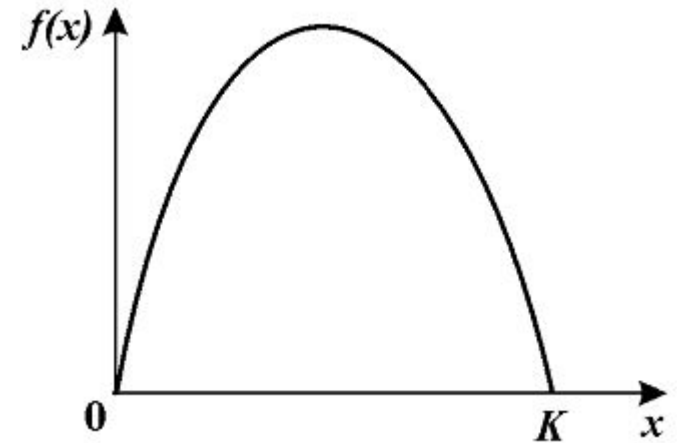
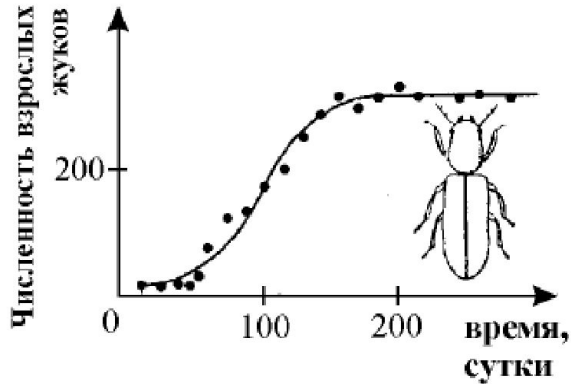


$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right)$$

$$x(t) = \frac{x_0 K e^{rt}}{K - x_0 + x_0 e^{rt}}$$

$$f(\bar{x}) = 0 \quad \bar{x}_1 = 0, \quad \bar{x}_2 = K$$

$$a = f'(x) \Big|_{x=\bar{x}} = r - \frac{2r\bar{x}}{K}$$



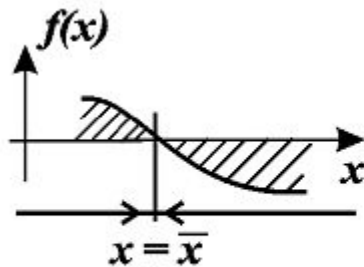
Дослідження стійкості стаціонарного стану

$$\frac{dx}{dt} = f(x). \rightarrow \frac{d(\bar{x} + \xi)}{dt} = f(\bar{x} + \xi). \quad \text{з урахуванням} \quad \left. \frac{dx}{dt} \right|_{x=\bar{x}} = 0 \quad \text{та} \quad \xi/\bar{x} \ll 1$$

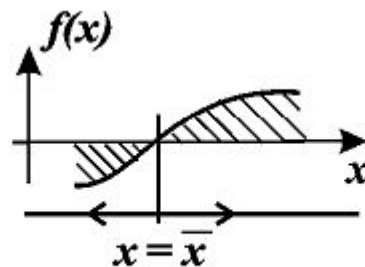
$$\frac{d\xi}{dt} = f(\bar{x}) + f'(\bar{x})\xi + \frac{1}{2}f''(\bar{x})\xi^2 + \dots \quad \text{або}$$

$$\frac{d\xi}{dt} = a_1\xi + a_2\xi^2 + \dots, \quad \text{де} \quad a_1 = f'(\bar{x}), \quad a_2 = \frac{1}{2}f''(\bar{x}), \quad \dots$$

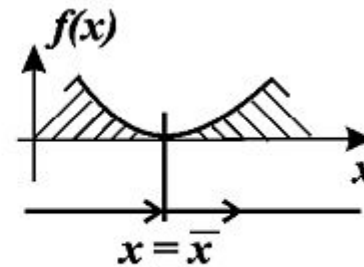
$$d\xi/dt = a_1\xi, \rightarrow \xi(t) = c \cdot \exp(\lambda t), \quad \text{де} \quad \lambda = a_1 = f'(\bar{x}), \quad c - \text{довільна стала}$$



a

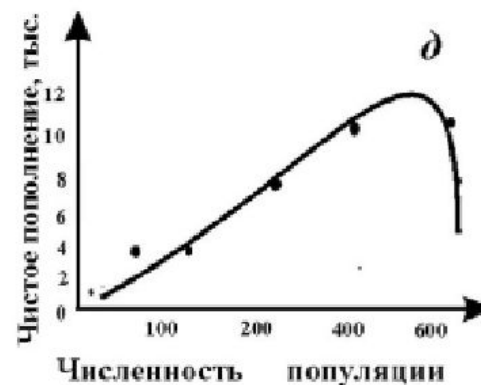
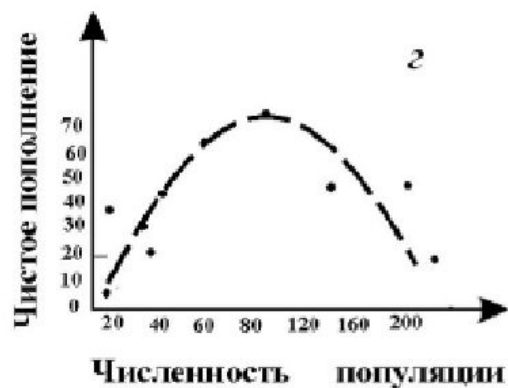
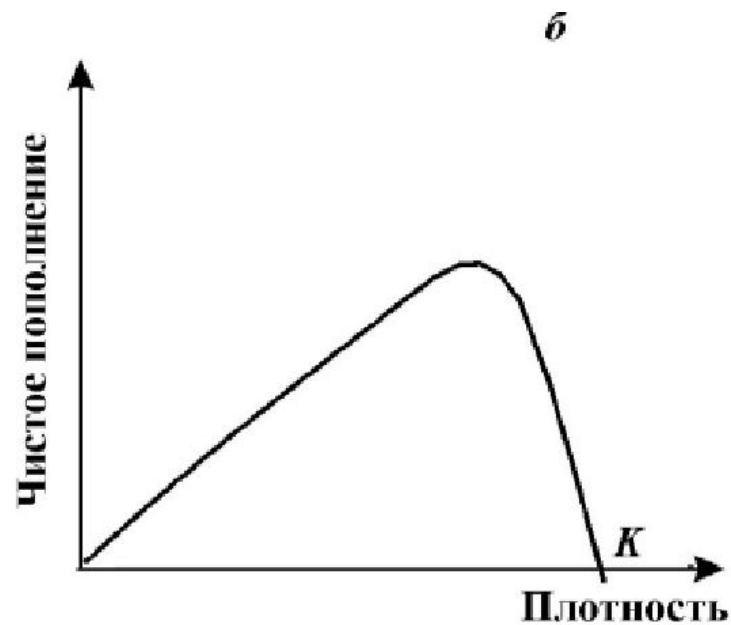
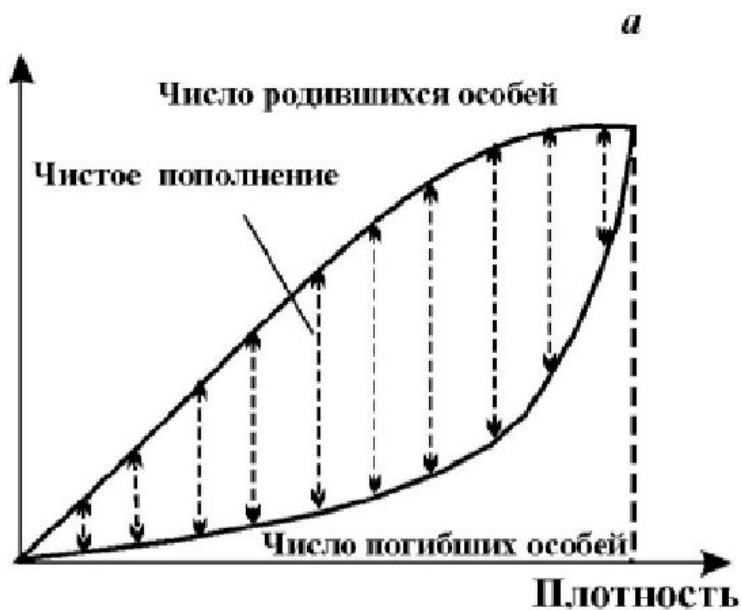


б



в

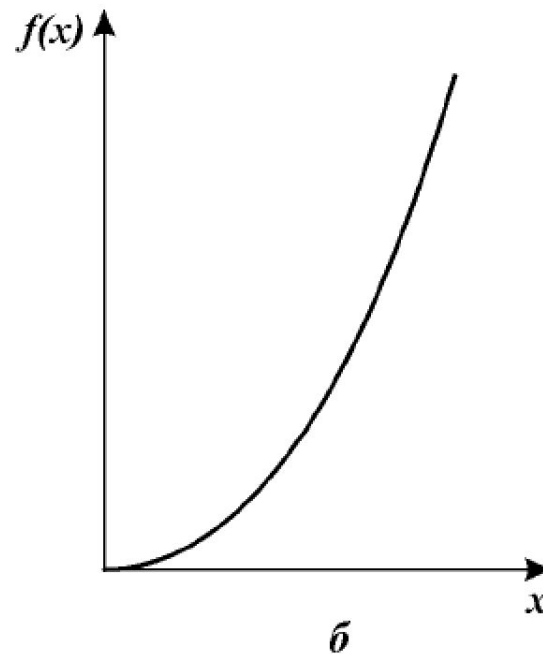
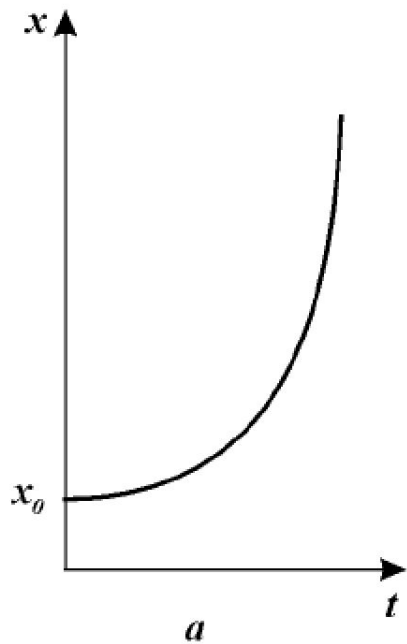
ПОПОВНЕННЯ



моделі з найменшою критичною чисельністю

$$\frac{dx}{dt} = rx^2$$

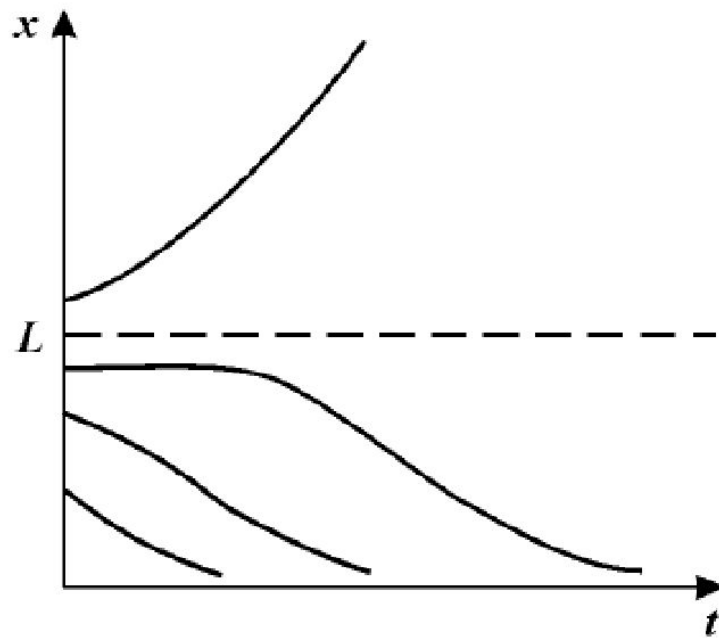
$$\frac{dx}{dt} = a \frac{\beta x^2}{\beta + \tau x}$$



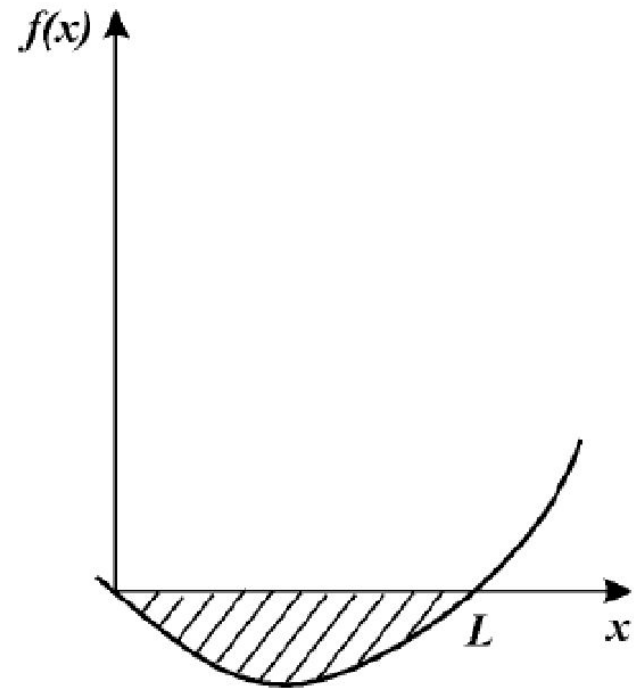
моделі з найменшою критичною чисельністю

$$\frac{dx}{dt} = a \frac{\beta x^2}{\beta + \tau x} - dx$$

$$\bar{x} = 0 \quad \bar{x} = d\beta(\alpha\beta - d\tau) = L$$



a

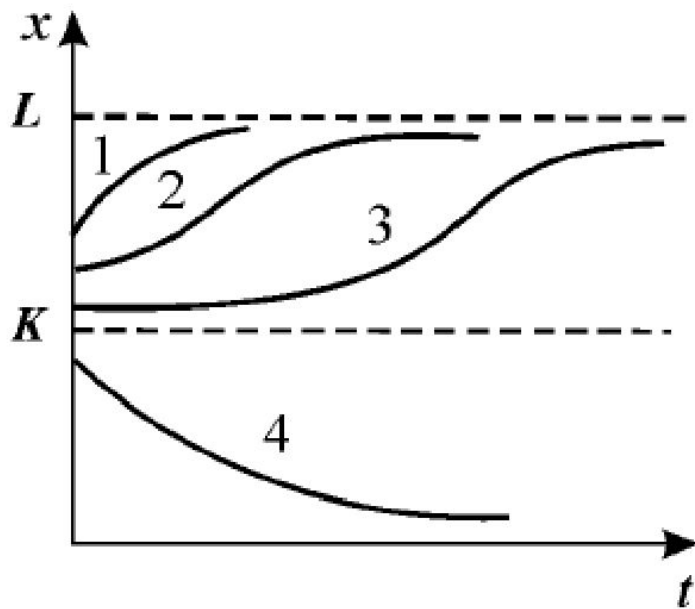


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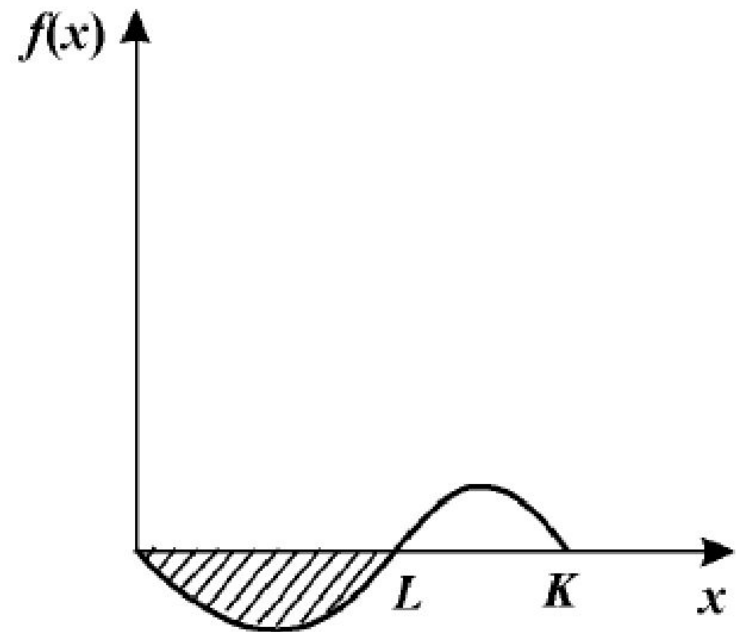
моделі з найменшою критичною чисельністю

$$\frac{dx}{dt} = a \frac{\beta x^2}{\beta + \tau x} - dx - \delta x^2$$

$$\bar{x} = 0 \quad \bar{x} = K \quad \bar{x} = L$$



a



б