

Integration by parts

- This is the name given to the process of integration which uses the differentiation product rule backwards

Integration by parts

- Reminder Product rule is ;

derivative of $uv = u \frac{dv}{dx} + v \frac{du}{dx}$

It follows then that $\int \left(u \frac{dv}{dx} + v \frac{du}{dx} \right) = uv + c$

Which can be written $\int u \frac{dv}{dx} + \int v \frac{du}{dx} = uv + c$

So $\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$

Integration by parts

$$\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$$

If we can use substitutions to express our integral as:

$$\int v \frac{du}{dx}$$

... we can perform more complex integration of stuff like by letting v equal the least complex function

$$\int x \cos 3x dx \quad \text{and} \quad \int x e^{4x} dx$$

Integration by parts

- Example $\int x \cos 3x dx$

- Let $v=x$ and $\frac{dv}{dx} = \cos 3x$

The
Formula:

$$\int v \frac{dv}{dx} = uv - \int u \frac{dv}{dx} + c$$

We need to know u and $\frac{dv}{dx}$

$$u = \frac{1}{3} \sin 3x \quad \text{and} \quad \frac{dv}{dx} = 1$$

Integration by parts

$$\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$$

$$\frac{du}{dx} = \cos 3x \quad u = \frac{1}{3} \sin 3x \quad v = x \quad \frac{dv}{dx} = 1$$

Substitute in

$$\int x \cos 3x = \frac{1}{3} x \sin 3x - \int \frac{1}{3} \sin 3x dx$$

Leads to

$$\int x \cos 3x = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + c$$

Integration by parts

$$\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$$

• Find $\int x e^{4x} dx$ by integration by parts.

• Use $v = x$ and $\frac{du}{dx} = e^{4x}$ that gives $u = \frac{1}{4} e^{4x}$ $\frac{dv}{dx} = 1$

$$\int x e^{4x} dx = \frac{1}{4} x e^{4x} - \int \frac{1}{4} e^{4x}$$

$$\int x e^{4x} dx = \frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + c$$

Why not “by substitution” ?

$$\int x e^{4x} dx$$

If $u = e^{4x}$ then $du/dx = 4 e^{4x}$

then $dx = du/(4 e^{4x})$

We get

$$\int x u \frac{du}{4 e^{4x}} = \frac{1}{4} \int \frac{x u}{e^{4x}} du$$

$$= \frac{1}{4} \int \frac{x u}{u} du = \frac{1}{4} \int x du \quad \text{GETTING NOWHERE}$$

Integration by parts of $\ln x$

- Example $\int \ln x dx$
- $\ln x$ is difficult to integrate so consider the function as $1 \ln x$ and use by parts.
- Let $v = \ln x$ and $\frac{du}{dx} = 1$
- So $\frac{dv}{dx} = \frac{1}{x}$ and $u = x$

Integration by parts of $\ln x$

$$\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$$

$$v = \ln x \quad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{x} \quad u = x$$

Substitute in

$$\int \ln x = x \ln x - \int x \frac{1}{x} dx$$

Leads to

$$\int \ln x = x \ln x - x + c$$

Integration by parts of arcsinx

- The same method for integrating $\ln x$ can be used to integrate arcsinx.
- So $v = \arcsin x$ and $\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$
- Therefore $\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $u = x$
- Substitute in:

$$\int \arcsin x = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Integration by parts of arcsinx

- Now use substitution to integrate $\int \frac{x}{\sqrt{1-x^2}} dx$
- Let $m = 1 - x^2$ so $dm = -2x dx$
- So

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{x}{\sqrt{m}} \frac{dm}{-2x} = -\frac{1}{2} \int \frac{1}{\sqrt{m}} dm \\ &= -\sqrt{m} + c = -\sqrt{1-x^2} + c\end{aligned}$$

Integration by parts of arcsinx

- So

$$\int \arcsin x = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

- Becomes

$$\int \arcsin x = x \arcsin x - -\sqrt{1-x^2} + c$$

$$= x \arcsin x + \sqrt{1-x^2} + c$$

Integration by parts of $e^x \cos x$

- This involves some algebraic manipulation since the second integral does not resolve into an easily integratable function.
- Let $v = e^x$ and $\frac{dv}{dx} = e^x$ and $u = \sin x$
- Therefore $\frac{du}{dx} = \cos x$ and $u = \sin x$
- So: $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$ [1]

Integration by parts of $e^x \cos x$

- Integrating by parts again for $e^x \sin x$ we get:

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

- Rearranging:

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx \quad [2]$$

Integration by parts of $e^x \cos x$

- Adding equations [1] and [2] we get:

$$2 \int e^x \cos x dx =$$

$$e^x \sin x + e^x \cos x - \int e^x \sin x + \int e^x \sin x dx$$

- Hence

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + c$$