Integration by parts

- This is the name given to the process of integration which uses the differentiation product rule backwards

Integration by parts

- Reminder Product rule is ;

derivative of $$uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

It follows then that $$\int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) = uv + c$$

Which can be written $$\int u\frac{dv}{dx} + \int v\frac{du}{dx} = uv + c$$

So $$\int v\frac{du}{dx} = uv - \int u\frac{dv}{dx} + c$$

Integration by parts

$$\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c$$

If we can use substitutions to express our integral as:

$$\int v \frac{du}{dx}$$

… we can perform more complex integration of stuff like by letting v equal the least complex function

$$\int x \cos 3x \, dx \quad \text{and} \quad \int x e^{4x} dx$$

Integration by parts

- Example $\int x \cos 3x \, dx$
- Let v=x and $\frac{du}{dx} = \cos 3x$

The Formula: $\boxed{\int v \frac{du}{dx} = uv - \int u \frac{dv}{dx} + c}$

We need to know u and $^{dv}/_{dx}$

$u = \frac{1}{3} \sin 3x$ and $\frac{dv}{dx} = 1$

Integration by parts

$$\boxed{\int v\frac{du}{dx} = uv - \int u\frac{dv}{dx} + c}$$

$$\frac{du}{dx} = \cos 3x \quad u = \frac{1}{3}\sin 3x \quad \mathrm{v = x} \quad \frac{dv}{dx} = 1$$

Substitute in

$$\int x\cos 3x = \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,dx$$

Leads to

$$\int x\cos 3x = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c$$

Integration by parts

$$\boxed{\int v\frac{du}{dx} = uv - \int u\frac{dv}{dx} + c}$$

- Find $\int xe^{4x}dx$ by integration by parts.
- Use v = x and $\frac{du}{dx} = e^{4x}$ that gives

$$u = \frac{1}{4}e^{4x} \quad \frac{dv}{dx} = 1$$

$$\int xe^{4x}dx = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}$$

$$\int xe^{4x}dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c$$

Why not “by substitution” ?

$$\int xe^{4x}dx$$

If $u = e^{4x}$ then $du/dx = 4\ e^{4x}$

then $dx = du/(4\ e^{4x})$

We get $\displaystyle\int xu\ \frac{du}{4e^{4x}} = \frac{1}{4}\int \frac{xu}{e^{4x}}du$

$$= \frac{1}{4}\int \frac{xu}{u}du = \frac{1}{4}\int xdu$$

GETTING NOWHERE

Integration by parts of lnx

- Example $\int \ln x \, dx$

- lnx is difficult to integrate so consider the function as 1lnx and use by parts.

- Let v = lnx and $\frac{du}{dx} = 1$

- So $\frac{dv}{dx} = \frac{1}{x}$ and u = x

Integration by parts of lnx

$$\boxed{\int v\frac{du}{dx} = uv - \int u\frac{dv}{dx} + c}$$

v = lnx $\quad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{x} \quad$ u = x

Substitute in

$$\int \ln x = x\ln x - \int x\frac{1}{x}dx$$

Leads to

$$\int \ln x = x\ln x - x + c$$

Integration by parts of arcsinx

- The same method for integrating lnx can be used to integrate arcsinx.
- So v = arcsin x and $\frac{du}{dx} = 1$
- Therefore $\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$ and u = x
- Substitute in:

$$\int \arcsin x = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Integration by parts of arcsinx

- Now use substitution to integrate $\int \frac{x}{\sqrt{1-x^2}}dx$
- Let m = 1 – x^2 so dm = -2xdx
- So

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{m}}\frac{dm}{-2x} = \frac{-1}{2}\int \frac{1}{\sqrt{m}}dm$$

$$= -\sqrt{m} + c = -\sqrt{1-x^2} + c$$

Integration by parts of arcsinx

- So

$$\int \arcsin x = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

- Becomes

$$\int \arcsin x = x \arcsin x - -\sqrt{1-x^2} + c$$

$$= x \arcsin x + \sqrt{1-x^2} + c$$

Integration by parts of $e^x\cos x$

- This involves some algebraic manipulation since the second integral does not resolve into an easily integratable function.
- Let $v = e^x$ and $\frac{du}{dx} = \cos x$
- Therefore $\frac{dv}{dx} = e^x$ and $u = \sin x$
- So: $$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx \quad [1]$$

Integration by parts of $e^x\cos x$

- Integrating by parts again for $e^x\sin x$ we get:

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

- Rearranging:

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx \quad [2]$$

Integration by parts of $e^x\cos x$

- Adding equations [1] and [2] we get:

$$2\int e^x \cos x dx =$$

$$e^x \sin x + e^x \cos x - \int e^x \sin x + \int e^x \sin x dx$$

- Hence

$$\int e^x \cos x dx = \frac{1}{2} e^x \left(\sin x + \cos x\right) + c$$