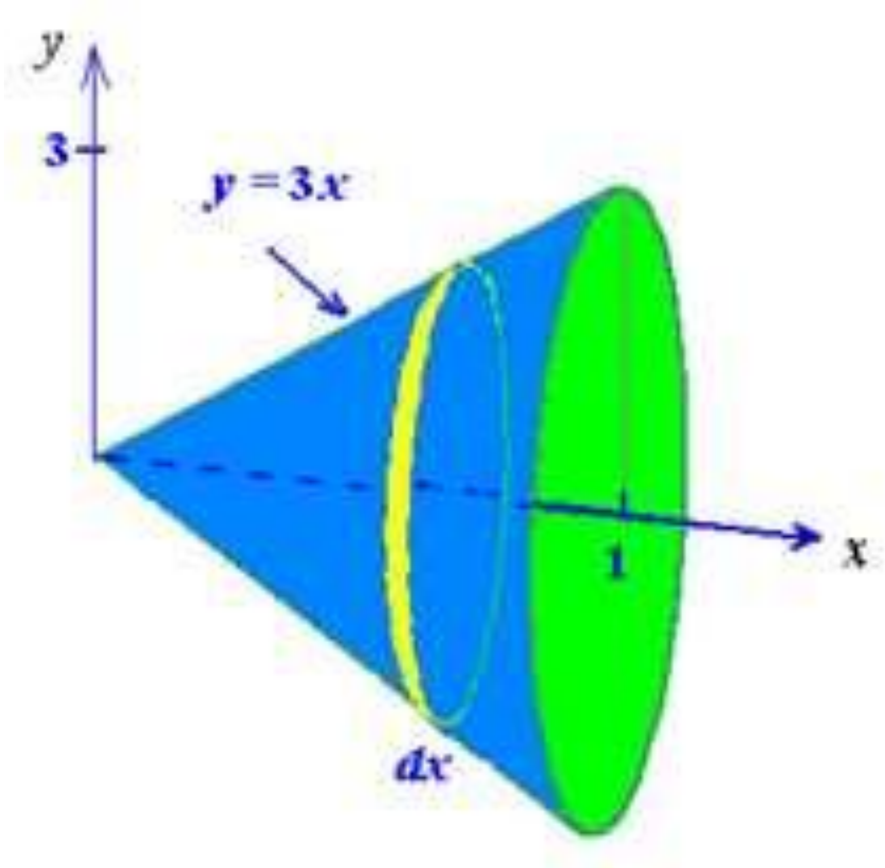
Volumes of Revolution

Consider the line y=3x

- Now rotate the line 360° about the x axis

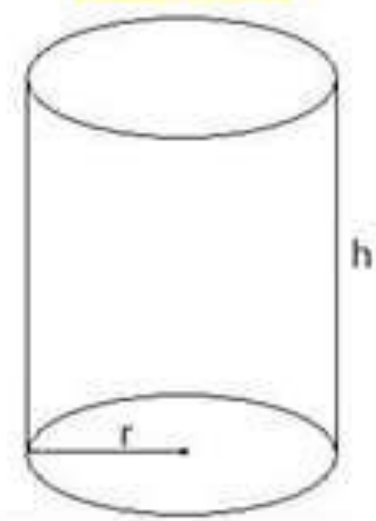


As you can see the result is a solid cone. The volume of the cone can be thought of as a series of discs or cylinders.

Volume of a cylinder

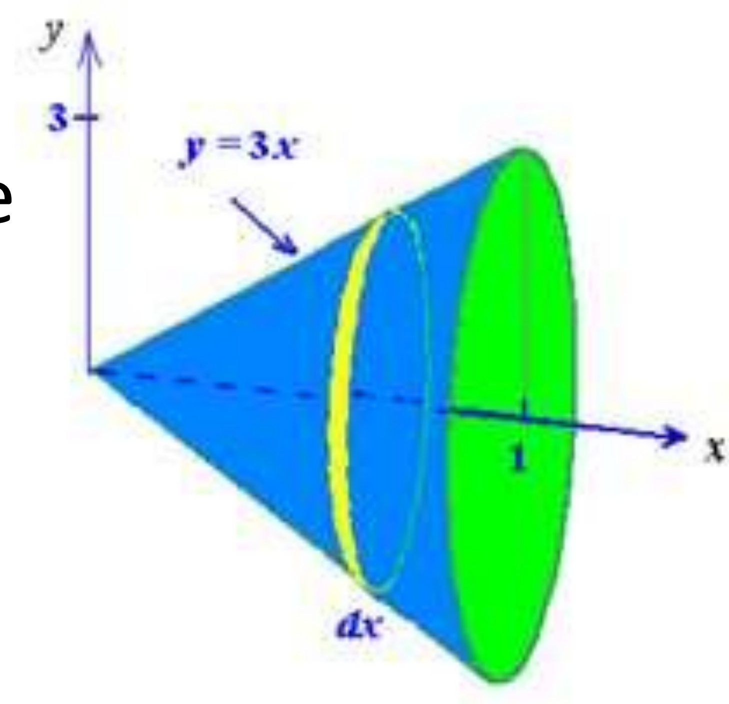
- The volume of a cylinder is the area of the circular cross-section multiplied by the height.
- The area of a circle is πr^2
- The height is h
- So $V = \pi r^2 h$
- We can think of a disc

as a very thin cylinder



Consider a small disc in the cone

- The volume of the disc is the area of the circular cross-section multiplied by the height (or length in our case).
- Now the radius of the circular cross-section is y and the length of the disc is δx.
- So $V=\pi y^2 \delta x$



Integration as a process of summation

- We have seen that integration is a process of summation that adds a series of very small strips to give an area.
- This process can also be used to add a series of very small discs.
- As with areas, as $\delta x \to 0$, the limit of the sum gives the result that the volume of revolution about the x axis is given by

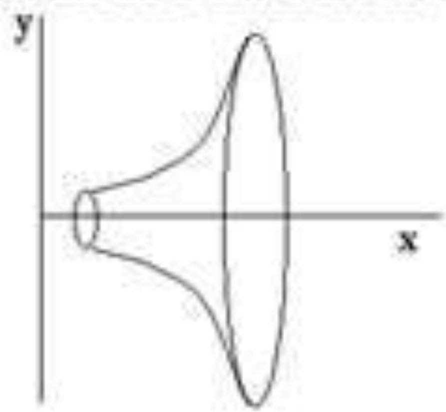
$$\int \pi y^2 dx = \pi \int y^2 dx$$

Volumes of revolution example

- As with area, vertical boundaries can be added in the form of limits:

$$\text{volume} = \int_a^b \pi y^2 \, dx$$

Example:
The graph of $y = x^2$ between $x = 1$ and $x = 3$ is rotated completely around the x-axis. Find the volume generated.



$$\text{volume} = \int_1^3 \pi x^4 \, dx$$

$$= \left[\frac{\pi x^5}{5} \right]_1^3$$

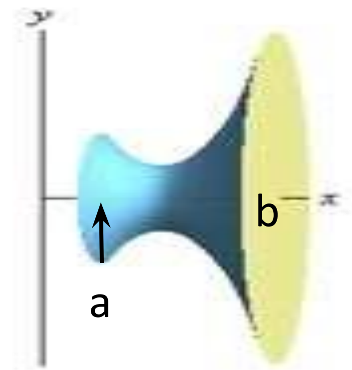
$$= \frac{243\pi}{5} - \frac{\pi}{5}$$

$$= \underline{\underline{48.4\pi}}$$

General formula

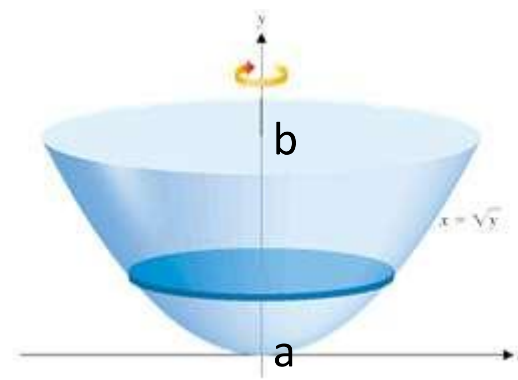
- So the Volume of Revolution between the limits of x=a and x=b about the x axis can be found with

$$\pi \int_a^b y^2 dx$$



- Similarly, the Volume of Revolution between the limits of y=a and y=b about the y axis can be found with

$$\pi \int_a^b x^2 dy$$



Example 1

- Find the volume generated when the area defined by the following inequality is rotated completely about the x axis:

$$0 \le y \le x(4 - x)$$

Example 1 Solution

- So our four boundary equations are:

- x-axis $y = 0$
- curve $y = x(x-4)$
- x-axis intercept $x = 0$
- x-axis intercept $x = 4$

Example 1 Solution

- Hence the volume of revolution is:

$$\pi\int_0^4 x^2(x-4)^2\,dx = \pi\int_0^4 x^4 - 8x^3 + 16x^2\,dx$$

$$= \pi\left[\frac{x^5}{5} - 2x^4 + \frac{16x^3}{3}\right]_0^4$$

$$= \pi\left[\frac{4^5}{5} - 2(4)^4 + \frac{16(4)^3}{3}\right] = \frac{512}{15}\pi$$

Example 2

- Find the volume generated when the area defined by the following inequalities is rotated completely about the y axis:

$$y \geq x^2 + 1,$$

$$x \geq 0,$$

$$y \leq 2$$

Example 2 Solution

- So our four boundary equations are:

- y-axis $x = 0$
- curve $y = x^2 + 1$
- y-axis intercept $y = 1$
- line $y = 2$

- Since we are integrating wrt y we need the equation of the curve in terms of x^2.

Example 2 Solution

- So, $\quad If \qquad y = x^2 + 1$

  $\qquad Then \quad x^2 = y - 1$

- Hence the volume of revolution is:

$$\pi \int_1^2 y - 1 dy == \pi \left[\frac{y^2}{2} - y \right]_1^2$$

$$= \pi \left[\left(\frac{2^2}{2} - 2 \right) - \left(\frac{1^2}{2} - 1 \right) \right] = \frac{1}{2} \pi$$