Reduction Method of Integration

Used to find

$$\int (f(x))^n dx$$

where n is large

Establish a reduction formula for $\int \cos^n x dx$

• Let
$$I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos^1 x dx$$

• Use by parts: $v = \cos^{n-1} x \Rightarrow \frac{dv}{dx} = -(n-1)\cos^{n-2} x \sin x$ $\frac{du}{dx} = \cos x \Rightarrow u = \sin x$

So,

$$I_n = \sin x \cos^{n-1} x - \int -(n-1)\cos^{n-2} x \sin x \sin x dx$$

Establish a reduction formula for $\int \cos^n x dx$

$$I_n = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

Establish a reduction formula for $\int \cos^n x dx$

So,

$$I_n + (n-1)I_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}$$

$$nI_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}$$

Hence,

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

Use a reduction formula to find $\int \cos^5 x dx$

Now,

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

So,

$$I_5 = \frac{1}{5}\sin x \cos^{5-1} x + \frac{(5-1)}{5}I_{5-2}$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}I_3$$

Use a reduction formula to find $\int \cos^5 x dx$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}I_3$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5} \left[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3}I_1 \right]$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5} \left[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3} \int \cos^1 x dx \right]$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5} \left| \frac{1}{3}\sin x \cos^2 x + \frac{2}{3}\sin x \right| + C$$

Establish a reduction formula for $\int \sin^n x dx$

• The same method can be applied to:

$$I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin^1 x dx$$

Hence,

$$I_n = \frac{(n-1)}{n} I_{n-2} - \frac{1}{n} \cos x \sin^{n-1} x$$

Establish a reduction formula for $\int \tan^n x dx$

• Let
$$I_n = \int \tan^n x dx = \int \tan^2 x \tan^{n-2} x dx$$

• Using $\tan^2 x = \sec^2 x - 1$

$$I_n = \int (\sec^2 x - 1) \tan^{n-2} x dx$$

$$= \int \sec^2 x \tan^{n-2} x dx - \int \tan^{n-2} x dx$$

$$=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}$$

Since
$$\int \sec^2 x \tan^n x dx = \frac{1}{n+1} \tan^{n+1} x$$

Establish a reduction formula for $\int x^n e^x dx$

• Let
$$I_n = \int x^n e^x dx$$

• Use by parts: $v = x^n \Rightarrow \frac{dv}{dx} = nx^{n-1}$

$$\frac{du}{dx} = e^x \implies u = e^x$$

So,

$$I_n = x^n e^x - \int nx^{n-1} e^x dx$$

$$I_n = x^n e^x - n \int x^{n-1} e^x dx$$

Establish a reduction formula for $\int x^n e^x dx$

• Hence,
$$I_n = x^n e^x - nI_{n-1}$$

Using a reduction formula find $\int x^4 e^x dx$

$$\int x^4 e^x dx$$

• Using: $I_n = x^n e^x - nI_{n-1}$

$$I_4 = x^4 e^x - 4I_3$$

$$I_4 = x^4 e^x - 4 \left[x^3 e^x - 3I_2 \right]$$

$$I_4 = x^4 e^x - 4 [x^3 e^x - 3 [x^2 e^x - 2I_1]]$$

$$I_4 = x^4 e^x - 4 \left[x^3 e^x - 3 \left[x^2 e^x - 2 \left[x e^x - I_0 \right] \right] \right]$$

$$I_4 = x^4 e^x - 4 \left[x^3 e^x - 3 \left[x^2 e^x - 2 \left[x e^x - \int x^0 e^x dx \right] \right] \right]$$

Using a reduction formula find $\int x^4 e^x dx$

$$\int x^4 e^x dx$$

Hence,

$$I_4 = x^4 e^x - 4 \left[x^3 e^x - 3 \left[x^2 e^x - 2 \left[x e^x - e^x \right] \right] + C$$