Reduction Method of Integration

Used to find

$$\int (f(x))^n dx$$

where n is large

Establish a reduction formula for $\int \cos^n x dx$

- Let $I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos^1 x dx$

- Use by parts: $v = \cos^{n-1} x \Rightarrow \dfrac{dv}{dx} = -(n-1)\cos^{n-2} x \sin x$

$$\frac{du}{dx} = \cos x \Rightarrow u = \sin x$$

- So,

$$I_n = \sin x \cos^{n-1} x - \int -(n-1)\cos^{n-2} x \sin x \sin x dx$$

Establish a reduction formula for $\int \cos^n x dx$

$$I_n = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

Establish a reduction formula for $\int \cos^n x dx$

$$So,$$

$$I_n + (n-1)I_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}$$

$$nI_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}$$

$$Hence,$$

$$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

Use a reduction formula to find $\int \cos^5 x dx$

$Now,$

$$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

$So,$

$$I_5 = \frac{1}{5}\sin x \cos^{5-1} x + \frac{(5-1)}{5} I_{5-2}$$

$$I_5 = \frac{1}{5}\sin x \cos^{4} x + \frac{4}{5} I_3$$

Use a reduction formula to find $\int \cos^5 x dx$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5} I_3$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}\left[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3} I_1\right]$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}\left[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3}\int \cos^1 x dx\right]$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}\left[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3}\sin x\right] + C$$

Establish a reduction formula for $\int \sin^n x dx$

- The same method can be applied to:

$$I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin^1 x dx$$

- Hence,

$$I_n = \frac{(n-1)}{n} I_{n-2} - \frac{1}{n} \cos x \sin^{n-1} x$$

Establish a reduction formula for $\int \tan^n x dx$

- Let $I_n = \int \tan^n x dx = \int \tan^2 x \tan^{n-2} x dx$

- Using $\tan^2 x = \sec^2 x - 1$

$$I_n = \int (\sec^2 x - 1) \tan^{n-2} x dx$$

$$= \int \sec^2 x \tan^{n-2} x dx - \int \tan^{n-2} x dx$$

$$= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$$

Since

$$\int \sec^2 x \tan^n x dx = \frac{1}{n+1} \tan^{n+1} x$$

Establish a reduction formula for $\int x^n e^x dx$

- Let $I_n = \int x^n e^x dx$
- Use by parts: $v = x^n \Rightarrow \dfrac{dv}{dx} = nx^{n-1}$

$$\frac{du}{dx} = e^x \Rightarrow u = e^x$$

- So,

$$I_n = x^n e^x - \int nx^{n-1} e^x dx$$

$$I_n = x^n e^x - n\int x^{n-1} e^x dx$$

Establish a reduction formula for $\int x^n e^x dx$

- Hence,

$$I_n = x^n e^x - nI_{n-1}$$

Using a reduction formula find $\int x^4 e^x dx$

- Using: $I_n = x^n e^x - nI_{n-1}$

$$I_4 = x^4 e^x - 4I_3$$

$$I_4 = x^4 e^x - 4\left[x^3 e^x - 3I_2\right]$$

$$I_4 = x^4 e^x - 4\left[x^3 e^x - 3\left[x^2 e^x - 2I_1\right]\right]$$

$$I_4 = x^4 e^x - 4\left[x^3 e^x - 3\left[x^2 e^x - 2\left[xe^x - I_0\right]\right]\right]$$

$$I_4 = x^4 e^x - 4\left[x^3 e^x - 3\left[x^2 e^x - 2\left[xe^x - \int x^0 e^x dx\right]\right]\right]$$

Using a reduction formula find $\int x^4 e^x dx$

- Hence,

$$I_4 = x^4e^x - 4\left[x^3e^x - 3\left[x^2e^x - 2\left[xe^x - e^x\right]\right]\right] + C$$