Definite Integration by Reduction Methods

Consider $$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$Now,$

$$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

$So,$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx = \left[\frac{1}{n}\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$$

Consider $$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$I_n = \frac{1}{n}\sin\frac{\pi}{2}\cos^{n-1}\frac{\pi}{2} - \frac{1}{n}\sin 0\cos^{n-1}0 + \frac{n-1}{n}I_{n-2}$$

$$I_n = \frac{1}{n}\sin\frac{\pi}{2}(0) - \frac{1}{n}(0)\cos^{n-1}0 + \frac{n-1}{n}I_{n-2}$$

$$Hence,\ I_n = \frac{n-1}{n}I_{n-2}$$

Now consider $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$

- Using exactly the same method obtains exactly the same result.
- Hence,

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \sin^n x dx = I_n = \frac{n-1}{n} I_{n-2}$$

Using this reduction formula successively

- Repeatedly applying the formula, we find that:

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) I_{n-4}$$

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) I_{n-6}$$

$etc.$

Using this reduction formula successively

- The final term will depend on if n is even or odd.
- If n is even:

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)I_0$$

- Also:

$$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x dx = \frac{\pi}{2} \qquad I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x dx = \frac{\pi}{2}$$

- Hence in both cases, when n is even:

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2}$$

Using this reduction formula successively

- If n is odd:

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)I_1$$

- Also:

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^1 x dx = 1 \qquad I_1 = \int_0^{\frac{\pi}{2}} \sin^1 x dx = 1$$

- Hence in both cases, when n is odd:

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$$

Example

- Evaluate $\int_0^{\frac{\pi}{2}} \cos^8 x dx$

- n is even so,

$$I_8 = \left(\frac{8-1}{8}\right)\left(\frac{8-3}{8-2}\right)\left(\frac{8-5}{8-4}\right)\left(\frac{8-7}{8-6}\right)\frac{\pi}{2}$$

$$I_8 = \left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2}$$

- Hence,

$$I_8 = \frac{35\pi}{256}$$

Example

- Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x dx$
- n is odd so,

$$I_7 = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)$$

$$I_7 = \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$$

- Hence,

$$I_7 = \frac{16}{35}$$