Definite Integration by Reduction Methods



Now,

$$I_{n} = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

So,

$$I_{n} = \int_{0}^{\frac{\pi}{2}} \cos^{n} x \, dx = \left[\frac{1}{n} \sin x \cos^{n-1} x\right]_{0}^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$$

Consider
$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$I_n = \frac{1}{n} \sin \frac{\pi}{2} \cos^{n-1} \frac{\pi}{2} - \frac{1}{n} \sin 0 \cos^{n-1} 0 + \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{1}{n} \sin \frac{\pi}{2} (0) - \frac{1}{n} (0) \cos^{n-1} 0 + \frac{n-1}{n} I_{n-2}$$

Hence,
$$I_n = \frac{n-1}{n} I_{n-2}$$

Now consider
$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$$

- Using exactly the same method obtains exactly the same result.
- Hence,

$$\int_{0}^{\frac{\pi}{2}} \cos^{n} x dx = \int_{0}^{\frac{\pi}{2}} \sin^{n} x dx = I_{n} = \frac{n-1}{n} I_{n-2}$$

Using this reduction formula successively

• Repeatedly applying the formula, we find that:

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) I_{n-4}$$

$$I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \left(\frac{n-5}{n-4}\right) I_{n-6}$$

Using this reduction formula successively

- The final term will depend on if n is even or odd.
- If n is even: $I_{n} = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \dots \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) I_{0}$
- Also:

$$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x \, dx = \frac{\pi}{2} \qquad I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = \frac{\pi}{2}$$

Hence in both cases, when n is even:

$$I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \dots \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) \frac{\pi}{2}$$

Using this reduction formula successively

• If n is odd: $I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \dots \left(\frac{4}{5}\right) \left(\frac{2}{3}\right) I_1$ • Also:

$$I_1 = \int_{0}^{\frac{\pi}{2}} \cos^1 x \, dx = 1 \qquad I_1 = \int_{0}^{\frac{\pi}{2}} \sin^1 x \, dx = 1$$

• Hence in both cases, when n is odd:

$$I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \dots \left(\frac{4}{5}\right) \left(\frac{2}{3}\right)$$

Example

- Evaluate $\int_{0}^{\frac{\pi}{2}} \cos^8 x dx$
- n is even so,

$$I_{8} = \left(\frac{8-1}{8}\right) \left(\frac{8-3}{8-2}\right) \left(\frac{8-5}{8-4}\right) \left(\frac{8-7}{8-6}\right) \frac{\pi}{2}$$
$$I_{8} = \left(\frac{7}{8}\right) \left(\frac{5}{6}\right) \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) \frac{\pi}{2}$$

• Hence,

$$I_8 = \frac{35\pi}{256}$$

Example

- Evaluate $\int_{0}^{\frac{\pi}{2}} \sin^7 x dx$
- n is odd so,

$$I_{7} = \left(\frac{7-1}{7}\right) \left(\frac{7-3}{7-2}\right) \left(\frac{7-5}{7-4}\right)$$
$$I_{7} = \left(\frac{6}{7}\right) \left(\frac{4}{5}\right) \left(\frac{2}{3}\right)$$

• Hence,

$$I_7 = \frac{16}{35}$$