Area Bounded Partly By A Curve

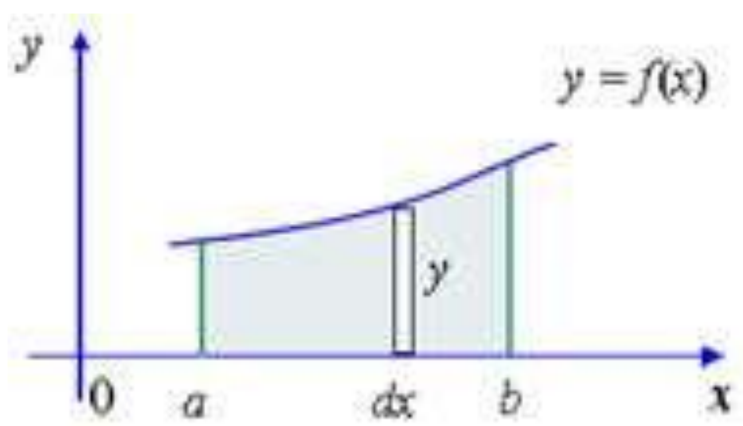
Considering areas under the x-axis
and areas bound by the y-axis

Recap

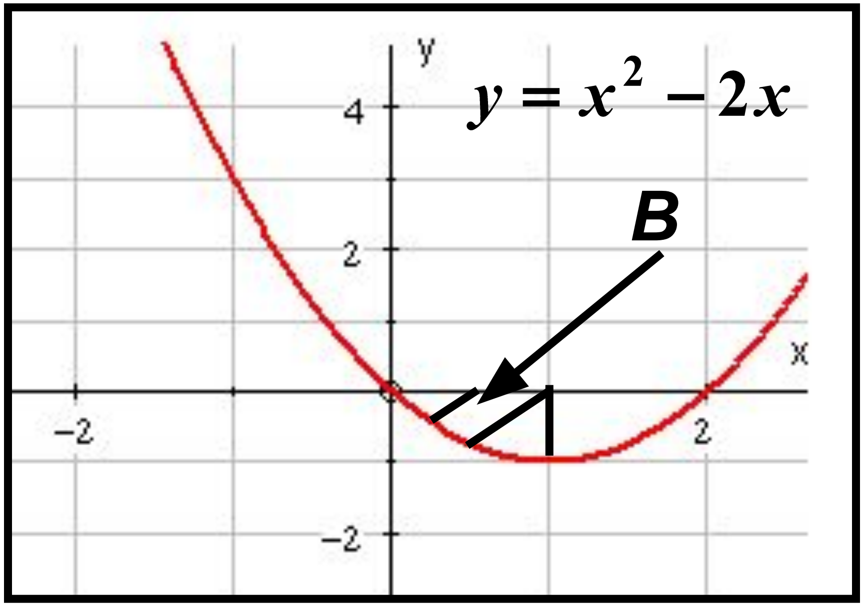
- We already know that evaluating the integral found by:

$$\int_a^b y\,dx$$

- Gives the area bound by x=a, x=b, y=f(x) and the x-axis, e.g.



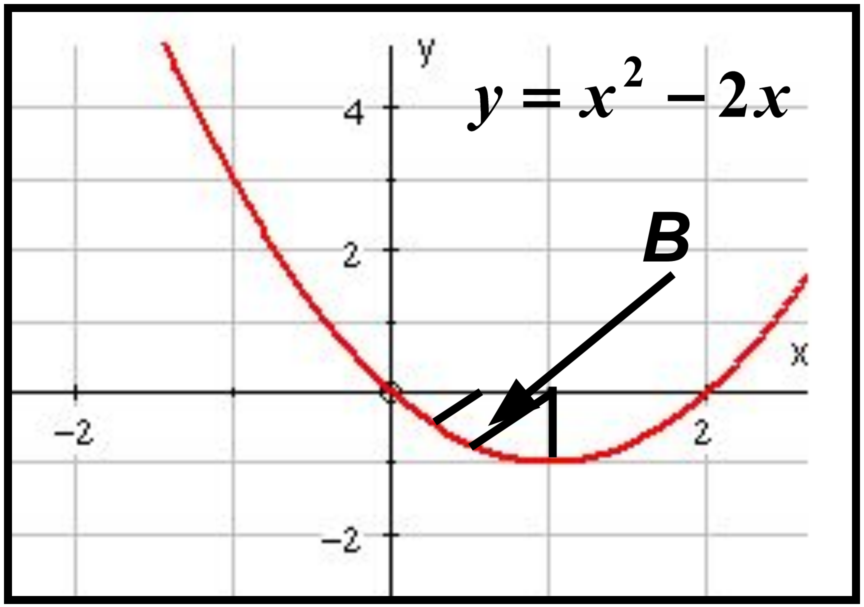
Areas below the x-axis



For parts of the curve below the *x*-axis, the definite integral is negative, so

$$\text{area } B = -\int_0^1 x^2 - 2x \; dx$$

Areas below the x-axis



$$B = -\int_0^1 x^2 - 2x \; dx$$

$$= -\left[\frac{x^3}{3} - x^2\right]_0^1$$

$$= -\left(\left[\frac{1}{3} - 1\right] - \left[\quad 0 \quad\right]\right)$$

$$= - -\frac{2}{3}$$

$\Rightarrow$ Area B $= \dfrac{\mathbf{2}}{\mathbf{3}}$

Areas below the x-axis

- An area is always positive.
- The definite integral is positive for areas above the *x*-axis but negative for areas below the axis.
- To find an area, we need to know whether the curve crosses the *x*-axis between the boundaries.
  - For areas above the axis, the definite integral gives the area.
  - For areas below the axis, we need to change the sign of the definite integral to find the area.

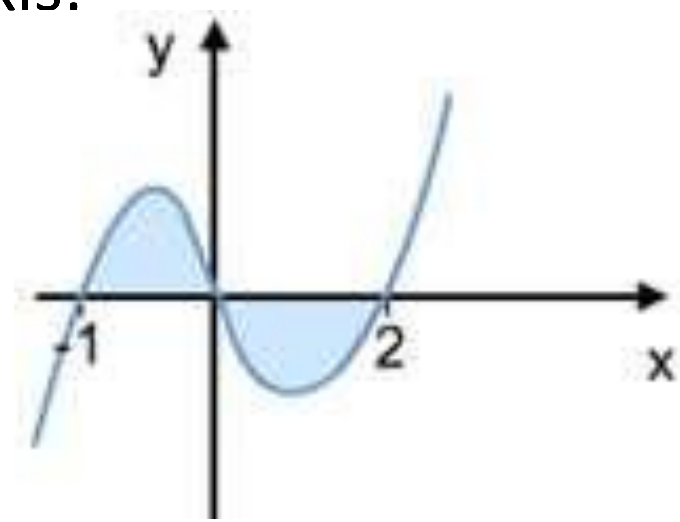
Example

- Evaluate: $$\int_{-1}^{2} x(x+1)(x-2)\,dx$$
- We can see that the function has 3 roots at:
  - x = -1, x = 0 and x = 2
  - So the function does cross the x-axis between the limits at x = 0.
  - Therefore we consider the question in two parts:

$$\int_{-1}^{0} x(x+1)(x-2)\,dx \qquad \mathit{and} \qquad \int_{0}^{2} x(x+1)(x-2)\,dx$$

Example

- Our knowledge of a positive cubic function tells us that the section between the limits -1 and 0 is above the x-axis and the section between the limits 0 and 2 is below the x-axis:



- So the first integral is positive but the second will give a negative result and so should be multiplied by -1.

Example

- So the total area is given by:

$$\int_{-1}^{0} x(x+1)(x-2)dx + (-1)\int_{0}^{2} x(x+1)(x-2)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= \left[0 - \left(\frac{1}{4} - \frac{-1}{3} - 1\right)\right] - \left[\left(\frac{16}{4} - \frac{8}{3} - 4\right) - 0\right]$$

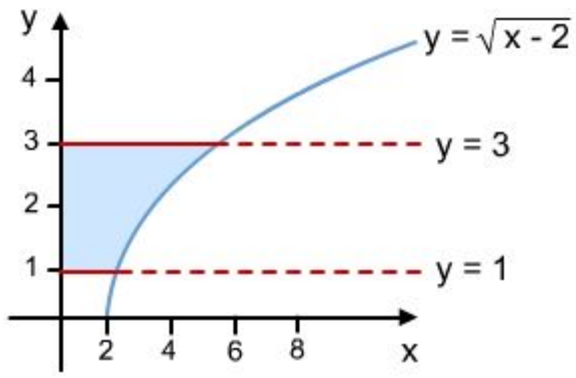
$$= \frac{5}{12} - \frac{-8}{3} = 3\frac{1}{12}$$

Areas bound with the y-axis

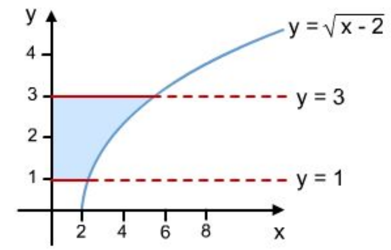
- We can also evaluate areas bound by y=a, y=b, x=f(y) and the y-axis by evaluating:

$$\int_a^b x\,dy$$

- An example would look like this:

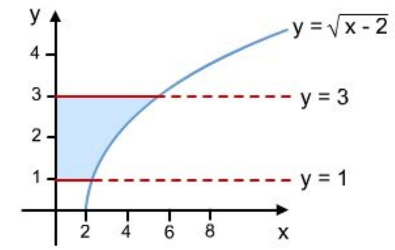


Example



- Find this area bound by y=1, y=3, y=√(x-2) and the y-axis.
- Firstly we need to make x the subject of the equation: $x = y^2 + 2$
- Now we have to evaluate: $\int_{1}^{3} y^2 + 2dy$

Example



$$\int_{1}^{3} y^2 + 2dy = \left[\frac{y^3}{3} + 2y\right]_1^3$$

$$= \left(\frac{3^3}{3} + 2\times 3\right) - \left(\frac{1^3}{3} + 2\times 1\right)$$

$$= 15 - \frac{7}{3} = \frac{38}{3}$$