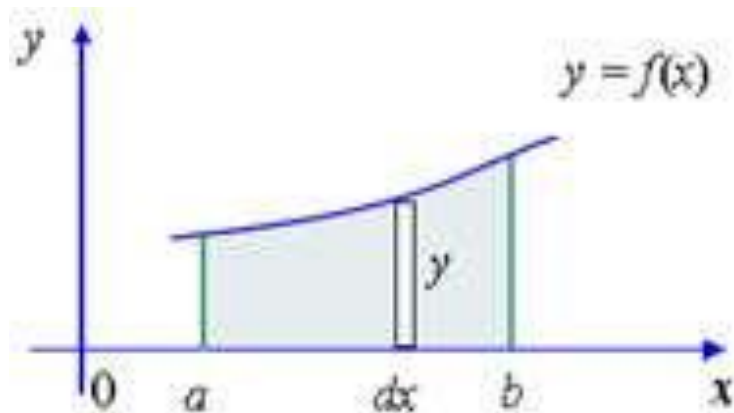


Area Bounded Partly By A Curve

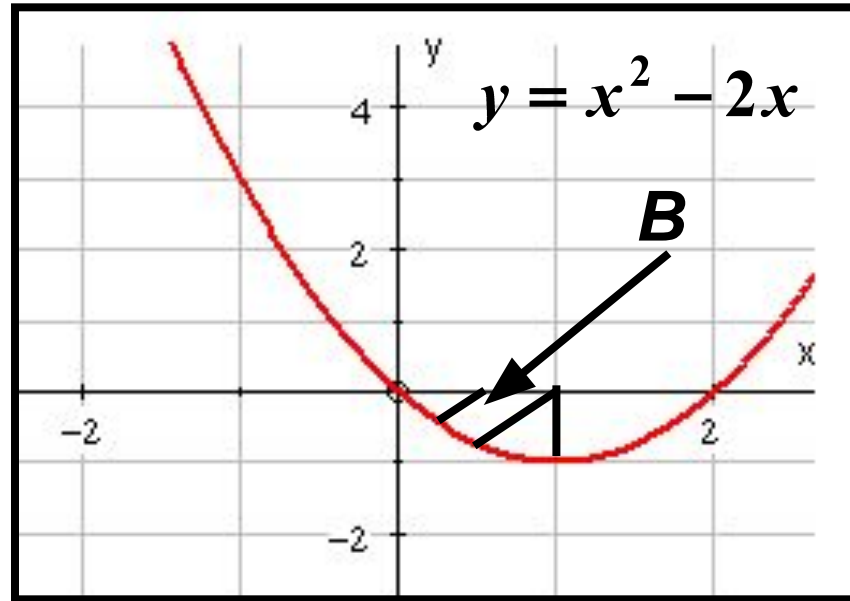
Considering areas under the x-axis
and areas bound by the y-axis

Recap

- We already know that evaluating the integral found by:
$$\int_a^b y dx$$
- Gives the area bound by $x=a$, $x=b$, $y=f(x)$ and the x -axis, e.g.



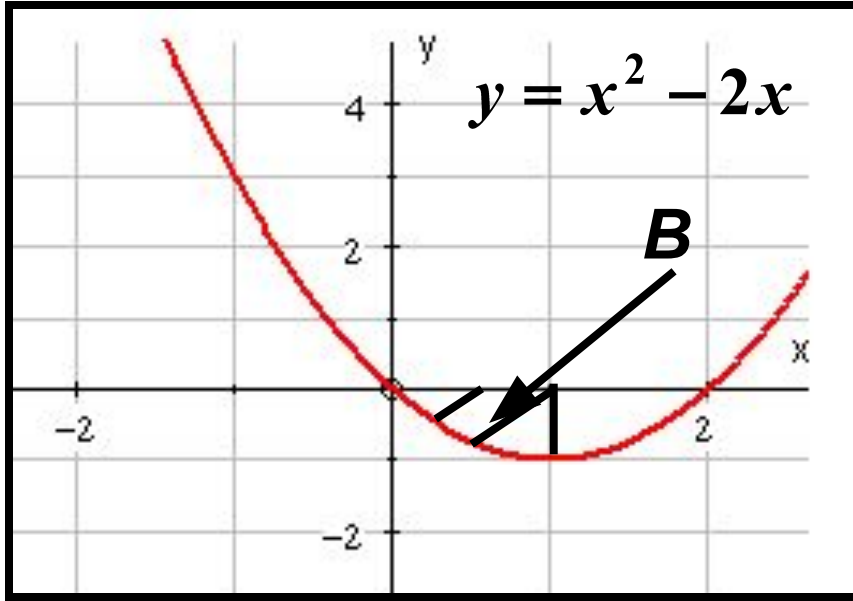
Areas below the x-axis



For parts of the curve below the x-axis, **the definite integral is negative**, so

$$\text{area } B = - \int_0^1 x^2 - 2x \, dx$$

Areas below the x-axis



$$\begin{aligned} B &= -\int_0^1 x^2 - 2x \, dx \\ &= -\left[\frac{x^3}{3} - x^2 \right]_0^1 \\ &= -\left(\left[\frac{1}{3} - 1 \right] - \left[0 \right] \right) \\ &= -\left(-\frac{2}{3} \right) \end{aligned}$$

$$\Rightarrow \text{Area } B = \frac{2}{3}$$

Areas below the x-axis

- An area is always positive.
- The definite integral is positive for areas above the x -axis but negative for areas below the axis.
- To find an area, we need to know whether the curve crosses the x -axis between the boundaries.
 - For areas above the axis, the definite integral gives the area.
 - For areas below the axis, we need to change the sign of the definite integral to find the area.

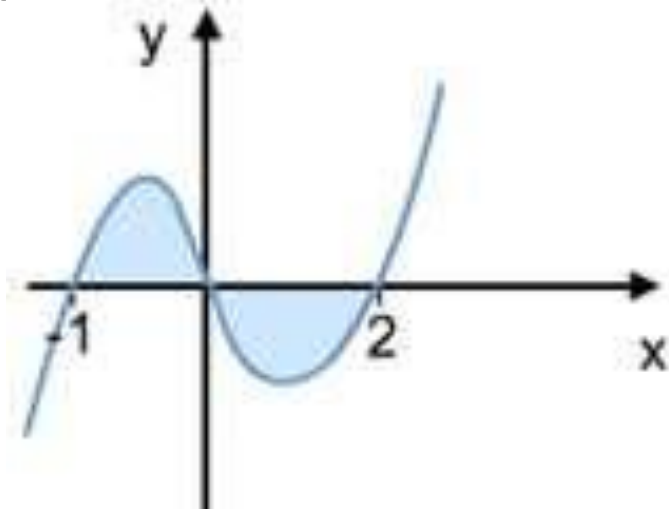
Example

- Evaluate: $\int_{-1}^2 x(x+1)(x-2)dx$
- We can see that the function has 3 roots at:
 - $x = -1$, $x = 0$ and $x = 2$
 - So the function does cross the x-axis between the limits at $x = 0$.
 - Therefore we consider the question in two parts:

$$\int_{-1}^0 x(x+1)(x-2)dx \quad \text{and} \quad \int_0^2 x(x+1)(x-2)dx$$

Example

- Our knowledge of a positive cubic function tells us that the section between the limits -1 and 0 is above the x -axis and the section between the limits 0 and 2 is below the x -axis:



- So the first integral is positive but the second will give a negative result and so should be multiplied by -1 .

Example

- So the total area is given by:

$$\int_{-1}^0 x(x+1)(x-2)dx + (-1) \int_0^2 x(x+1)(x-2)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 - \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} - \frac{-1}{3} - 1 \right) \right] - \left[\left(\frac{16}{4} - \frac{8}{3} - 4 \right) - 0 \right]$$

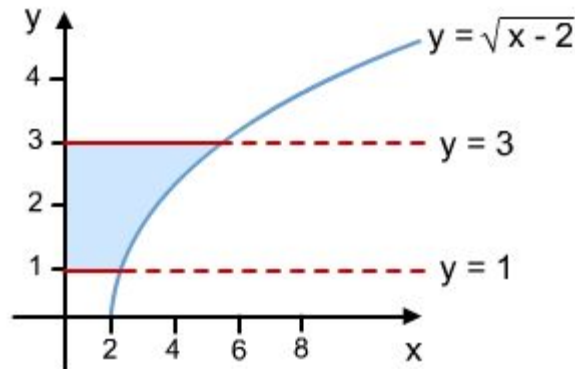
$$= \frac{5}{12} - \frac{-8}{3} = 3\frac{1}{12}$$

Areas bound with the y-axis

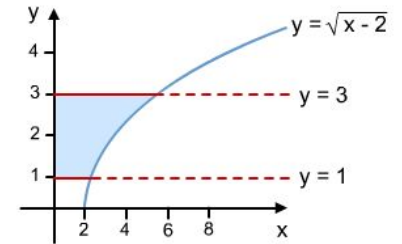
- We can also evaluate areas bound by $y=a$, $y=b$, $x=f(y)$ and the y-axis by evaluating:

$$\int_a^b x dy$$

- An example would look like this:

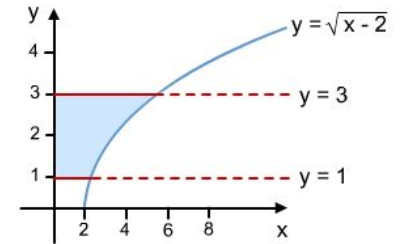


Example



- Find this area bound by $y=1$, $y=3$, $y=\sqrt{(x-2)}$ and the y -axis.
- Firstly we need to make x the subject of the equation: $x = y^2 + 2$
- Now we have to evaluate: $\int_1^3 y^2 + 2 dy$

Example



$$\begin{aligned} \int_1^3 y^2 + 2 dy &= \left[\frac{y^3}{3} + 2y \right]_1^3 \\ &= \left(\frac{3^3}{3} + 2 \times 3 \right) - \left(\frac{1^3}{3} + 2 \times 1 \right) \\ &= 15 - \frac{7}{3} = \frac{38}{3} \end{aligned}$$