

# Maxwell's equations

## PLAN OF LECTURE

- 1. Rotational electrical field.*
- 2. Displacement current.*
- 3. Maxwell's equations. Integral form.*

## ROTATIONAL ELECTRICAL FIELD

Variation of magnetic field in time causes the appearance of external forces in the electrical contour. These forces act on the electric charges – carriers of current. Maxwell supposed that *varying magnetic field gave a birth to electrical field*. In result the induced current appears even in unmovable contour. This current is caused by *rotational electric field*.

### Properties of rotational electrical field.

Definition of ElectroMotive Force (EMF). In the electrostatic field it was a *circulation of the vector of the field strength along the closed contour*:

$$\oint_L \left( \overset{\nabla}{E} d\overset{\nabla}{l} \right) = \mathbf{E}$$

## ROTATIONAL ELECTRICAL FIELD

Following to Maxwell idea, the variation of magnetic field in time gives birth to the rotation electrical field which is the reason of the EMF existence in the electric contour:

$$\mathcal{E}_i = - \frac{\partial \Phi_B}{\partial t} = \oint_L \left( \vec{E}_B, d\vec{l} \right) = \oint_L E_{Bl} dl$$

где  $E_{Bl}$  is a projection of vector  $\vec{E}_B$  on direction of local element of contour length  $dl$ .

The Flux of magnetic induction vector (or **MAGNETIC FLUX**) across the surface  $S$  restricted by some contour – that is magnitude

$$\Phi_B = \int_S \left( \vec{B}, d\vec{S} \right) = \int_S B_n dS.$$

This definition leads to expression:

$$\oint_L \left( \vec{E}_B, d\vec{l} \right) = - \frac{d}{dt} \int_S \left( \vec{B}, d\vec{S} \right)$$

## ROTATIONAL ELECTRICAL FIELD

$$\oint_L (\vec{E}_B \cdot d\vec{l}) = - \frac{d}{dt} \int_S (\vec{B} \cdot d\vec{S})$$

Replace of differentiation and integration operations leads to:

$$\oint_L (\vec{E}_B \cdot d\vec{l}) = - \frac{d}{dt} \int_S (\vec{B} \cdot d\vec{S}) = - \int_S \left( \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \right)$$

Use of partial derivatives means that in general the vector  $\vec{B}$  is a function both time and coordinates.

Let us make a short return to the Electrostatics:

In the electrostatic field the EMF in the closed contour is equal to zero. By other words, a circulation of the vector of electrostatic field strength  $\vec{E}_q$  along the closed contour is equal to zero:

$$\oint_L (\vec{E}_q \cdot d\vec{l}) = 0$$

## *ROTATIONAL ELECTRICAL FIELD*

$$\oint_L (\vec{E}_q \cdot d\vec{l}) = 0$$

Zero circulation of vector  $\vec{E}_q$  in Electrostatics means that force lines of electrostatic field can not be closed,

They have beginning on the positive charges and have the end on the negative charges or walk away up to infinity. It means that electrostatic field has a potential.

$$\oint_L (\vec{E}_q \cdot d\vec{l}) = 0$$

$$\oint_L (\vec{E}_B \cdot d\vec{l}) = - \int_S \left( \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \right)$$

Let us compare the expressions for circulation of the electrical field strength in electrostatic field  $\vec{E}_q$  and in rotational electrical field  $\vec{E}_B$ .

In the rotational electric field the circulation of the strength of field  $\vec{E}_B$  is not equal to zero.

Therefore the electrical field  $\vec{E}_B$  excited by the varying magnetic field is a **rotational** field likely to magnetic field. That is why we consider the force line of rotational electric field  $\vec{E}_B$  as closed lines.

## *ROTATIONAL ELECTRICAL FIELD*

Thus, it was shown above that the varying in time magnetic field yields to the appearance of rotational electrical field in the area of magnetic field existence.

In general case the electrical field can be created jointly by the quasi-stationary electric charges and by the variable magnetic field.

At this situation the electrical field can have the component  $E_q$  created by charges and the component  $E_B$  stipulated by the varying in time magnetic field.

## *DISPLACEMENT CURRENT*

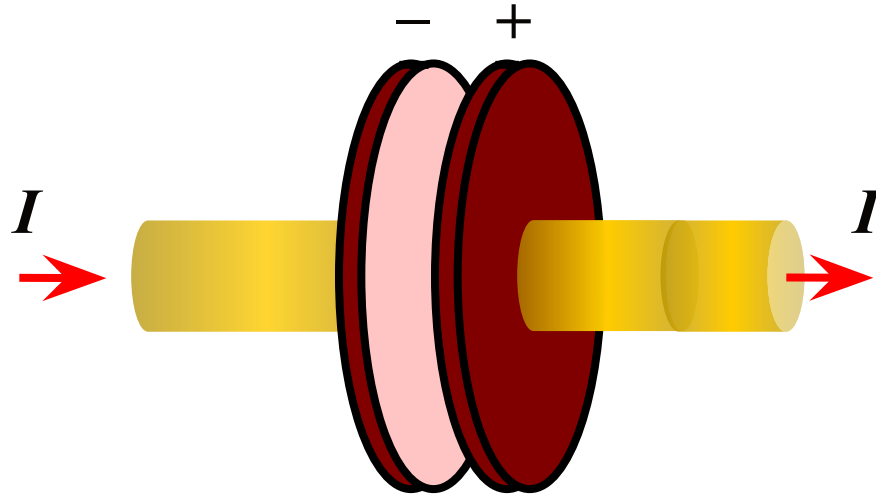
J. K. Maxwell created the completed theory of electrical and magnetic phenomena. The foundation of theory consists in the mutual dependence between electrical and magnetic fields. .

One of important suppositions of Maxwell says that if the varying magnetic field  $\partial B / \partial t$  creates the electrical field the variable electrical field  $\partial E / \partial t$  must create by similar way the magnetic field due to special component of current which does not need a presence of free charges for its passage. Maxwell introduced into physics this special kind of current under name “**DISPLACEMENT CURRENT**”.

Use of displacement current gave a possibility for Maxwell to get the correct quantitative correlation between varying electrical field and the current caused by this field jointly with corresponding magnetic field.

## DISPLACEMENT CURRENT

The necessity of displacement current to be taken into consideration can be explained on the passage of variable current across the capacitor.



*(The circulation of vector  $\vec{H}$  along the any closed contour is equal to sum of currents flowing across the square restricted by contour.)*

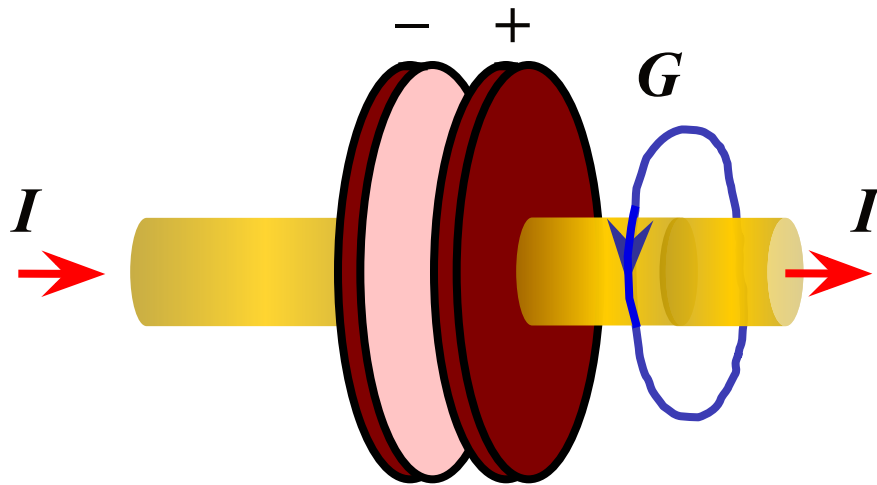
Let the capacitor was initially charged. At its connection to some external resistor the electrical current  $I$  will flow along the resistor and connectors.

We can apply the theorem about circulation of magnetic strength  $\vec{H}$  vector:

$$\oint_L (\vec{H} d\vec{l}) = \int_S (\vec{j} d\vec{S}) = I$$



## *DISPLACEMENT CURRENT*

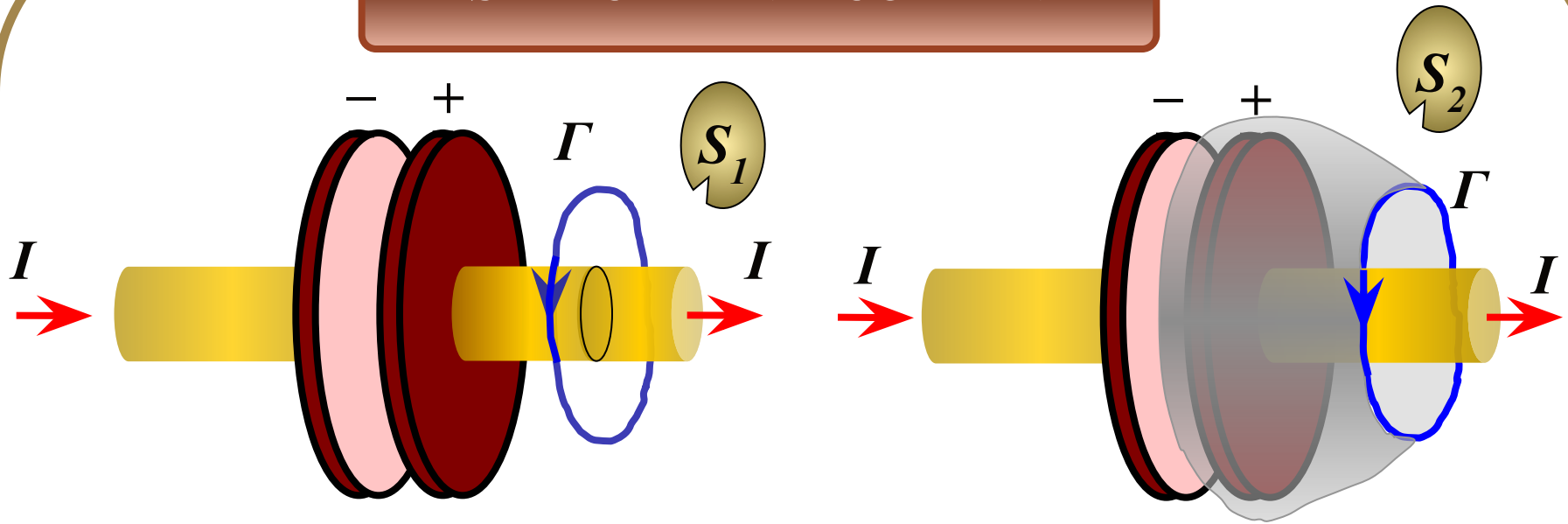


We draw the closed contour  $G$  around the current-carrying connector. Direction of current is shown by arrow..

For application of theorem about circulation of vector  $\vec{H}$  it is necessary to choose a surface restricted by this contour  $G$ .

The circulation of vector  $\vec{H}$  does not depend on the form of chosen surface. It is possible to consider two variants of surface which tightens up the contour.

## DISPLACEMENT CURRENT

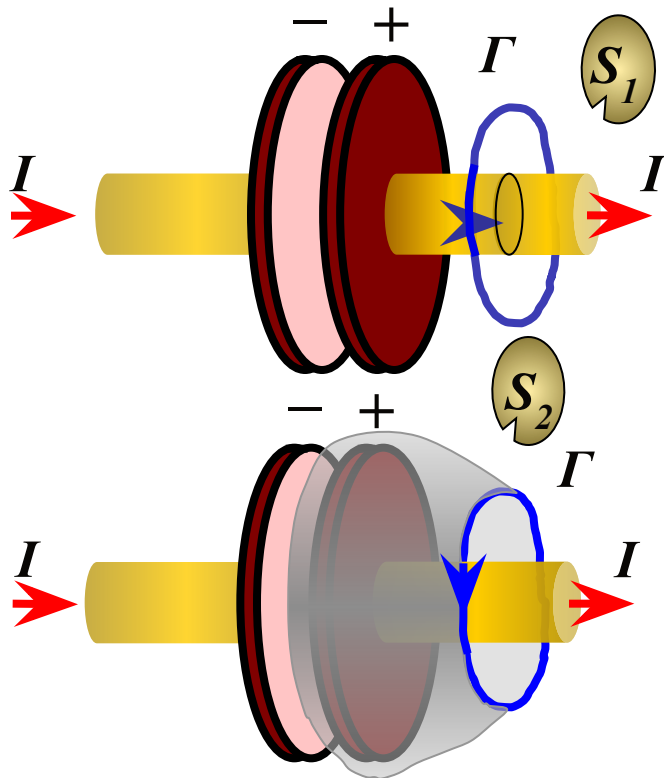


a) The surface  $S_1$  has a cross section with current-carrying conductor.

b) The surface  $S_2$  has not cross section with current-carrying conductor.

It is seen that current of conductance  $I$  is flowing across the surface  $S_1$  while current across the surface  $S_2$  is absent through the absence of conductor and any free charges in the air gap between the plates of capacitor.

## DISPLACEMENT CURRENT



We can do a false conclusion that the circulation of vector  $\mathbf{H}$  depends on the form of surface restricted by contour. It is a full contradiction with theorem about of vector  $\mathbf{H}$  circulation.

Is this theorem staying correct in case of field and current variable in time?

~~$$\oint_L (\mathbf{H}, d\mathbf{l}) = \int_S (\mathbf{j}, d\mathbf{S}) = I \quad ???$$~~

To resolve this contradiction Maxwell added the new component into the right side of this equation as the *displacement current* with corresponding *density of this current*.

## *DISPLACEMENT CURRENT*

### Expression for definition of displacement current.

Remember that  $I = dq/dt$ ,  $j = dI/dS$ , thus  $j = dq/(dS dt)$ .

Take into account that surface  $S_2$  has crossing only with vectors of electrical field strength.

According to Gauss' theorem the flux of electric induction vector  $\mathbf{D} = \epsilon\mathbf{E}$  in the stationary electric field across the some closed surface  $S$  is equal to (1):

$$\oint_S (\mathbf{D} d\mathbf{S}) = q \quad (1)$$

For the variable electrical field it is possible to write from Gauss' theorem equation (2):

$$\oint_S \left( \frac{\partial \mathbf{D}}{\partial t} d\mathbf{S} \right) = \frac{\partial q}{\partial t} \quad (2)$$

For the current of conductivity which is flowing opposite to the normal unit vector of surface  $S$  the equation of current density continuity can be written as (3):

$$\oint_S (\mathbf{j} d\mathbf{S}) = -\frac{\partial q}{\partial t} \quad (3)$$

## DISPLACEMENT CURRENT

$$\begin{aligned}
 & \oint_S \left( \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} \right) = \frac{\partial q}{\partial t} \\
 + & \oint_S (\vec{j} \cdot d\vec{S}) = -\frac{\partial q}{\partial t}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \oint_S \left( \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} \right) = \frac{\partial q}{\partial t} \\ \oint_S (\vec{j} \cdot d\vec{S}) = -\frac{\partial q}{\partial t} \end{aligned}} \right\} \text{Addition of left and right sides yields to:}$$

$$\oint_S \left( \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} \right) = 0$$

This equation is similar to equation of direct current continuity. Beside of density of current of conductivity  $\vec{j}$  it includes once more summand  $\partial \vec{D} / \partial t$  which has dimension of current density.

This is a ***density of displacement current***:

$$\boxed{j_d = \partial \vec{D} / \partial t}$$

The sum of conductivity current and displacement current is called as ***total current***:

$$I_{total} = \int_S \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

## DISPLACEMENT CURRENT

$\vec{j}_{TOTAL} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$  is a density of total current.

Following to expression  $\oint_S \left( \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S} \right) = 0$  the lines

of total current are continuous. The currents of conductivity, if they have not a conductor for closing, can be closed by displacement current.

Due to introduction of displacement current into Maxwell's equations the problem of correct application of theorem about circulation of vector  $\vec{H}$  has got a resolution for variable electrical fields.

In general case of presence both current of conductivity and displacement current this theorem has a view:

$$\oint_L (\vec{H} d\vec{l}) = \int_S \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$$

$$\vec{j}_d = \partial \vec{D} / \partial t$$

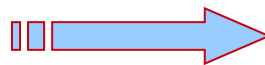
## DISPLACEMENT CURRENT

The term “displacement current” is formal. In fact that is a way to take into account the varying in time electrical field which is able to create the additional variable magnetic field. Only one property is the same for conductivity current and displacement current, namely, the ability of both to create the magnetic field.

Displacement currents exist only in the presence of varying in time (rotational) electrical field.

Displacement currents in dielectrics:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$



$$\vec{j}_{cm} = \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$$

$\vec{P}$  is a polarization vector of dielectrics

$\epsilon_0 \frac{\partial \vec{E}}{\partial t}$  is a “true” displacement current

$\frac{\partial \vec{P}}{\partial t}$  is a current of polarization stipulated by the local motion of fixed charges in dielectrics

## DISPLACEMENT CURRENT

Likely to the “true” displacement current  $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ ,  
 the polarization current  $\frac{\partial \vec{P}}{\partial t}$  takes part in the creation of varying  
 in time magnetic field.

The “true” displacement current is not connected with passage of free charges but connected only with variation (or oscillation) in time of charges which create the electrical field at the borders of dielectrics. The polarization current is connected with presence of fixed charges inside of dielectrics and their oscillating under influence of external field.

*Discovery of displacement current has been done by Maxwell in theory, but this discovery had extremely important significance not only for development of electromagnetic field theory but also for practical applications of electromagnetism.*



## *MAXWELL'S EQUATIONS*

Due to discovery of displacement current Maxwell could built the completed theory of electrical and magnetic phenomena which is presented today as the **macroscopic theory of electromagnetic field**.

Theory of Maxwell gave the explanation of many phenomena of electromagnetism and allowed to predict some unknown before electromagnetic processes.

The foundation of this theory consists of 4 main equations. In the theory of electromagnetism they play the role which is similar to role of Newton in mechanics or to role of main laws of thermodynamics.

The solution of Maxwell's equations gives a possibility to find a distribution in space for electrical and magnetic fields and to define they as the function of time.

## MAXWELL'S EQUATIONS

### Integral form of Maxwell's equations

$$1. \quad \oint_L (\vec{E} \cdot d\vec{l}) = - \int_S \left( \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \right)$$

*A circulation of the vector  $\vec{E}$  along any closed contour is equal with minus to time derivative of magnetic flux across the surface restricted by chosen contour.*

In the first equation  $\vec{E} = \vec{E}_q + \vec{E}_B$  because the electrical field can be sum of electrostatic field  $\vec{E}_q$  and rotational electrical field  $\vec{E}_B$ .

The first equation shows that not only electrical charges but also varying in time magnetic fields can be the sources of electrical fields.

The first equation is known as **Faradey** law.

## MAXWELL'S EQUATIONS

### Integral form of Maxwell's equations

2.  $\oint_S (\mathbf{B} \cdot d\mathbf{S}) = 0$       Theorem of Gauss for magnetic field.

*The flux of magnetic induction vector across the any closed surface is equal to zero.*

Magnetic field has not the outflow and sources, the field lines have not the end or beginning. Magnetic field was called ***solenoidal*** or ***rotational*** one (terms are taken from the mathematic theory of vector fields).

## *MAXWELL'S EQUATIONS*

### *Integral form of Maxwell's equations*

$$3. \oint_L (\vec{H}, d\vec{l}) = \int_S \left( \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right), d\vec{S} \right) \quad \begin{array}{l} \text{Account} \\ \text{current} \end{array} \quad \text{of} \quad \text{displacement}$$

*A circulation of the vector  $\vec{H}$  along the any closed contour is equal to the total current across the surface restricted by the chosen contour.*

There is the total current implies as the sum of current of conductivity and displacement current.

This equation shows that the magnetic fields can be excited by the moving electrical charges (i.e. by current) or by the variable electrical field.

## MAXWELL's EQUATIONS

### Integral form of Maxwell's equations

4. 
$$\oint_S (\vec{D} \cdot d\vec{S}) = \int_V \rho dV$$

Gauss theorem for the vector of electrical induction (displacement vector).

*The flux of a displacement vector across any closed surface is equal to electrical charge enclosed inside of this surface.*

This is a Maxwell postulate which presents the law of electrical field creation at any medium as result of the charge presence inside of chosen closed surface at known volume density  $\rho$  of this charge.

## MAXWELL'S EQUATIONS

Consequences of Maxwell's equations:

- as electrical charges and varying in time magnetic fields can be the sources of electrical field;
- the moving electrical charges (i.e. electrical currents) as well as the varying electrical fields are the reason of magnetic field excitation.

The Maxwell's equations do not contain a symmetry with respect to magnetic and electrical fields. It corresponds to natural fact about existence of local electrical charges and impossibility of magnetic charges existence.

For the stationary fields ( $\mathbf{E} = \text{const}$  and  $\mathbf{B} = \text{const}$ ) the Maxwell's equations have a view:

$$\oint_L (\mathbf{E} d\mathbf{l}) = 0; \quad \oint_S (\mathbf{B} d\mathbf{S}) = 0; \quad \oint_L (\mathbf{H} d\mathbf{l}) = I; \quad \oint_S (\mathbf{D} d\mathbf{S}) = q.$$