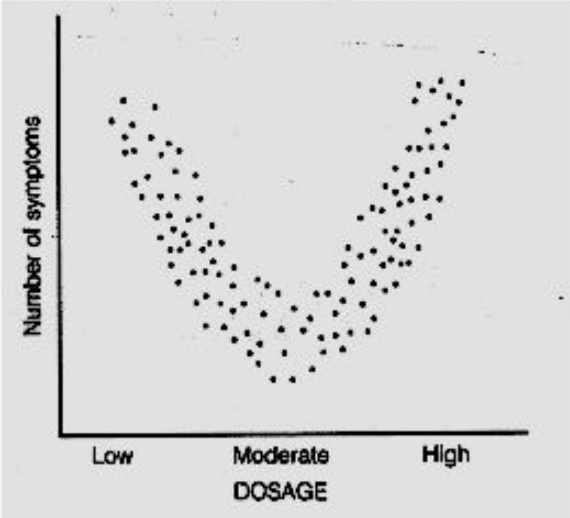
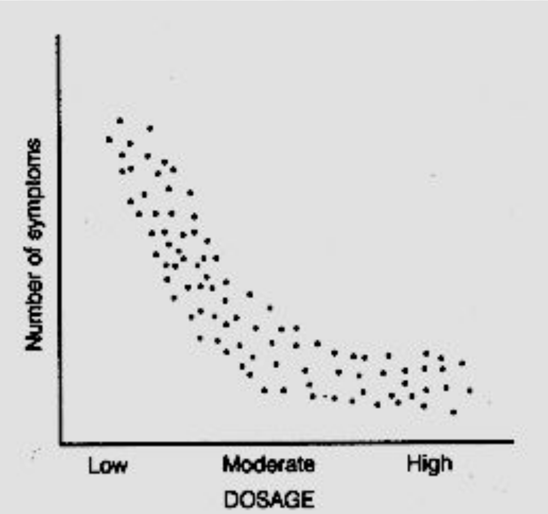
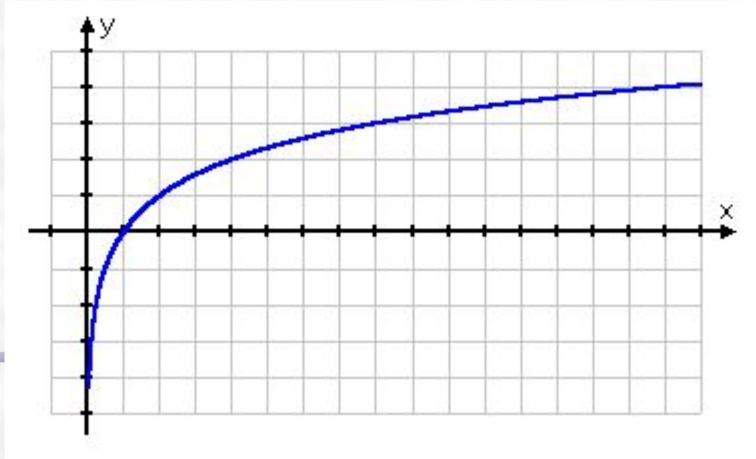
Choice of the functional form

What if…

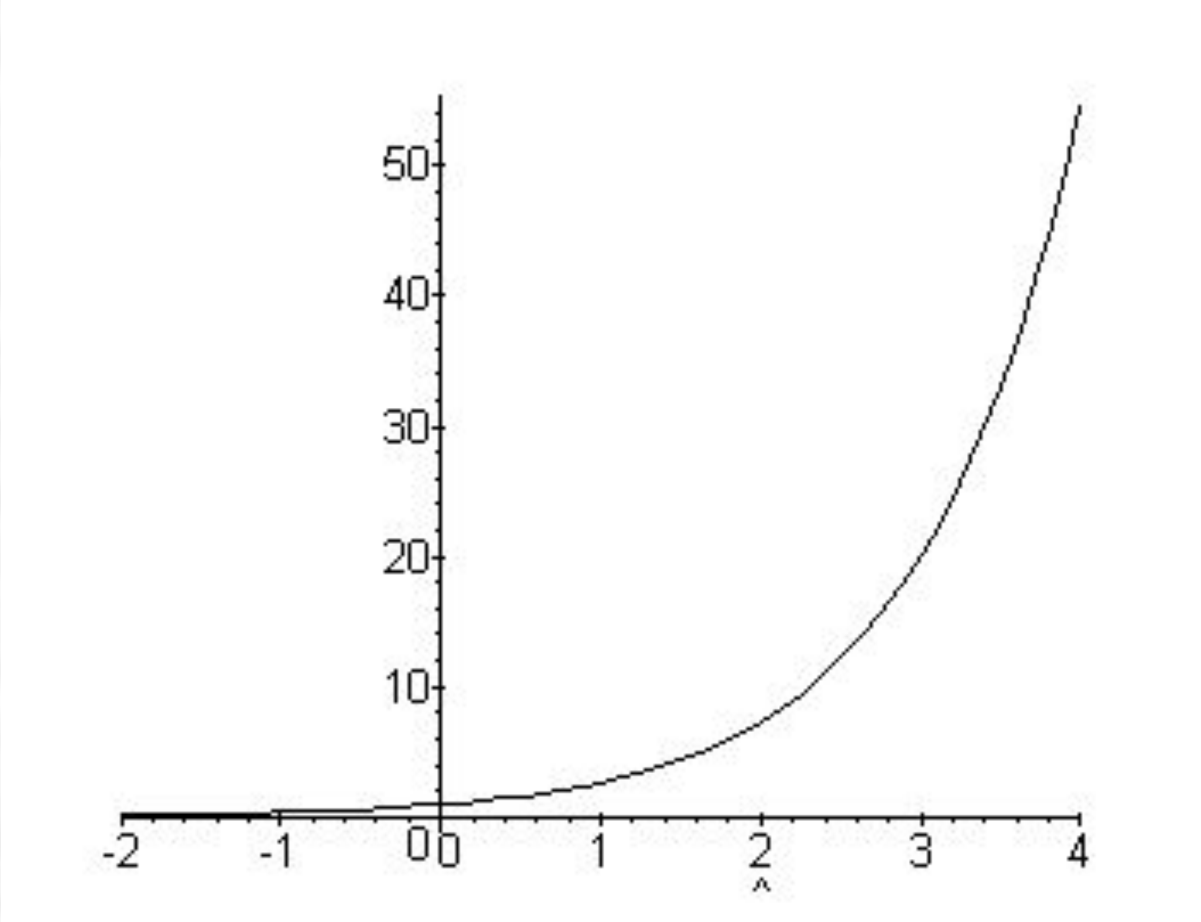


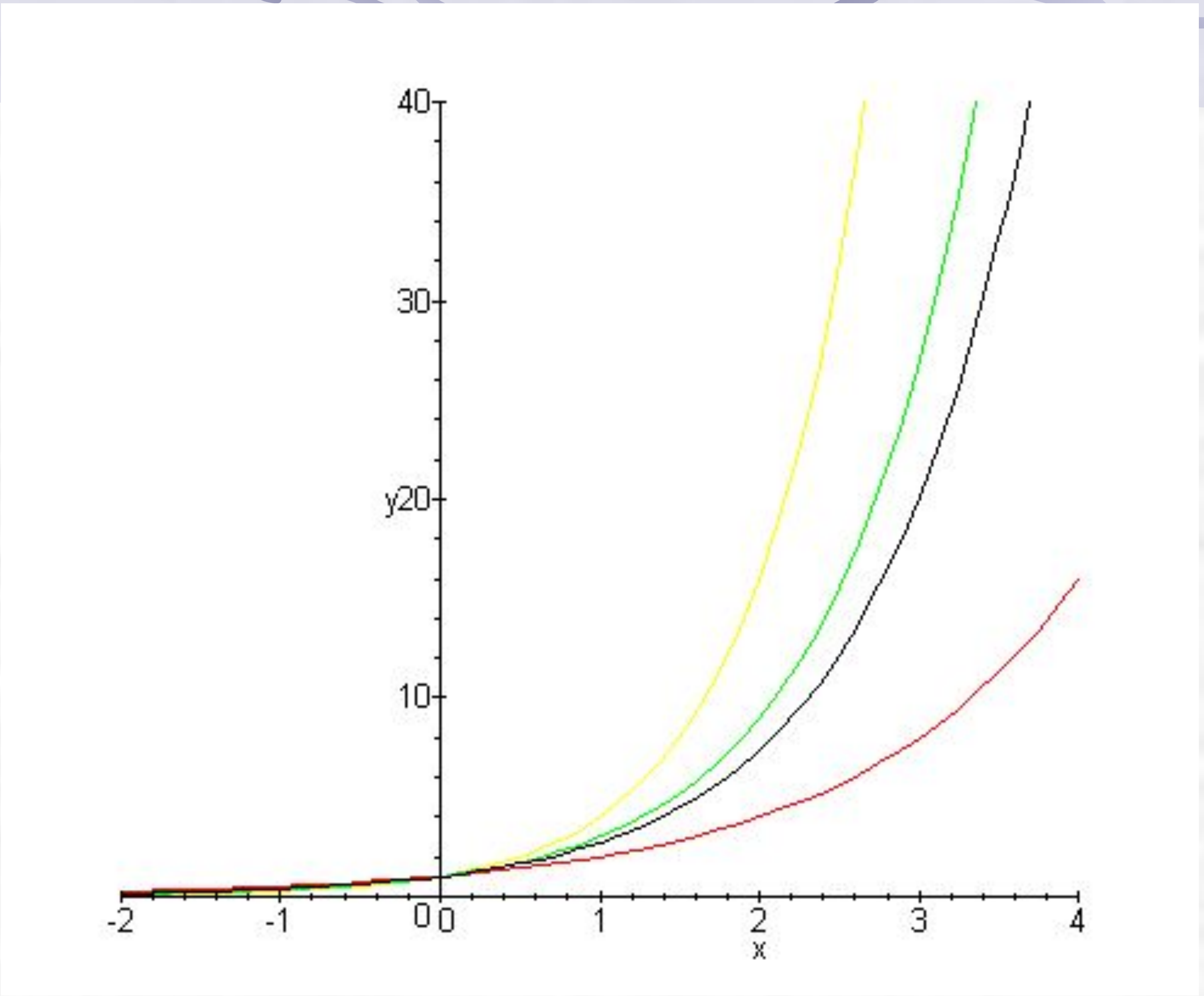




Exponential function

Exponential functions are functions which can be represented by graphs similar to the graph on the right



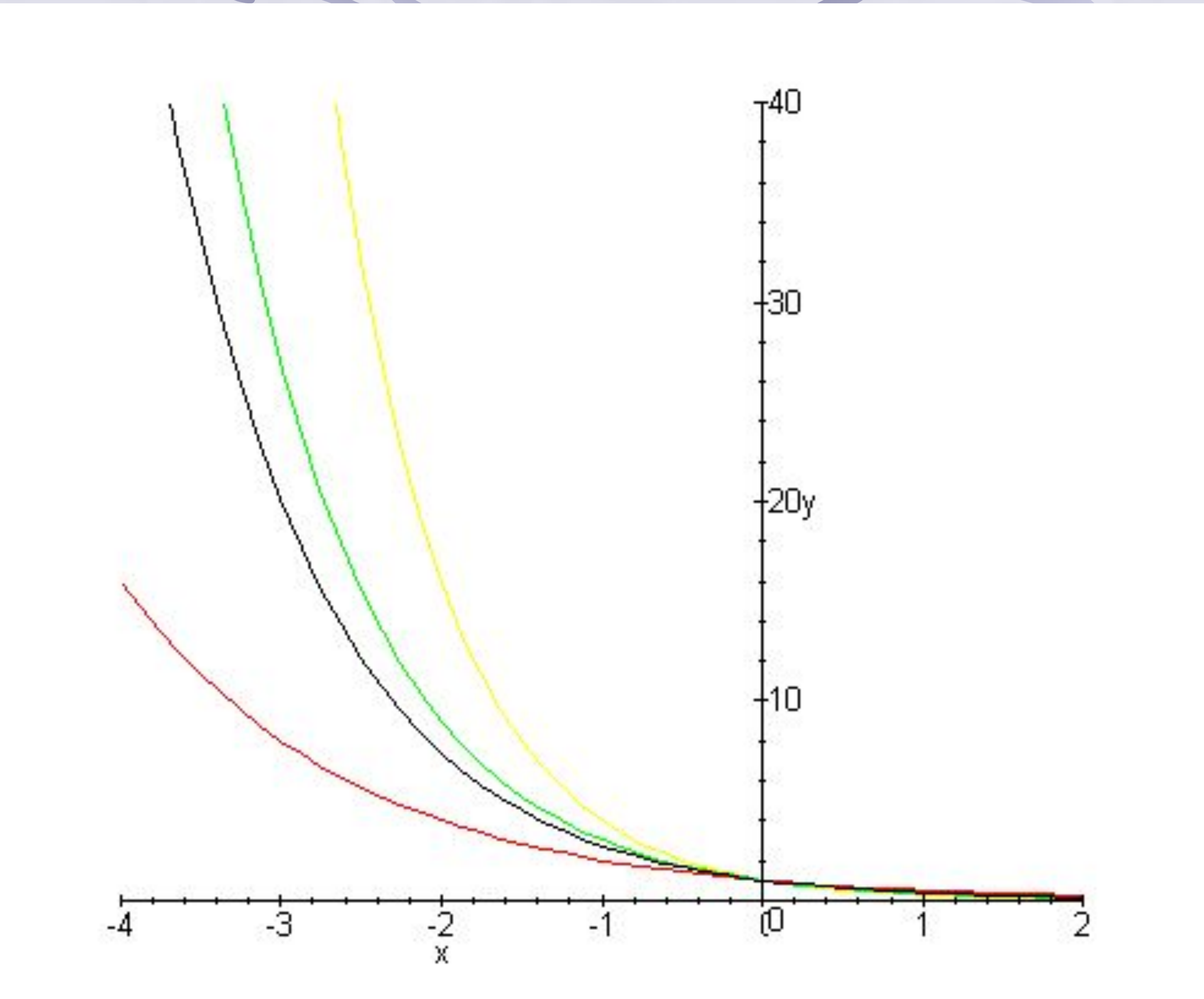


Yellow = 4^x Green = e^x

Black = 3^x Red = 2^x

As you could see in the graph, the larger the base, the faster the function increased

If we place a negative sign in front of the x, the graphs will be reflected(flipped) across the y-axis



Yellow = 4^{-x}

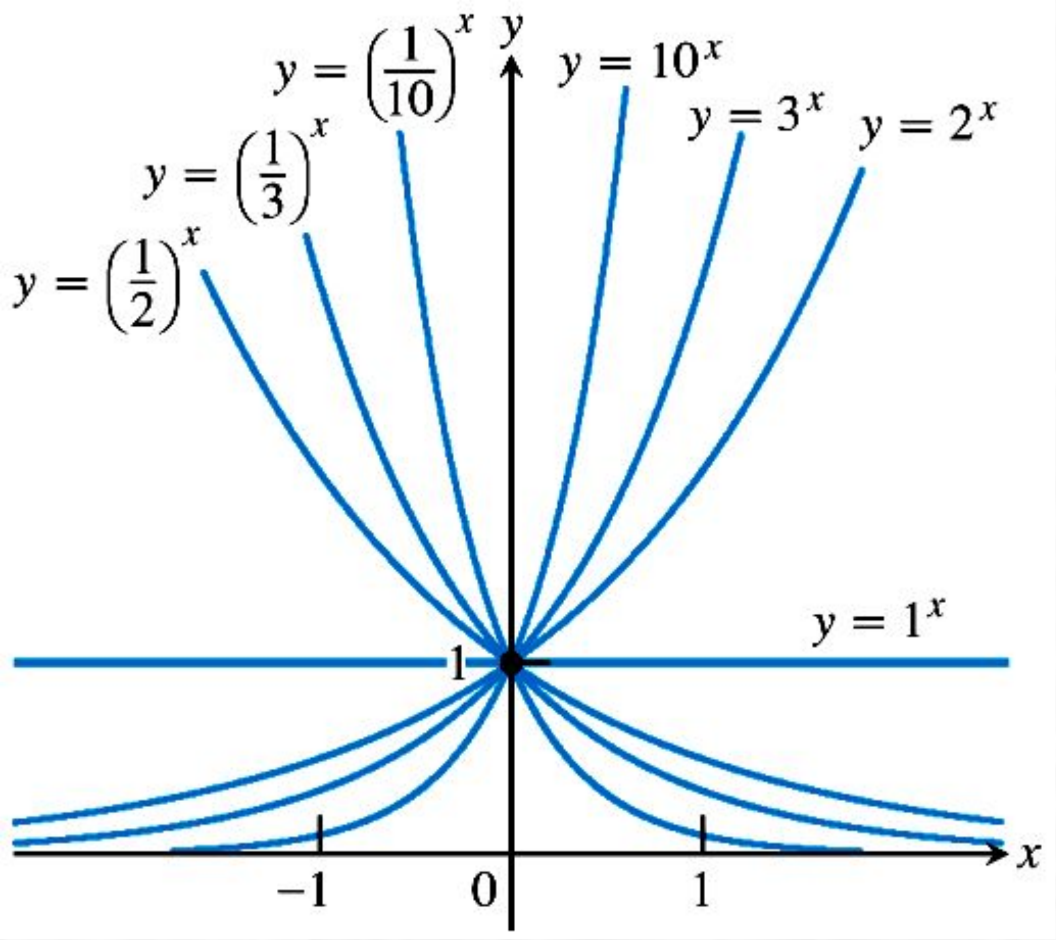
Green = e^{-x}

Black = 3^{-x}

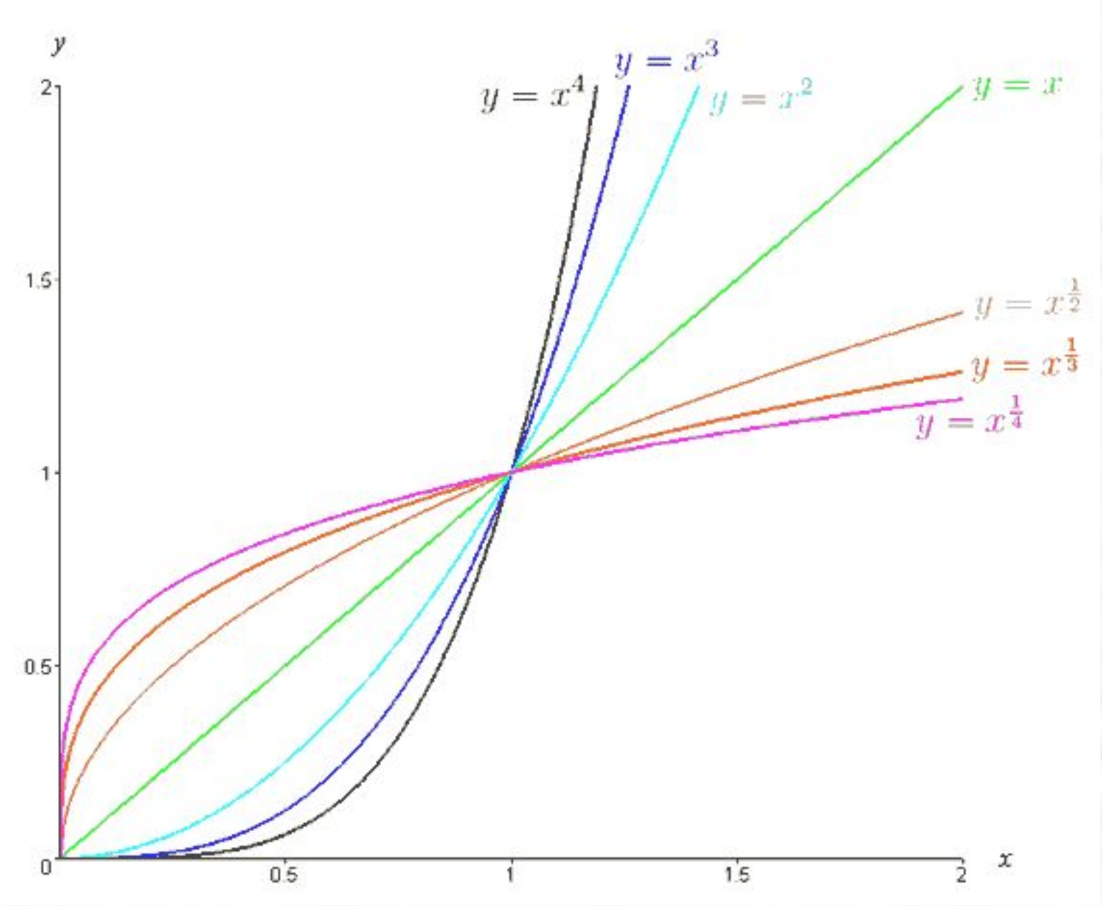
Red = 2^{-x}

Exponential function

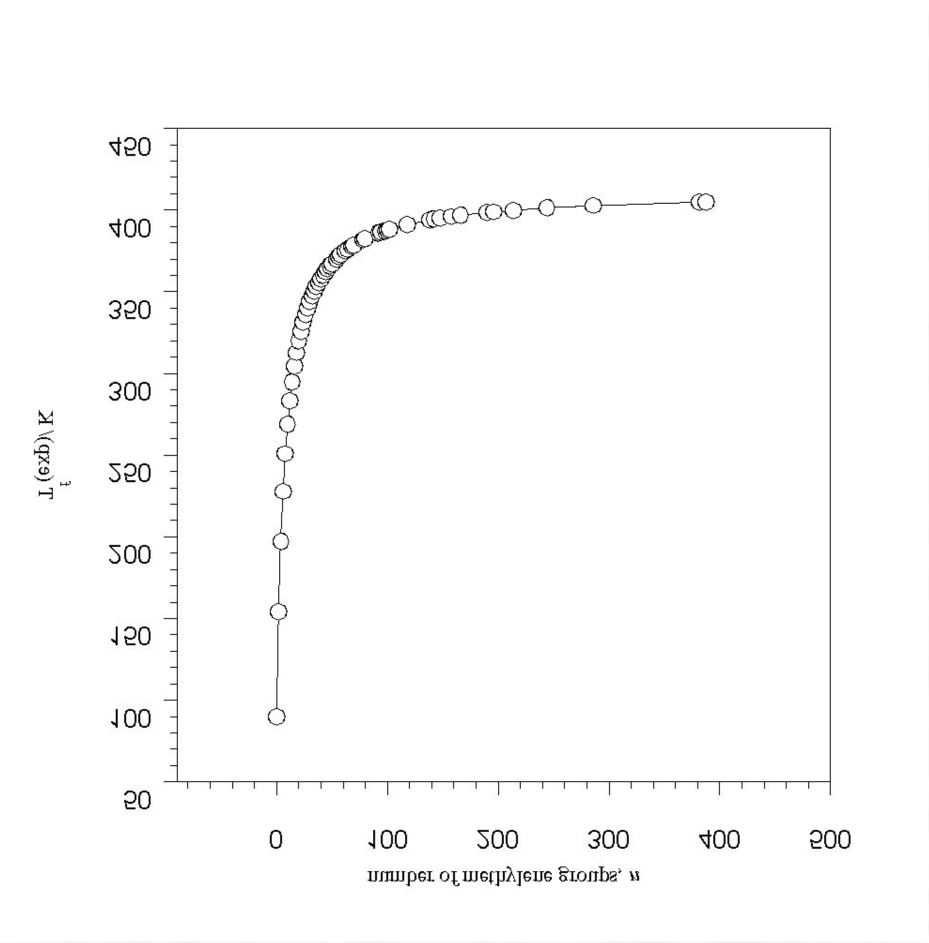
- Exponential functions decrease if $0 < b_1 < 1$ and increase if $b_1 > 1$



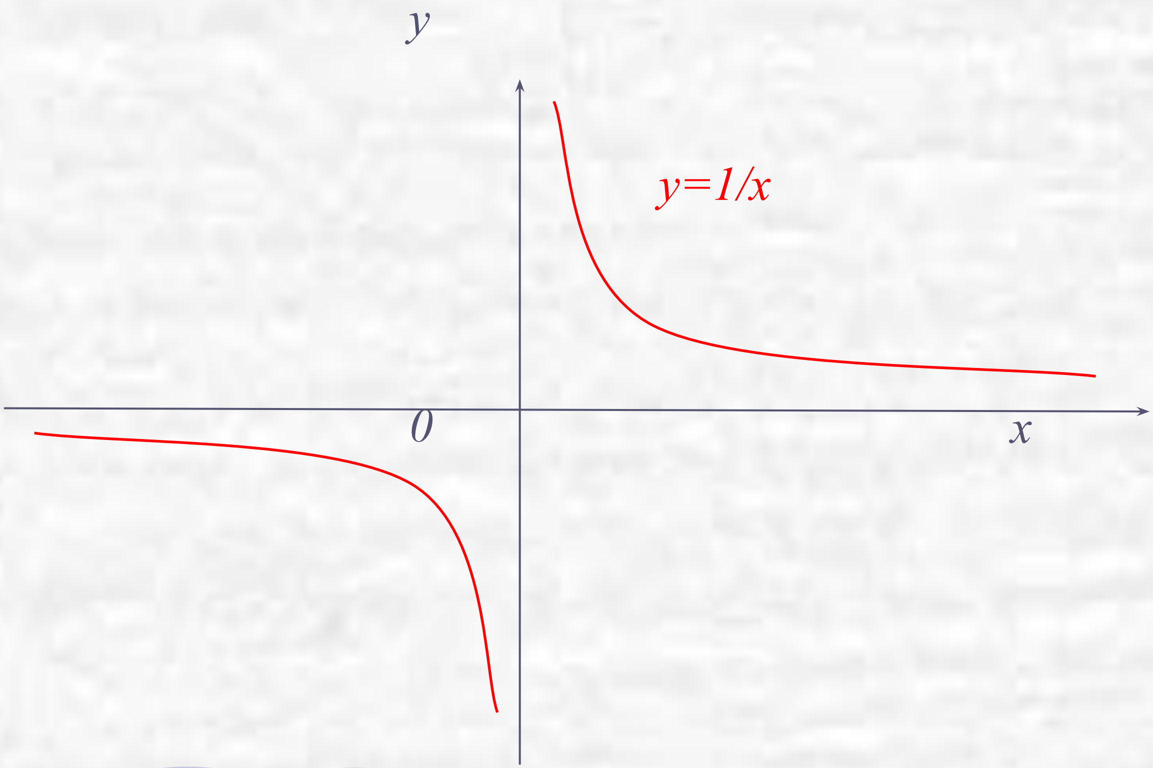
Power function



Logarithmic function

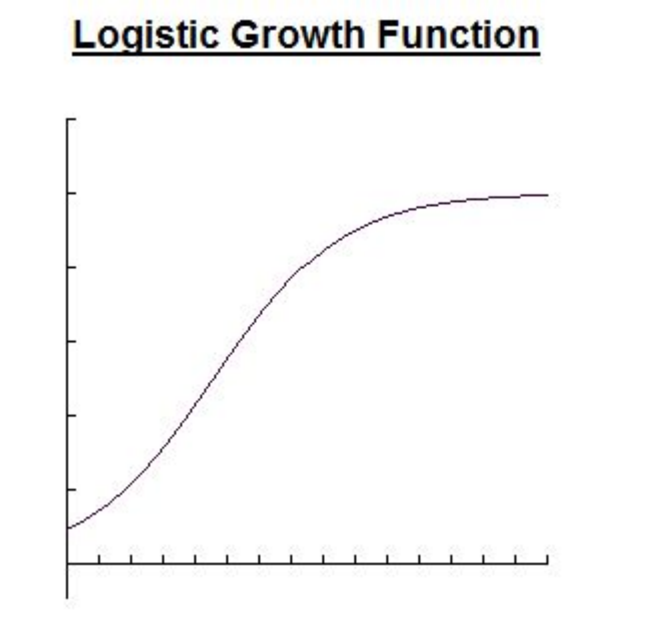


Hyperbolic function



Quadratic function

Logistic function



If growth begins slowly, then increases rapidly and eventually levels off, the data often can be model by an “S-curve”, or a logistic function.

General information

- NONLINEAR MODELS OFTEN ARE USED FOR SITUATION IN WHICH THE RATE OF INCREASE OR DECREASE IN THE DEPENDENT VARIABLE (WHEN PLOTTED AGAINST A PARTICULAR INDEPENDENT VARIABLE) IS NOT CONSTANT.

General information

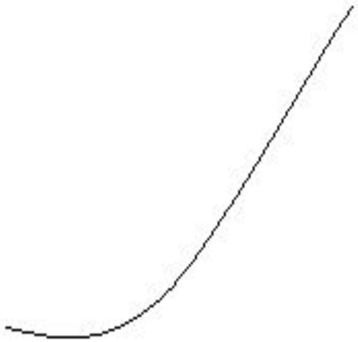
- SOME OF THESE MODELS REQUIRED A TRANSFORMATION TO THE INDEPENDENT VARIABLE.

Transformation

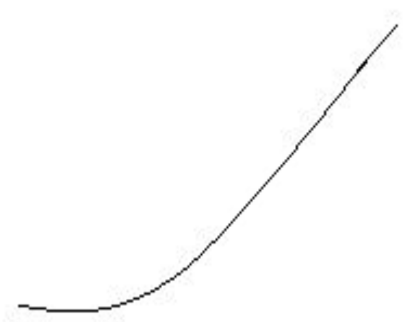
- Logarithms
- Substitution

Data transformations can be used to convert an equation into a linear form

Exponential function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 b_1^{\ x}$ exponential function		Logarithms	$y' = b_0' + b_1 x'$ where y'=log y b_0'=log b_0 b_1'=log b_1

Power function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 x^{b1}$ power function		Logarithms	$y' = b_0' + b_1 x'$ where y'=log y x'=log x b_0'=log b_0

Quadratic function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 + b_1 x + b_2 x^2$ quadratic function		Substitution	$y = b_0 + b_1 x_1 + b_2 x_2$ where $x_1 = x$ $x_2 = x^2$

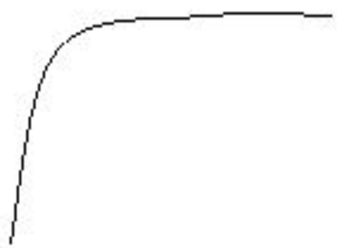
Polynomial function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_k x^k$ polynomial function		Substitution	$y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots + b_k x_k$ where $x_1 = x$ $x_2 = x^2$ $x_3 = x^3$ $x_k = x^k$

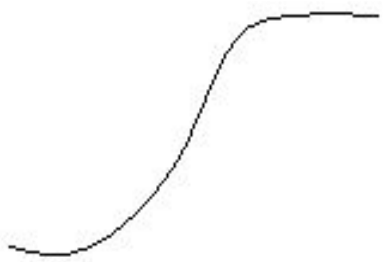
Hyperbolic function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 + b_1 \frac{1}{x}$ hyperbolic function		Substitution	$y = b_0 + b_1 x'$ where $x' = 1/x$

Logarithmic function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = b_0 + b_1 \ln x$ logarithmic function		Substitution	$y = b_0 + b_1 x'$ where $x' = \ln x$

Logistic function

Model	Scatter plot	Transformation	Model after transformation
$\hat{y} = \dfrac{b_0}{1 + b_1 e^{-x}}$ logistic function		Substitution	$y' = b_0' + b_1' x'$ where $y' = 1/y$ $x' = e^{-x}$ $b_0' = 1/b_0$ $b_1' = b_1/b_0$

Linear function

X	Y	Δ X	ΔY	ΔY/ΔX
1,2	3,08	-	-	-
1,4	3,58	0,2	0,5	2,50
1,6	4,08	0,2	0,5	2,50
1,8	4,58	0,2	0,5	2,50

$$b_1 = 2,50$$

Exponential function

X	Y	Δ X	ΔY	ΔY/ΔX	log y	log y/x
1,2	4,963	-	-	-	0,6957	0,580
1,4	6,482	0,2	1,519	7,595	0,8117	0,580
1,8	11,086	0,4	4,604	11,51	1,0448	0,580
3	54,872	1,2	43,786	36,4883	1,7394	0,580

$$b_1=10^{0,580}=3,8$$

Power function

X	Y	ΔX	ΔY	ΔY/ΔX	log y	log y/x	log x	log y/log x
1,4	2,027	-	-	-	0,3069	0,219	0,146	2,10
2	4,28	0,6	2,253	3,755	0,6314	0,316	0,301	2,10
5	29,37	3	25,085	8,36167	1,4678	0,294	0,699	2,10
9	100,9	4	71,535	17,88375	2,0039	0,223	0,954	2,10

$$\boxed{b_1 = 2{,}10}$$

Comparison

EXPONENTIAL	POWER
Independent variable is a power **exponent**	Independent variable is a power **base**
Form of model:	
$\hat{y} = b_0 \cdot b_1^{\,x}$	$\hat{y} = b_0 \cdot x^{b_1}$
Interpretation of the coefficients	
b_0- is the value of Y if independent variable is equal to **zero**.	b_0- is the value of Y if independent variable is equal to **one**
b_1- is the growth rate Y. If the independent variable increases 1 unit, the dependent variable will change (increase, if $b_1>1$, or decrease, if $b_1<1$) b_1 times, on average {or $(b_1-1)\cdot 100[\%]$, on average}.	b_1 - is the elasticity Y. If the independent variable increases 1 %, the dependent variable will change (increase, if $b_1>0$, or decrease, if $b_1<0$) b_1%, on average.

Comparison

EXPONENTIAL	POWER
Linear transformation - logarithms	
$\log\hat{y} = \log b_0 + x\cdot\log b_1$ **Linear form**	$\log\hat{y} = \log b_0 + b_1\cdot\log x$
Parameters estimation – OLS:	
$\begin{bmatrix}\log b_0\\ \log b_1\end{bmatrix} = (X^T X)^{-1} X^T \log Y$	$\begin{bmatrix}\log b_0\\ b_1\end{bmatrix} = (\log X^T \log X)^{-1} \log X^T \log Y$
Matrix and vector:	
$X^T X = \begin{bmatrix} n & \sum x\\ \sum x & \sum x^2\end{bmatrix}$	$\log X^T \log X = \begin{bmatrix} n & \sum \log x\\ \sum \log x & \sum (\log x)^2\end{bmatrix}$
$X^T \log Y = \begin{bmatrix}\sum \log y\\ \sum x\cdot\log y\end{bmatrix}$	$\log X^T \log Y = \begin{bmatrix}\sum \log y\\ \sum \log x\cdot\log y\end{bmatrix}$

Comparison

EXPONENTIAL	POWER
After log b_0 and log b_1 are estimated we should check goodness of fit (standard error of the estimate, indetermination coefficient, determination coefficient, test parameters individually and check residuals' characteristics – at least linearity) for the linear form.	
$\log \hat{y} = \log b_0 + x \cdot \log b_1$	$\log \hat{y} = \log b_0 + b_1 \cdot \log x$
$S_e = \sqrt{\dfrac{\sum (\log y_i - \log \hat{y}_i)^2}{n-k-1}}$	$S_e = \sqrt{\dfrac{\sum (\log y_i - \log \hat{y}_i)^2}{n-k-1}}$
To interpret the results, antilog b_0 and b_1 should be calculated	To interpret the results, antilog b_0 should be calculated