

# Internal Loadings

Characteristics of External  
Loading

# Introduction

- One of the most important applications of statics, in the analysis of problems involving mechanics of materials is to be able to determine the internal resultant force developed within the body, which are necessary to hold the body together when the body is subjected to external loads.
- Mathematically, this requires

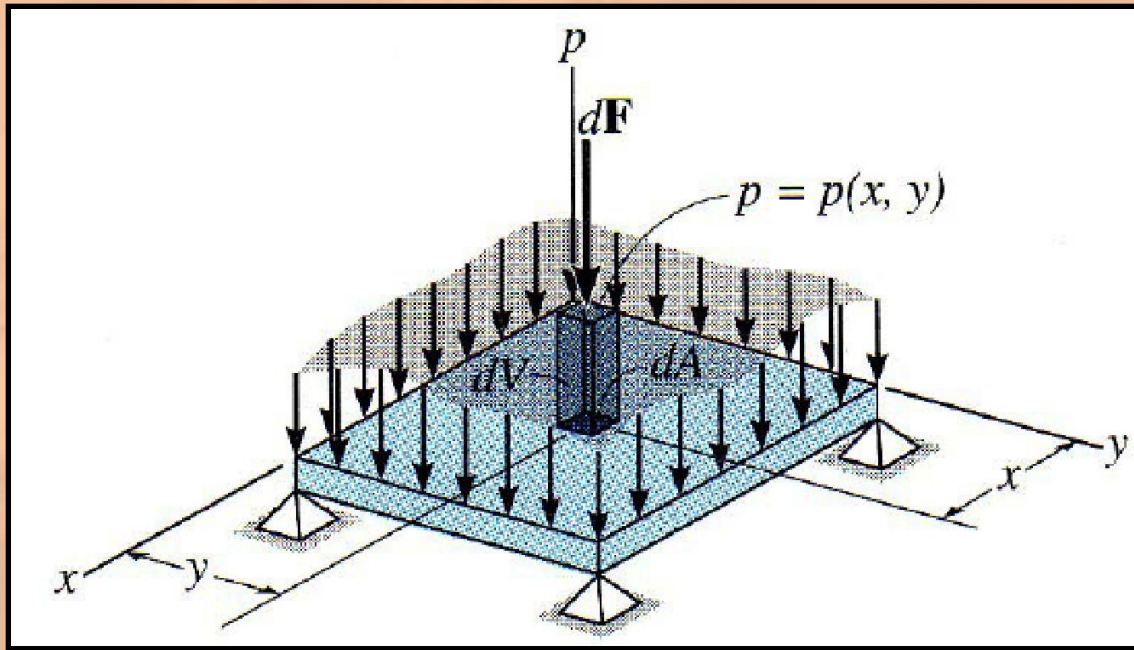
$$\mathbf{F}_{\text{INT}} = \mathbf{F}_{\text{EXT}}$$

- Objectives:
  - i. Determine the characteristics of *external forces*.
  - ii. Develop a technique to analyze the resultant *internal forces* within the body.
  - iii. Learn how to apply shear and moment equations and plot the resultant forces.
  - iv. Know the relationship between distributed load, shear and moment

# External Forces

- In reality, the structures under investigation may be subjected to very complicated loadings in a form of distribution.
- How is it possible to reduce a very complicated external loading to a characteristics or equivalent loading.

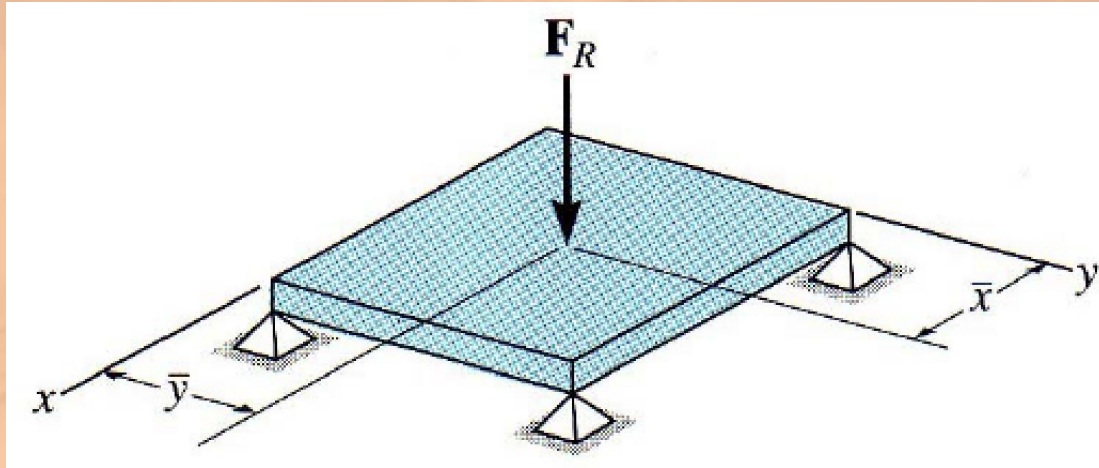
# Pressure Distribution Over a Flat Surface



- The external force is described by the load function  $p = p(x, y)$ . For an infinitesimal area,  $\delta A$ , the force is given by
$$\delta F = p(x, y) \delta A$$
- The resultant loading is therefore given by

$$F_R = \int dF = \int_A p(x, y) dA$$

- This resultant force can be represented as shown below:



The location where  $F_R$  acts on the surface can be evaluated using the formulae

$$\bar{x} = \frac{\int x p(x, y) dA}{\int_A p(x, y) dA}; \quad \bar{y} = \frac{\int y p(x, y) dA}{\int_A p(x, y) dA}$$

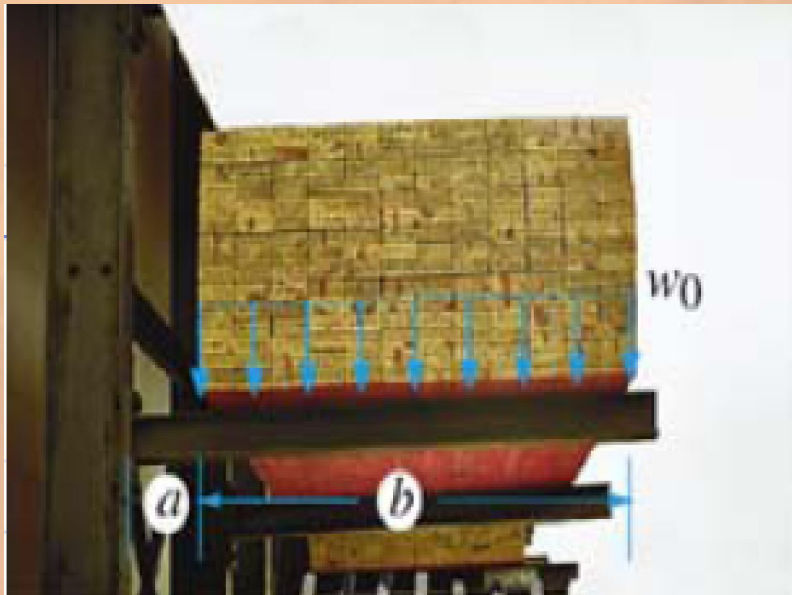
where  $(\bar{x}, \bar{y})$  is the geometric center of the volume under the distributed load.

Note: Observe that  $p(x, y) dA = dV$ , the infinitesimal volume of the  $p - A$  diagram.



- Pressure distribution can be caused by wind, fluid, weights of objects etc.
- The intensity of the pressure distribution is measured in SI as  $\text{N/m}^2$ .
- In Imperial System, it is measured in  $\text{lb/ft}^2$  (psf) or  $\text{lb/in}^2$  (psi).
- Sometimes, kips is used for kilo-pound per sq. inch.

- The following example shows a typical distributed loading.

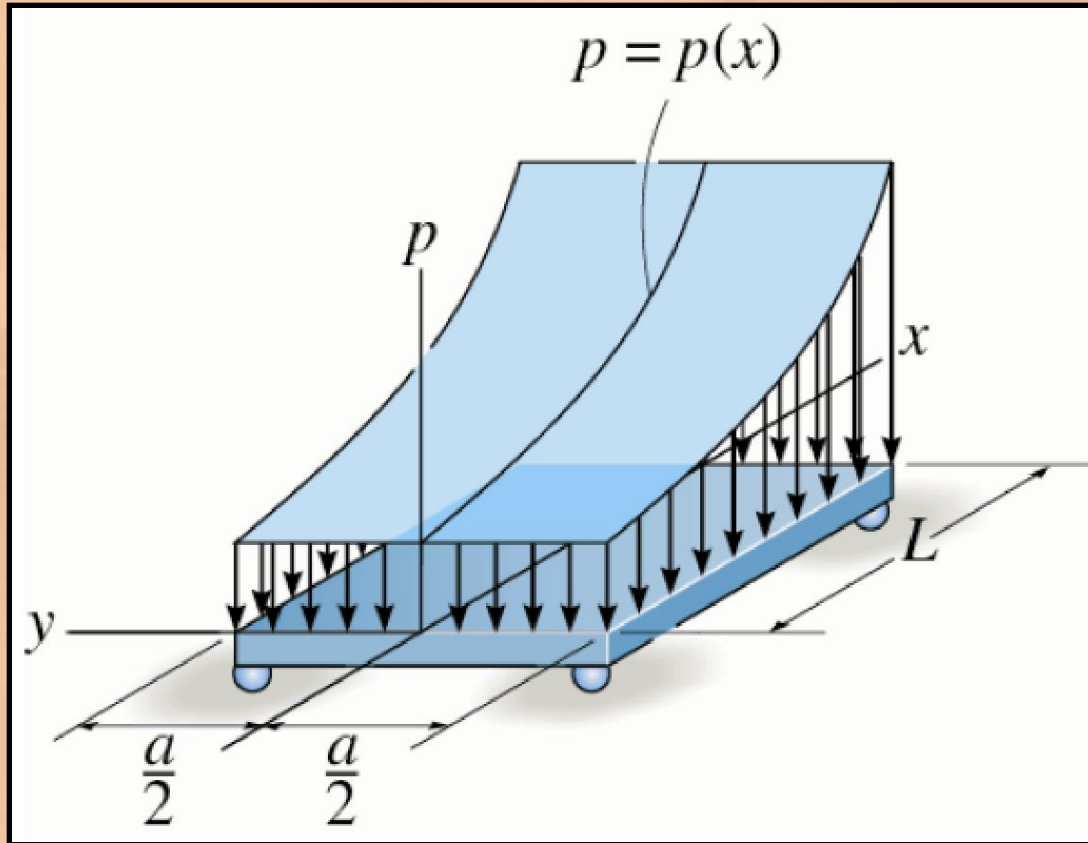


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# Linear Distribution of Load Along a Straight Line



Consider the loading function shown. Here,

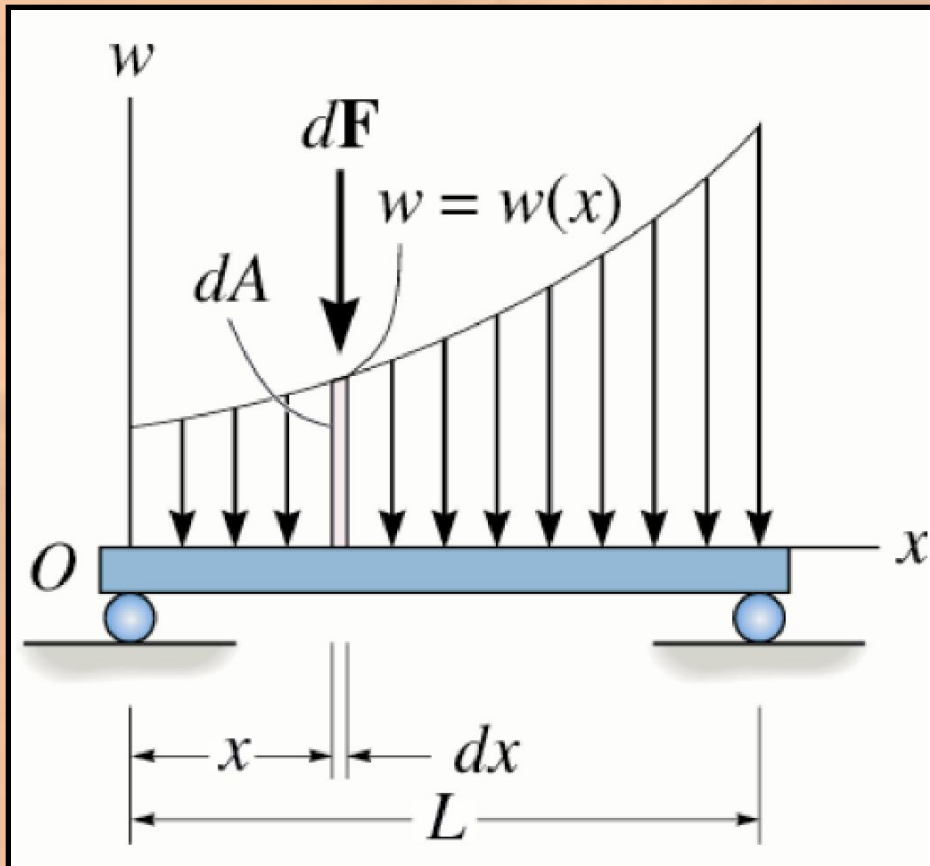
$$p = p(x)$$

Because the pressure is uniform along the  $y$ -axis. We can write

$$w = p(x)a \text{ [N/m]},$$

which is the load per unit length along the line  $y = 0$ .

- The three-dimensional problem is now reduced to a two-dimensional problem, as shown.



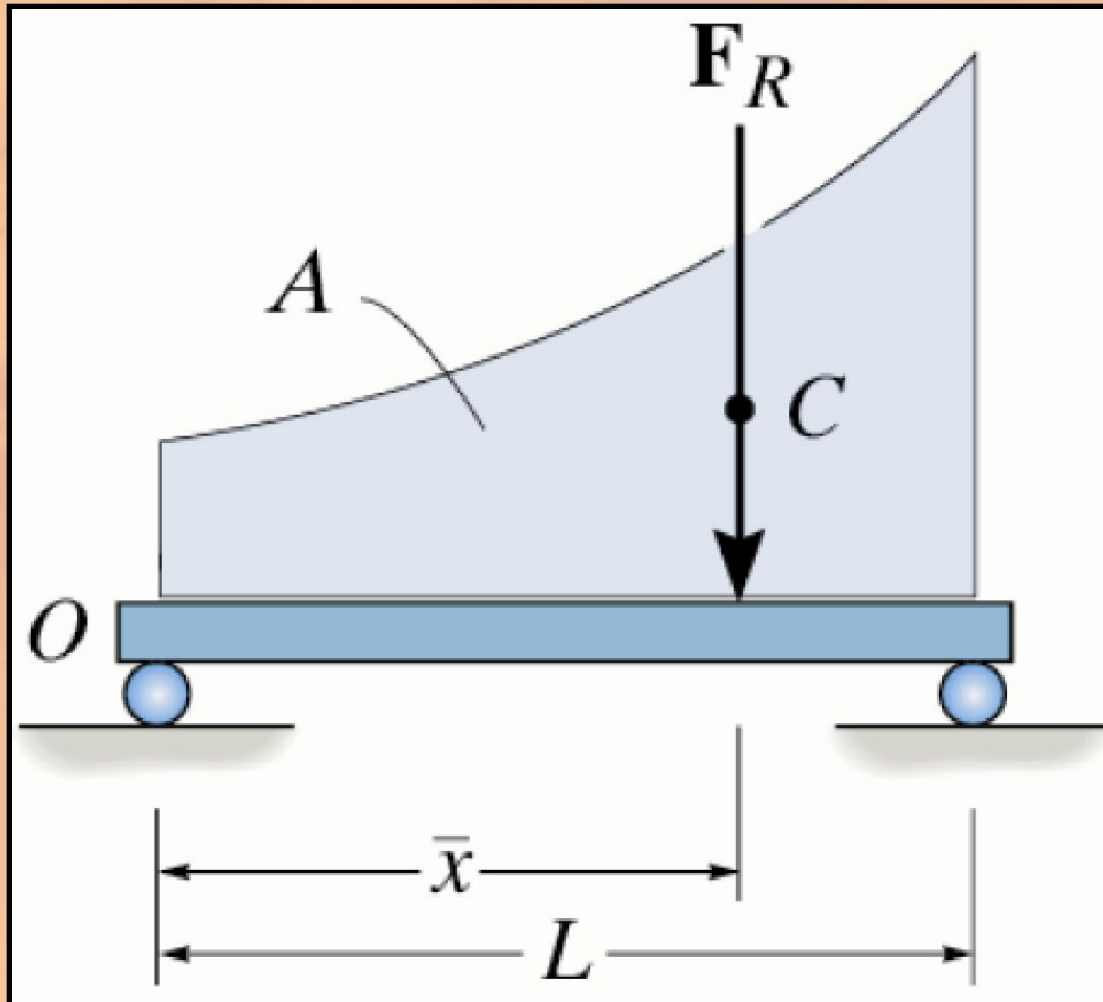
The resultant force is calculated by

$$F_R = \int_L w(x) dx$$

and the location  $x$  of the line of action of  $F_R$  is

$$\bar{x} = \frac{\int_L xw(x) dx}{\int_L w(x) dx} = \frac{\int_A x dA}{\int_A dA}$$

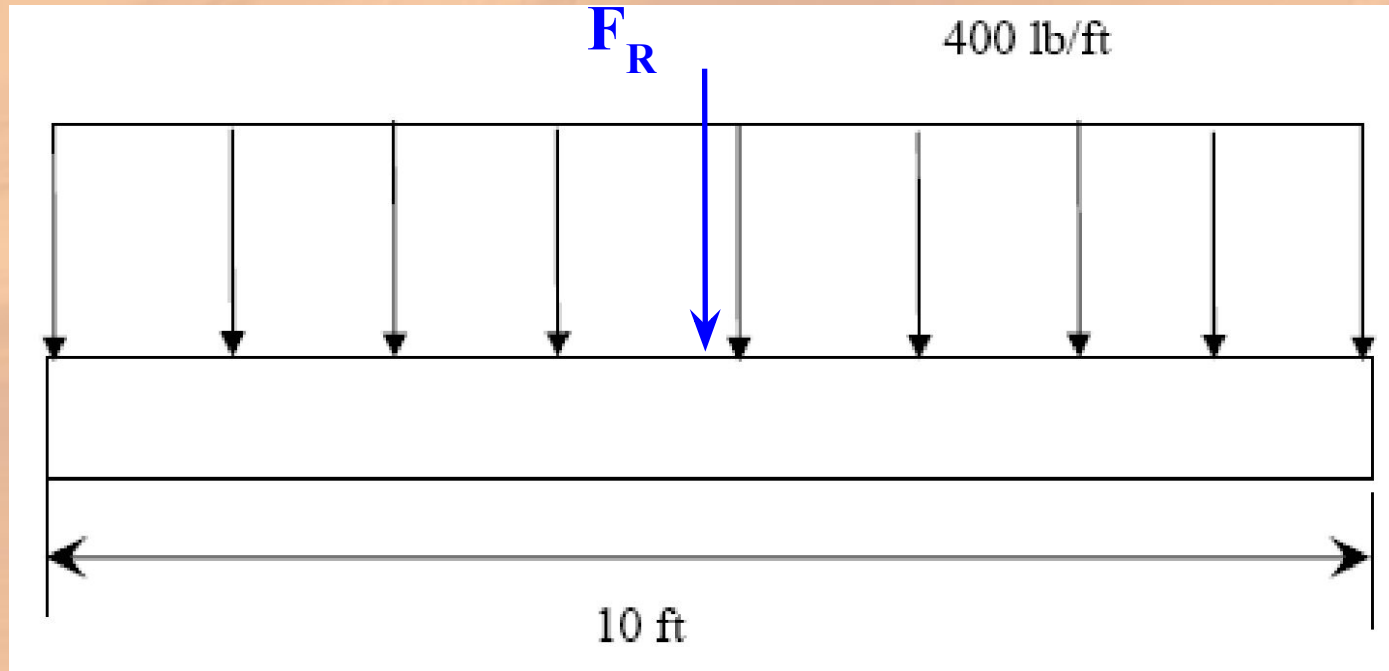
- The equivalent force is thus as shown.



Examples:

Determine the magnitude and location of the resultant of the distributed load shown.

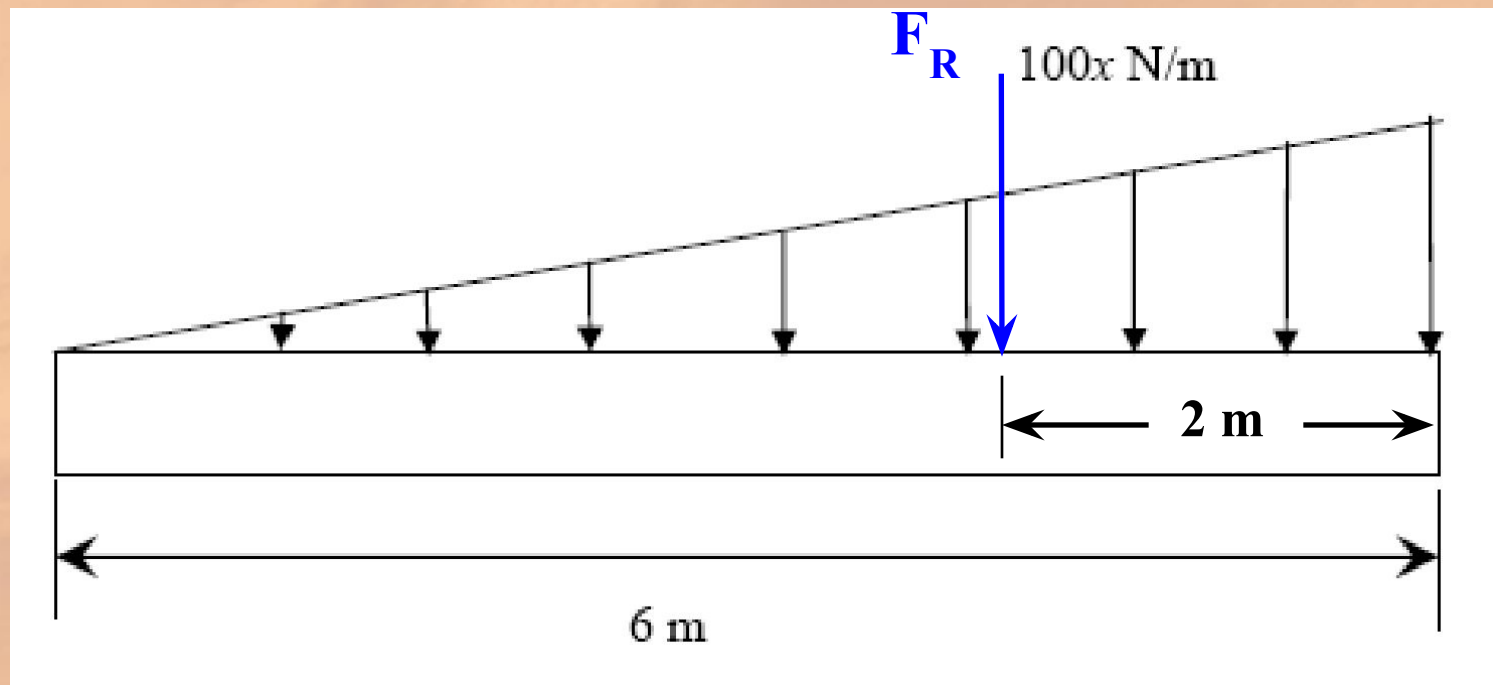
i)



$$F_R = \int w dx = 400(10) = 4000 \text{ lb}$$

$$\bar{x} = 5 \text{ ft}$$

ii)

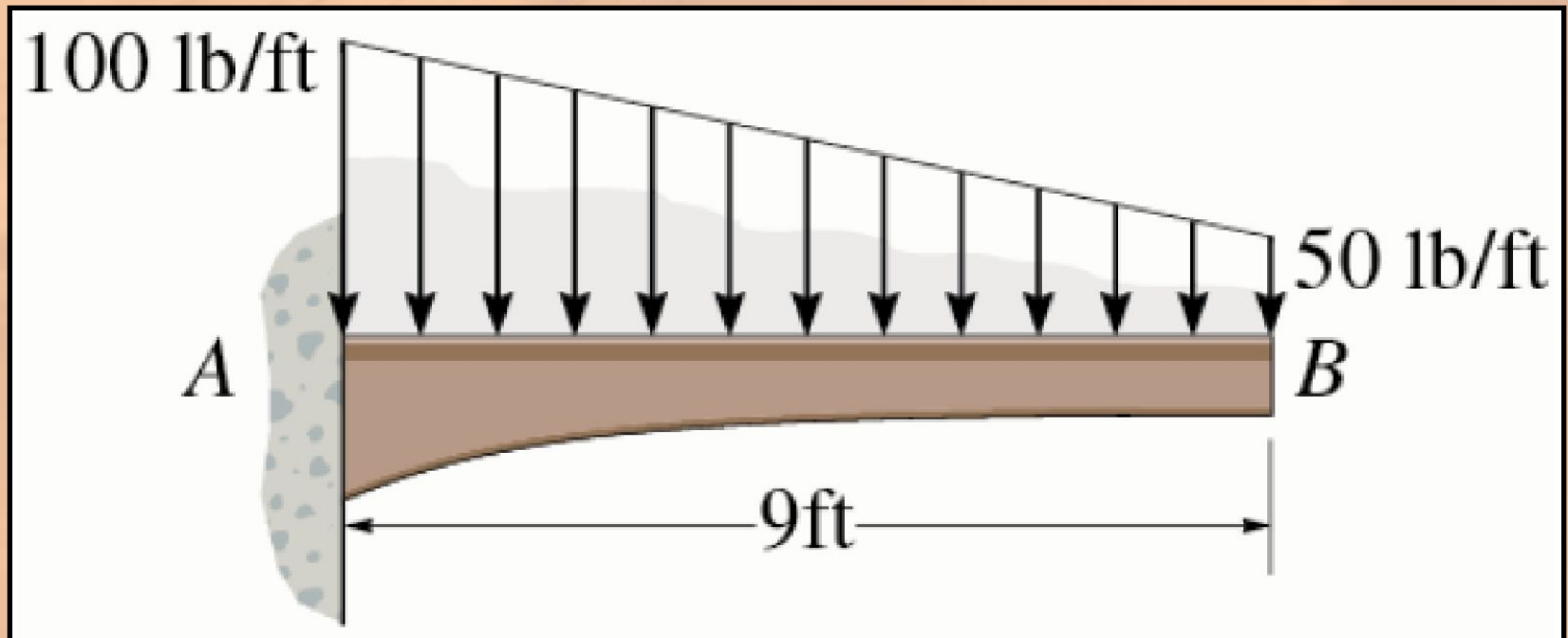


$$F_R = \frac{1}{2}(600)(6) = 1800 N$$

$$\bar{x} = \frac{1}{3}(6) = 2$$

Example:

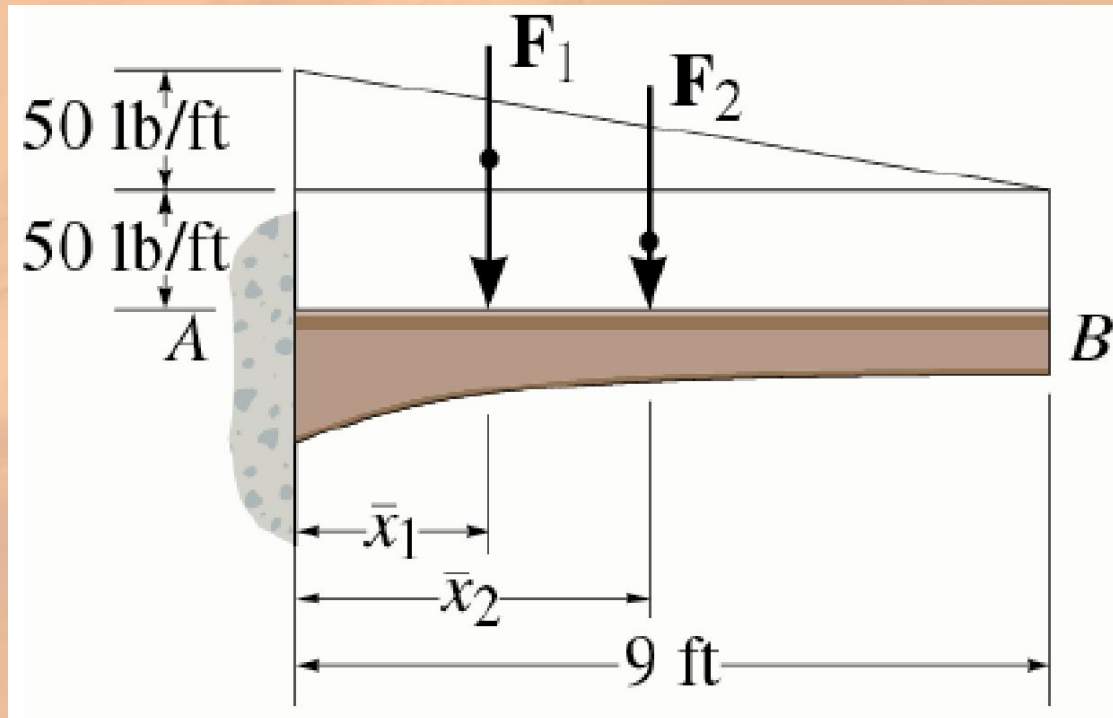
Determine the magnitude and resultant of the loading acting on the beam.





Solution:

Divide the trapezoidal loading into rectangle and triangle loading.



The equivalent loads are:

$$F_1 = \frac{1}{2}(9)(50) = 225 \text{ lb};$$

$$F_2 = 9(50) = 450 \text{ lb}$$

The equivalent position can be found as

$$\bar{x}_1 = \frac{1}{3}(9) = 3 \text{ ft};$$

$$\bar{x}_2 = \frac{1}{2}(9) = 4.5 \text{ ft}$$

Vertical force equilibrium requires:

$$\sum F_y = F_R = 225 + 450 = 675 \text{ lb}$$

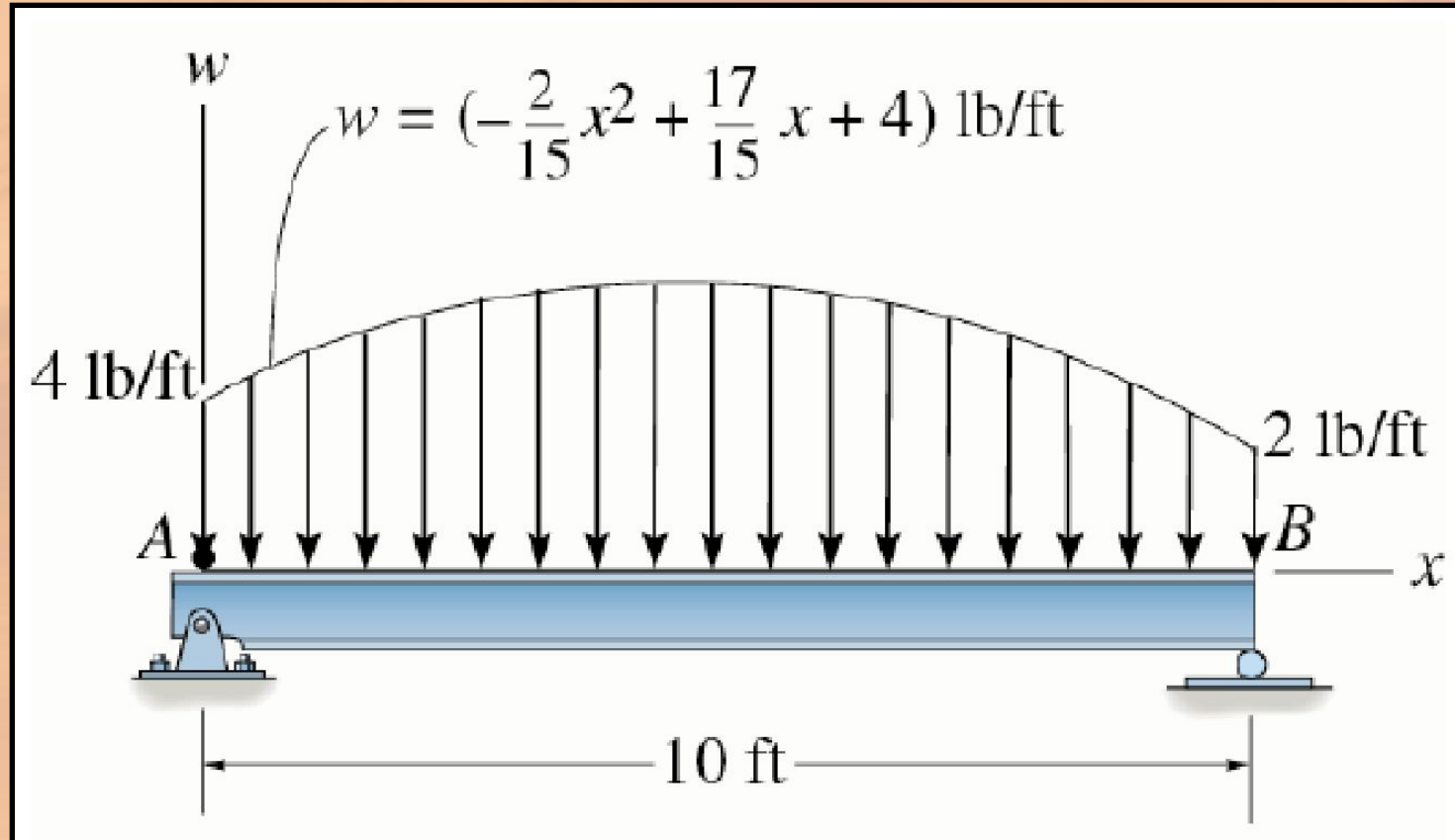
To find the equivalent point of action, we can take moment about A, thus

$$\sum M_A = 0 : \bar{x}(675) = 3(225) + 4.5(450)$$

$$\therefore \bar{x} = 4 \text{ ft}$$

Example:

Determine the location and resultant force on the beam.



$$F_R = \int_L w(x)dx = \int_0^{10} \left(-\frac{2}{15}x^2 + \frac{17}{15}x + 4\right)dx = \frac{470}{9}$$

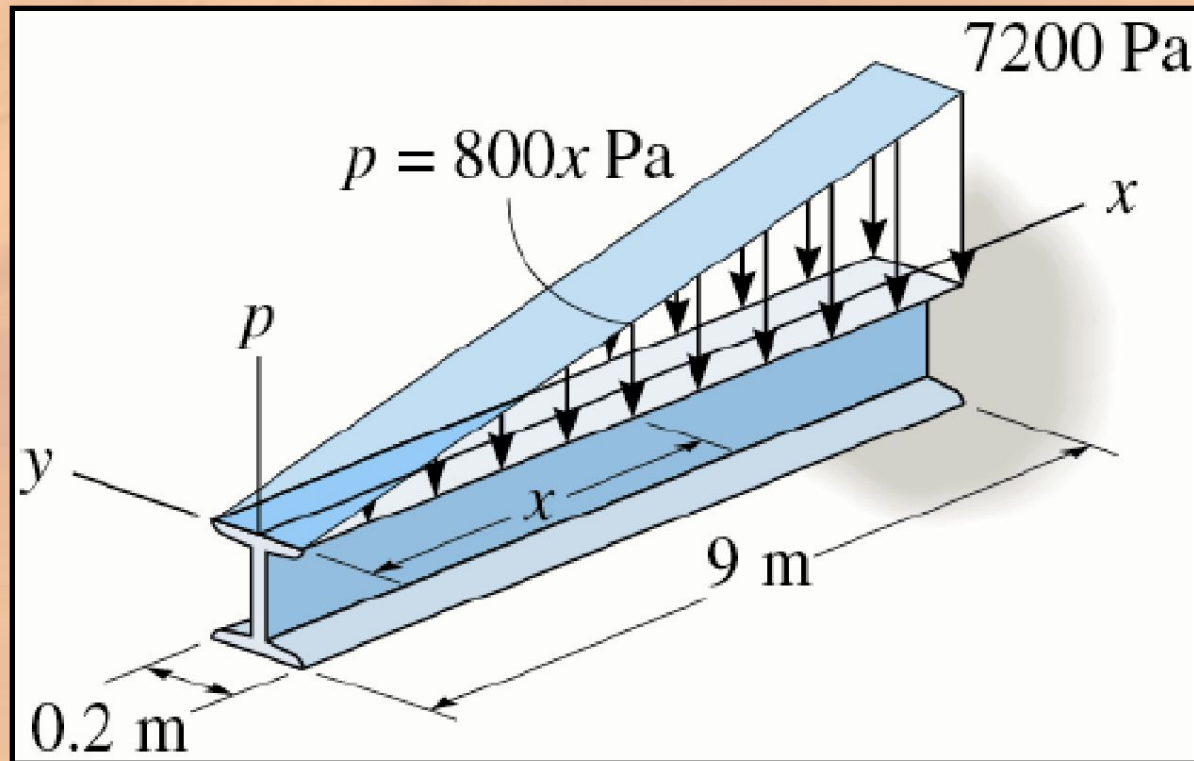
$$F_R = 52.222 \text{ lb}$$

$$\bar{x} = \frac{\int_L xw(x)dx}{\int_L w(x)dx} = \frac{\int_0^{10} x\left(-\frac{2}{15}x^2 + \frac{17}{15}x + 4\right)dx}{\int_0^{10} \left(-\frac{2}{15}x^2 + \frac{17}{15}x + 4\right)dx} = \frac{220}{47}$$

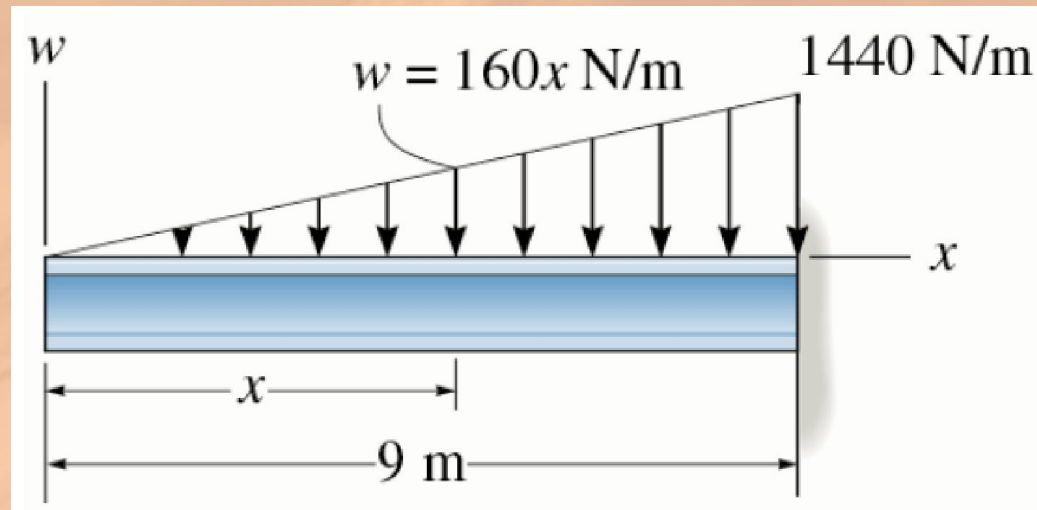
$$\bar{x} = 4.681 \text{ ft}$$

Example:

Determine the magnitude and location of the resultant force.



Solution:



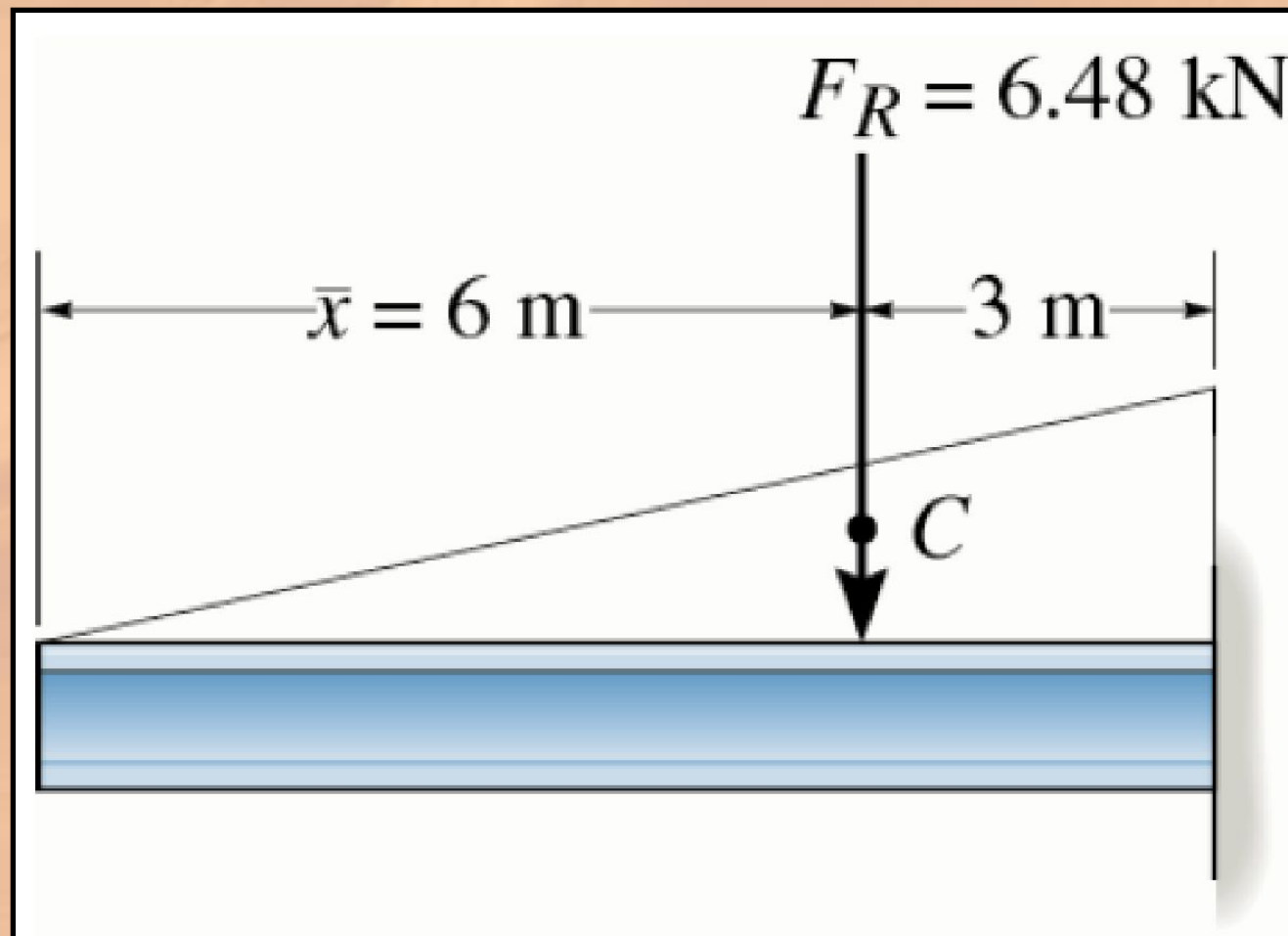
Even though the loading is 3D, the distribution is only a function of  $x$ . Thus

$$w = 800x(0.2) = 160x \text{ N/m}$$

The magnitude of the resultant force

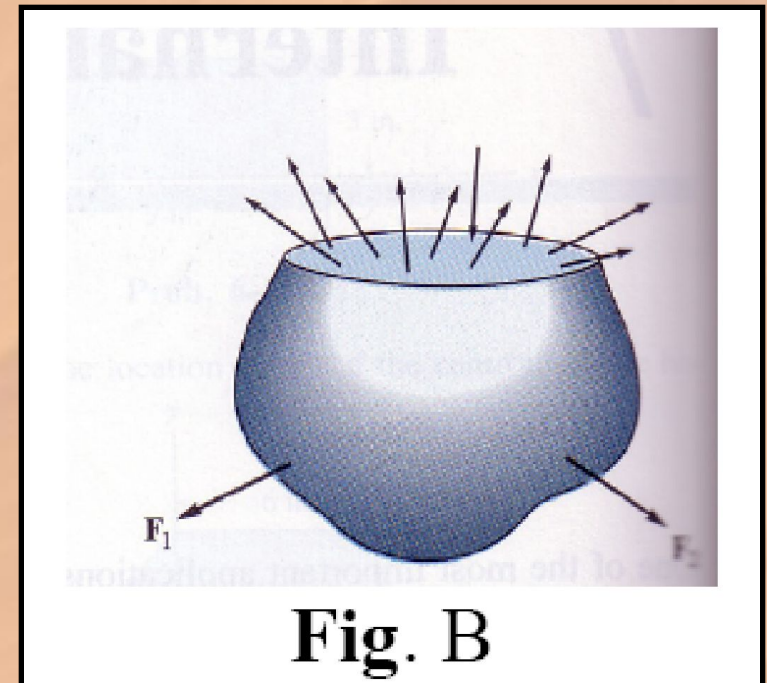
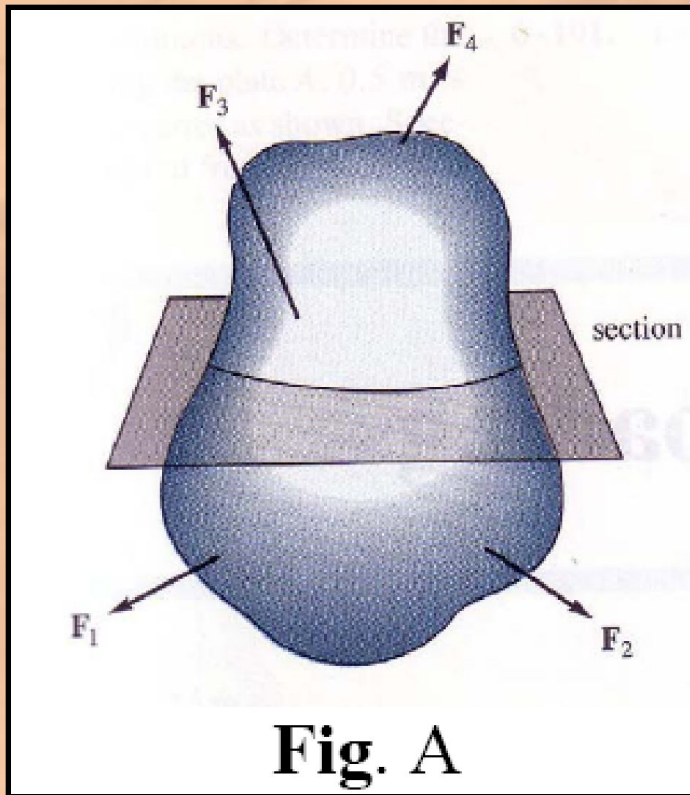
$$F_R = \frac{1}{2}(9)(1440) = 6.48 \text{ kN}$$



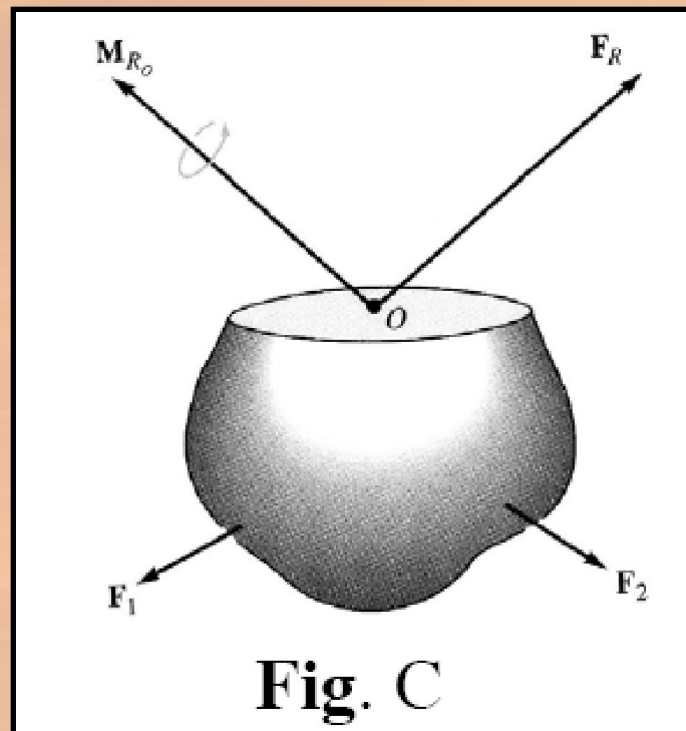


# Internal Forces

- Consider an arbitrary body subjected to the external forces as shown in Fig. A.
- If we make an imaginary cut through the body, we can see that there is a distribution of internal force acting on the 'exposed' area; as shown in Fig. B.



- These forces represent the effect of the material of the top part of the body acting on the adjacent material of the bottom part.
- Although the exact distribution is unknown, we can use statics to determine the resultant internal force ( $F_R$ ) and moment ( $M_R$ ) and at any arbitrary point.



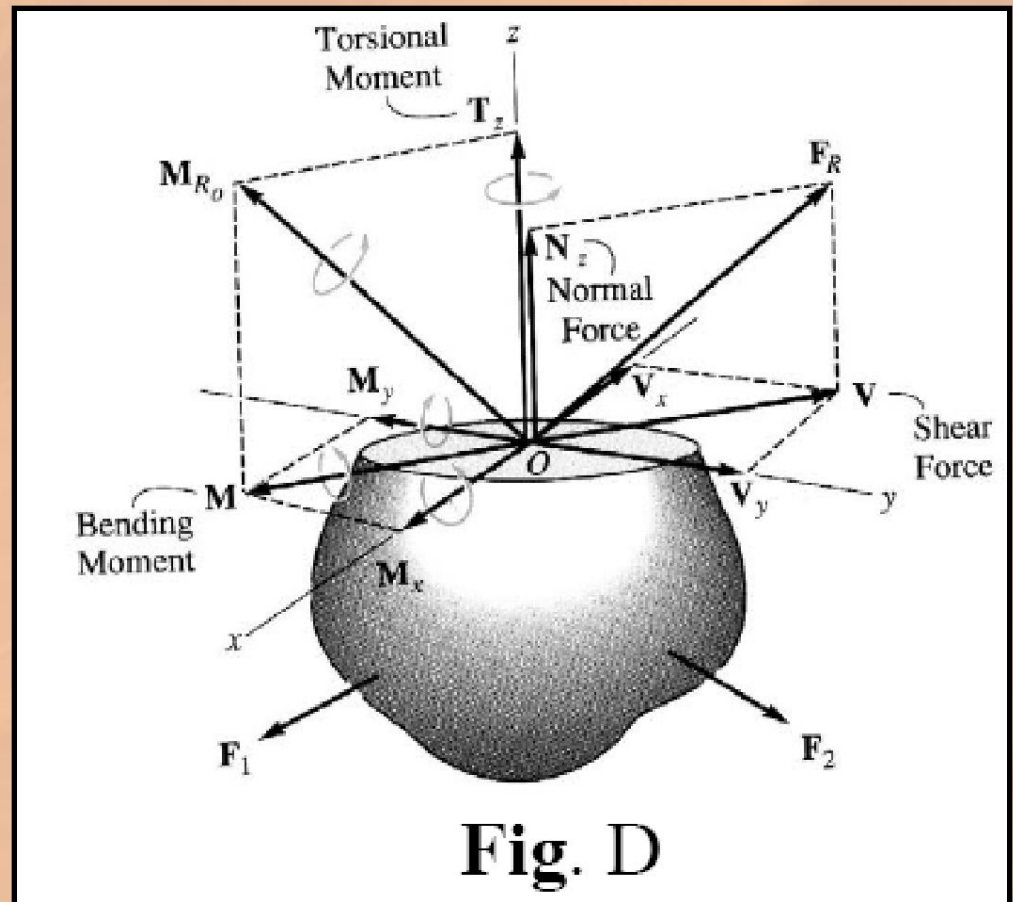
- In fact, these resultant forces,  $F_R$  and  $M_R$ , can be resolved into four components each; as shown in Fig. D, where

N – Normal force

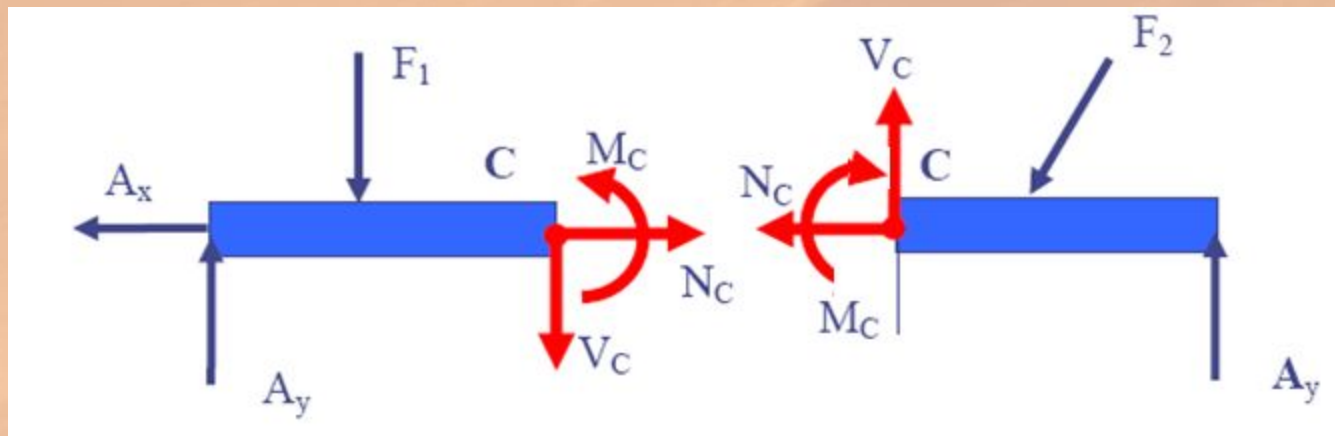
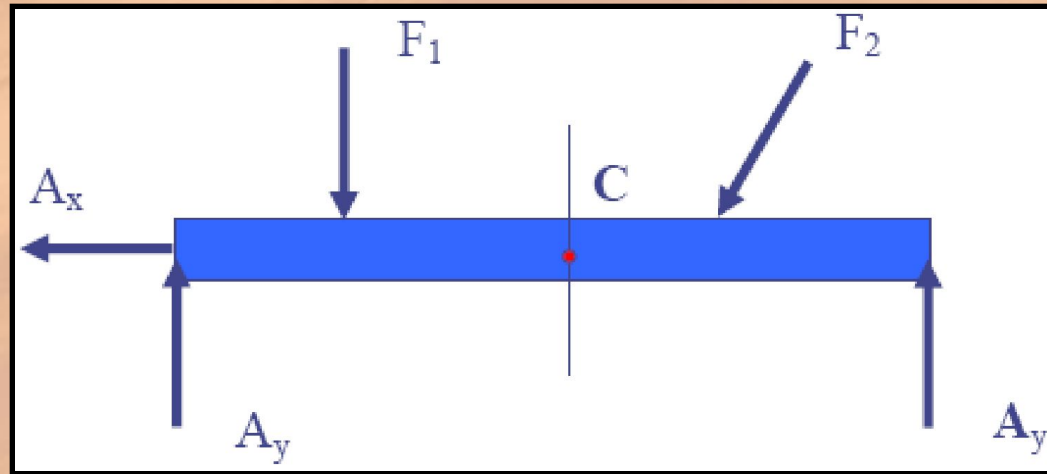
V – Shear force

T – Torsional moment

M – Bending moment



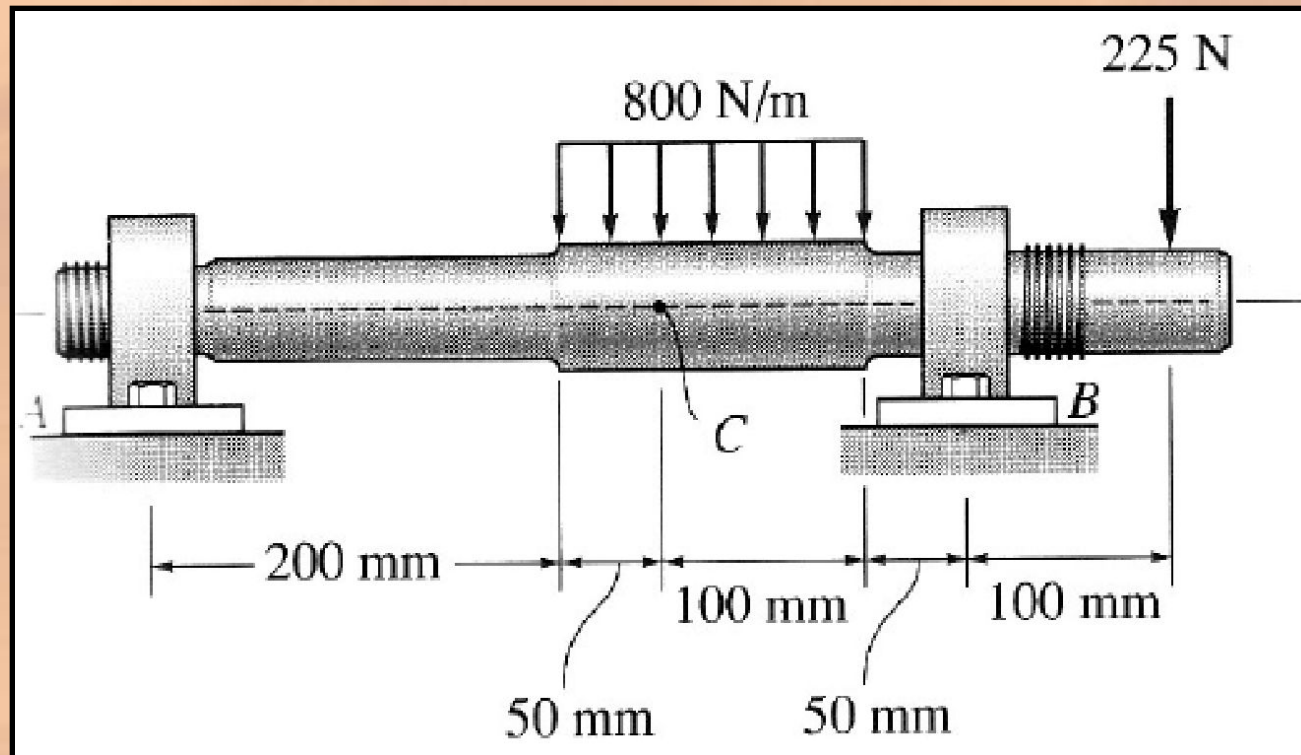
- For analysis of two dimensional bodies, it is sufficient to consider the *normal force*, *shear force* and *bending moment*, i.e. three components only.





Example:

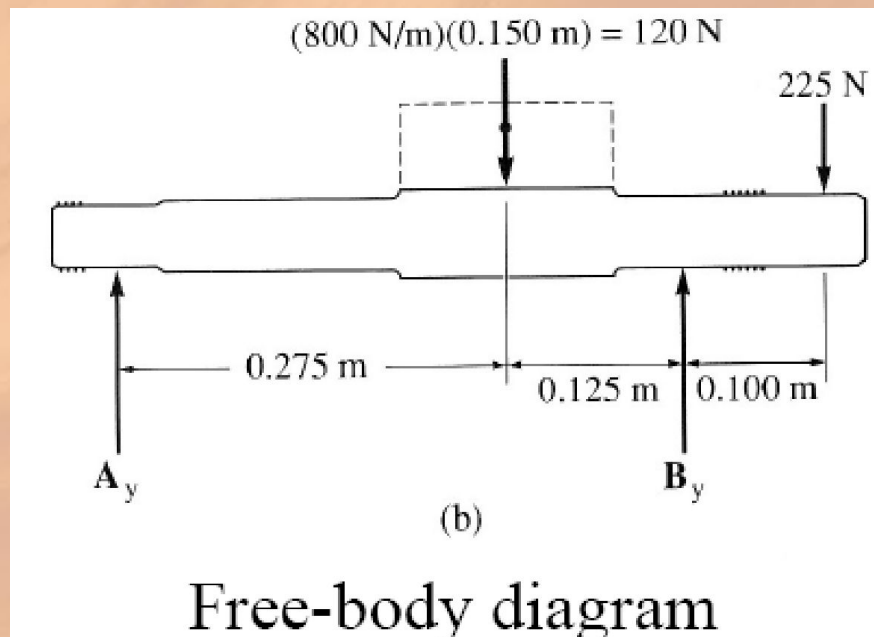
Determine the resultant internal loadings acting on the cross section at  $C$  of the machine shaft shown. The shaft is supported by bearings at  $A$  and  $B$ , which exert only vertical forces on the shaft.





Solution:

First we need to obtain the reaction at the supports.



Equilibrium of moment at A:

$$\sum M_A = 0 = -120(0.275) + 0.4B_y - 225(0.5)$$

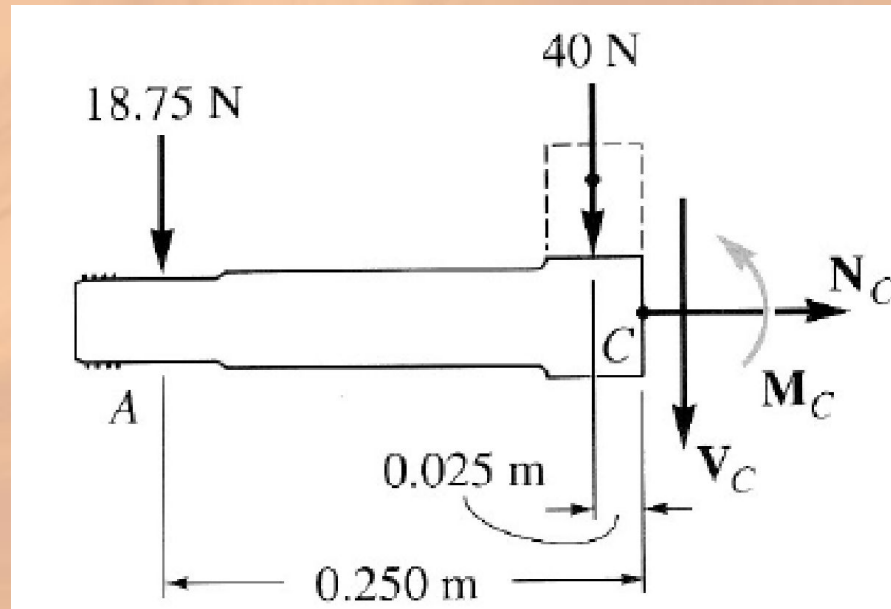
$$\therefore B_y = 363.75 \text{ N}$$

Equilibrium force vertically:

$$\sum F_y = 0 = A_y - 120 + 363.75 - 225$$

$$\therefore A_y = -18.75 \text{ N}$$

In order to analyze the resultant internal loading at C, we section the beam at C.



$$\sum F_x = 0 = N_C$$

$$\sum F_y = 0 = -18.75 - 40 - V_C$$

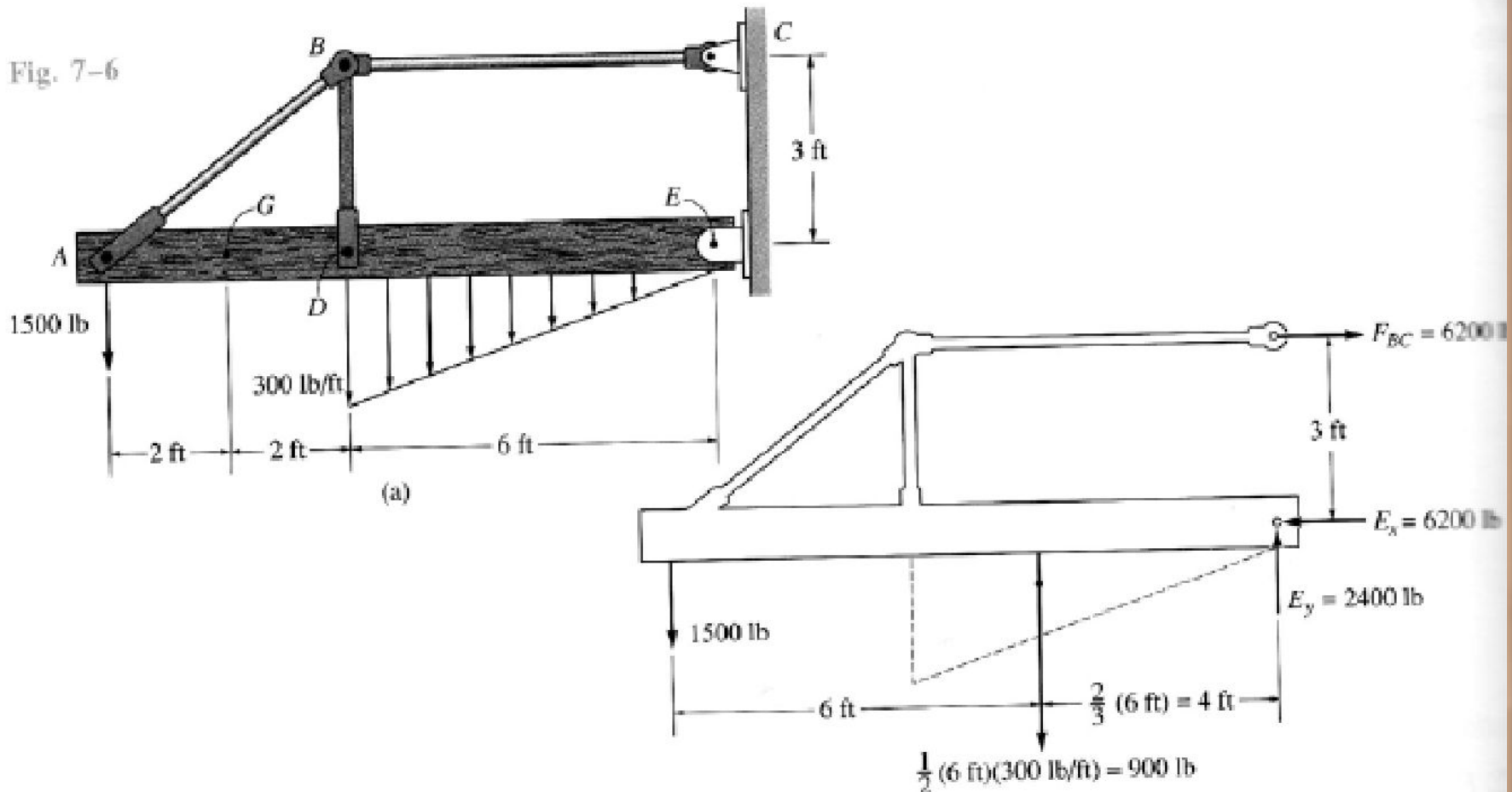
$$\therefore V_C = -58.8 \text{ N}$$

$$\sum M_C = 0 = M_C + 40(0.025) + 18.75(0.250)$$

$$\therefore M_C = -5.69 \text{ Nm}$$

Example:

Determine the resultant internal loadings acting on the cross-section at G of the beam shown. Assume the joints at A, B, C, D and E are pin-connected.



Soln:

First we need to compute the reactions

$$\sum F_x = 0 = -C_x - E_x$$

$$\Rightarrow C_x = -E_x$$

$$\sum F_y = 0 = C_y + E_y - 1500 - 900$$

$$\Rightarrow C_y + E_y = 2400$$

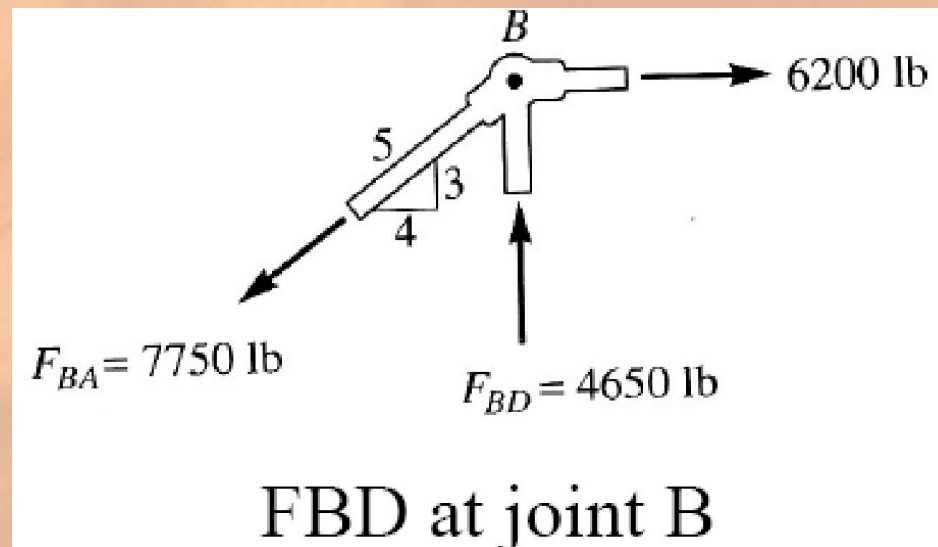
$$\sum M_E = 0 = 900(4) + 1500(10) + 3C_x$$

$$\Rightarrow C_x = 6200\text{lb}$$

$$\text{And } E_x = 6200\text{lb}$$

But we cannot solve  $C_y$  and  $E_y$ . To proceed, if we look at beam AE, it is seen that  $E_y \neq 0$  due to the horizontal external force. The member BC is a two-force member and therefore  $C_y = 0$ . Thus we have  $E_y = 2400$ .

Next, we proceed to analyze joint B.



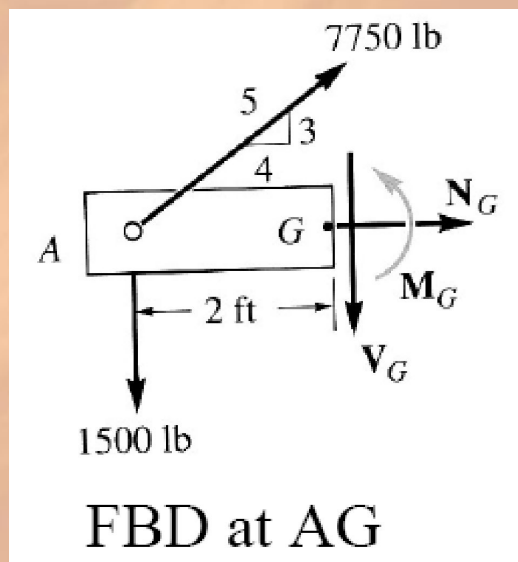
$$\sum F_x = 0 = 6200 - \frac{4}{5} F_{AB}$$

$$\Rightarrow F_{AB} = 7750 \text{ lb}$$

$$\sum F_y = 0 = -\frac{3}{5} F_{AB} + F_{BD}$$

$$\Rightarrow F_{BD} = 4650 \text{ lb}$$

Finally we analyze segment AG



$$\sum F_x = 0 = 7750\left(\frac{4}{5}\right) + N_G$$

$$\Rightarrow N_G = -6200\text{lb}$$

$$\sum F_y = 0 = -1500 + \frac{3}{5}7750 - V_G$$

$$\Rightarrow V_G = 3150\text{lb}$$

$$\sum M_G = 0 = M - \left(\frac{3}{5}\right)(7750)(2) + 1500(2)$$

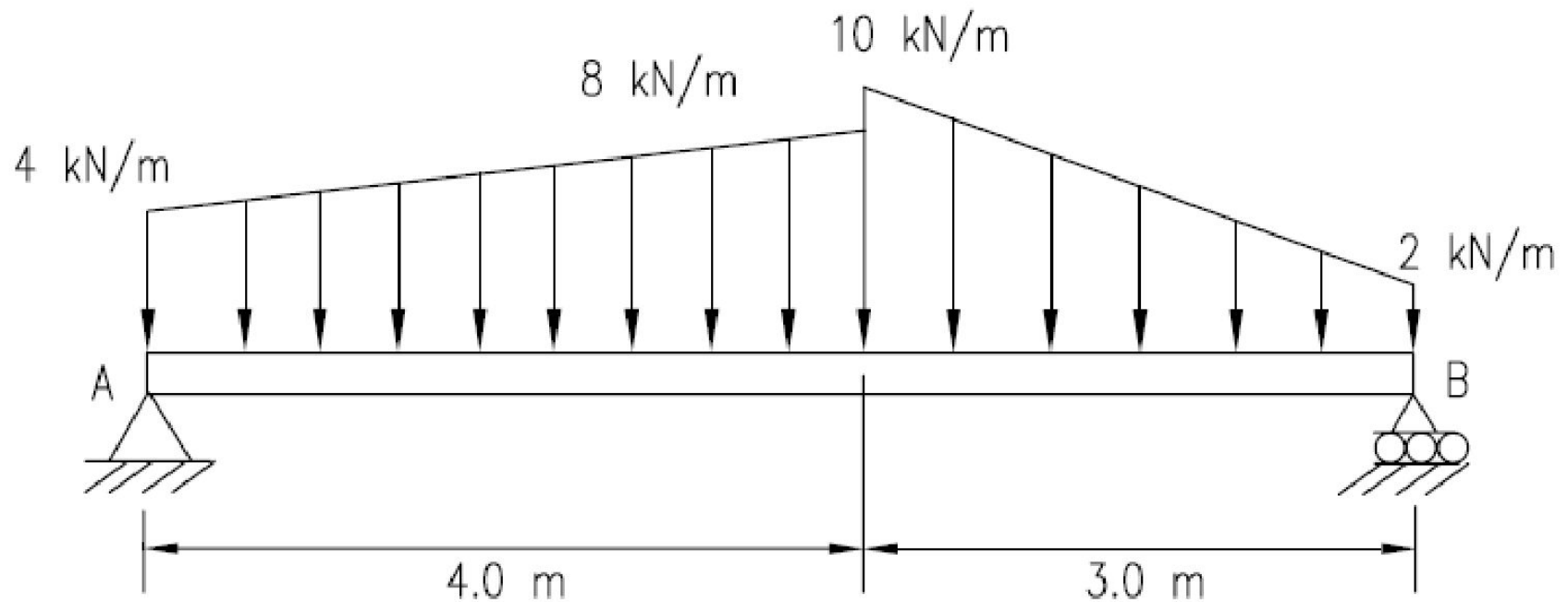
$$\Rightarrow M = 6300 \text{ lb} \cdot \text{ft}$$



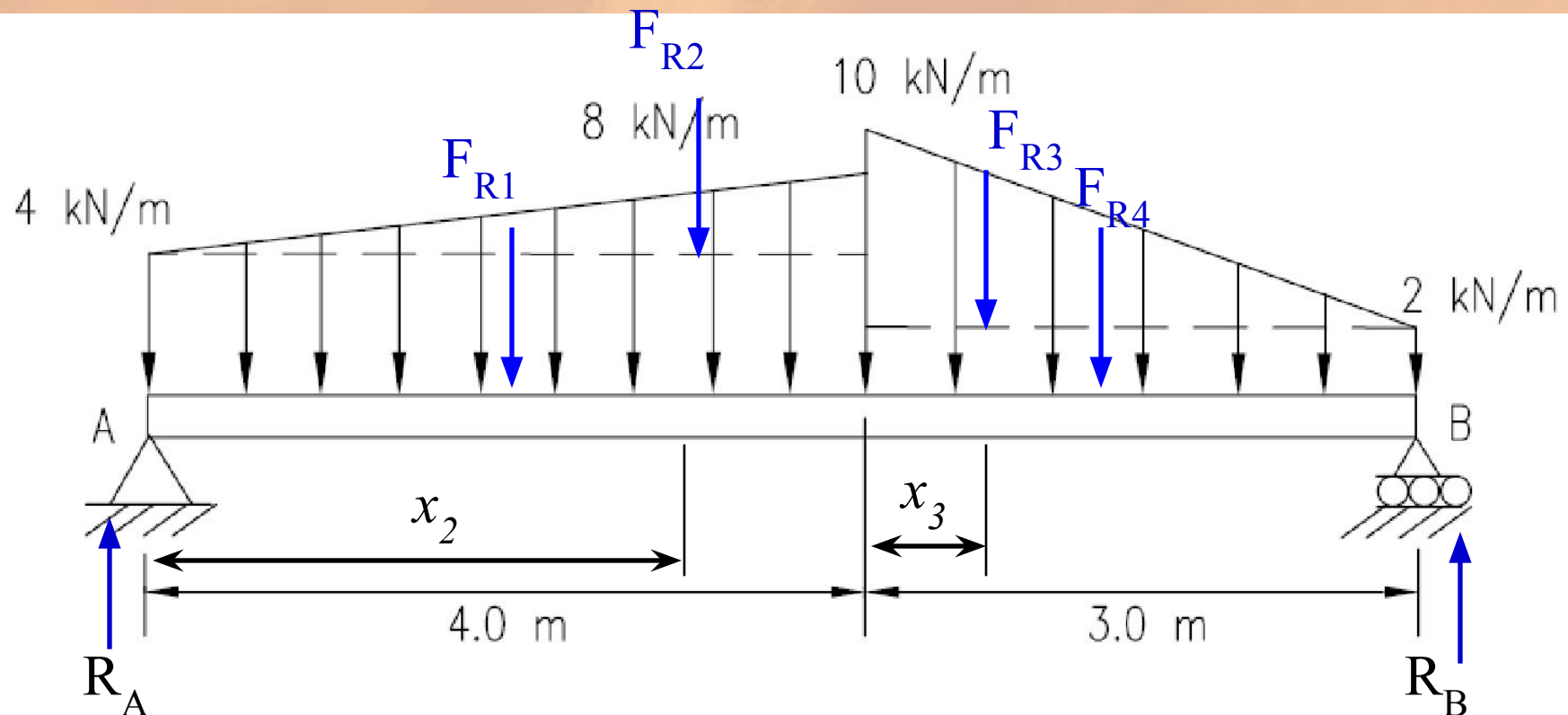
# Extra Example

## **Characteristics of External Loading**

Find the reactions at the supports.

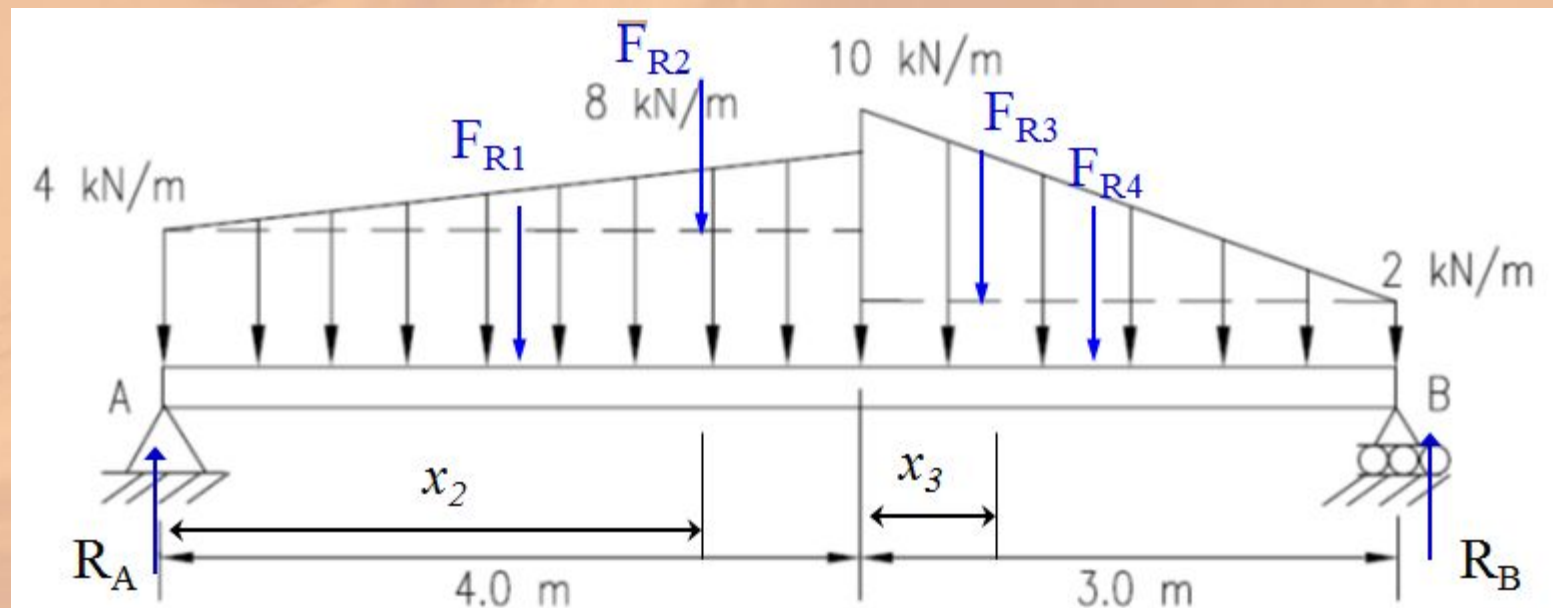


# Characteristics of External Loading



$$\begin{aligned} F_{R1} &= 4 \times 4 = 16 \text{ kN} & x_1 &= 4/2 = 2 \text{ m} \\ F_{R2} &= 0.5 \times 4 \times 4 = 8 \text{ kN} & x_2 &= 2/3 \times 4 = 2.67 \text{ m} \\ F_{R3} &= 0.5 \times 3 \times 8 = 12 \text{ kN} & x_3 &= 1/3 \times 3 = 1 \text{ m} \\ F_{R4} &= 2 \times 3 = 6 \text{ kN} & x_4 &= 3/2 = 1.5 \text{ m} \end{aligned}$$

# Characteristics of External Loading



$$\sum M_A = 0 \quad 7R_B - 16(2) - 8(2.67) - 12(4 + 1) - 6(4 + 1.5) = 0$$

$$R_B = 20.91 \text{ kN}$$

$$\sum F_y = 0 \quad R_B + R_A - 16 - 8 - 12 - 6 = 0$$

$$20.91 + R_A - 42 = 0$$

$$R_A = 21.09 \text{ kN}$$