Cross Sections

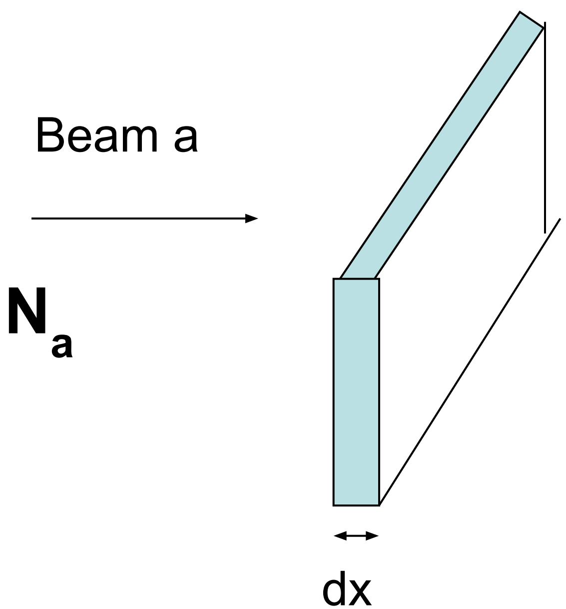
- Definition of Cross Section
  - Why its useful.
- Breit-Wigner Resonances
- Rutherford Scattering

Cross-Sections

- Why concept is important
  - Learn about dynamics of interaction and/or constituents (cf Feynman's watches).
  - Needed for practical calculations.
- Experimental Definition
- How to calculate σ
  - Fermi Golden Rule
  - Breit-Wigner Resonances
  - QM calculation of Rutherford Scattering

Definition of σ

- a+b□x
- Effective area for reaction to occur is σ



$N_a(0)$ particles type a/unit time hit target b

N_b atoms b/unit volume

Number /unit area= N_b dx

Probability interaction = $\sigma N_b dx$

$dN_a = -N_a N_b \, dx \, \sigma$

$N_a(x) = N_a(0) \exp(-x/\lambda)$; $\lambda = 1/(N_b \sigma)$

Reaction Rates

- **N_a beam particles/unit volume, speed v**
- **Flux $F= N_a\ v$**
- **Rate/target b atom $R=F\sigma$**
- **Thin target $x<<\lambda$: $R=(N_b{}^T)\ F\ \sigma^{Total}$**
- This is total cross section. Can also define differential cross sections, as a function of reaction product, energy, transverse momentum, angle etc.
- **dR(a+b□c+d)/dE=($N_b{}^T$) F dσ(a+b□c+d) /dE**

Breit-Wigner Line Shape

- **Start with NR Schrödinger equation:**

$$i\hbar\frac{\partial\psi}{dt} = H\psi \quad ;\psi(t) = \sum_{n=0}^{\infty} a_n(t)\exp(-iE_n t/\hbar)\varphi_n$$

$$i\hbar\dot{a}_n \exp(-iE_n t/\hbar)\,\varphi_n + a_n E_n \exp(-iE_n t/\hbar)\,\varphi_n = \sum_m a_m H \exp(-iE_m t/\hbar)\,\varphi_n$$

X by φ^*_n and integrate $\int \phi_m^{\ *}\phi_n d^3r = \delta_{nm} \quad ; \quad H_{mn} = \int \phi_m^{\ *} H\phi_n d^3r$

$$i\hbar\dot{a}_n \exp(-iE_n t/\hbar) + a_n E_n \exp(-iE_n t/\hbar) = \sum_m a_m H_{mn} \exp(-iE_m t/\hbar)$$

Start in state m $\Rightarrow$ exponential decay $\quad a_m(t) = \exp(-\Gamma t/2\hbar)$

$$|a_m(t)|^2 = \exp(-\Gamma t/\hbar)$$

Breit-Wigner Line Shape - 2

$$i\hbar\dot{a}_n = H_{mn}\exp\{[-i(E_n - E_m) - \Gamma/2]t/\hbar\}$$

$$i\hbar a_n(t) = \int_0^t dt\, H_{mn}\exp[-i(E_n - E_m) - \Gamma/2]t/\hbar$$

$$a_n(t) = \left[\frac{H_{mn}\exp[-i(E_n - E_m) - \Gamma/2]t/\hbar}{-i(E_n - E_m) - \Gamma/2}\right]_0^t$$

For $t >> \hbar/\Gamma$

$$a_n(t) = \frac{H_{mn}}{i(E_n - E_m) + \Gamma/2}$$

Breit-Wigner Line Shape -3

$$|a_n(t)|^2 = \frac{|H_{mn}|^2}{(E_m - E_n)^2 + \Gamma^2/4}$$

$$|a_n(t)|^2 = \frac{2\pi}{\Gamma}|H_{mn}|^2 P(E_m - E_n)$$

Normalised Breit-Wigner line shape

$$P(E_m - E_n) = \frac{\Gamma}{2\pi}\frac{1}{(E_m - E_n)^2 + \Gamma^2/4}$$

Q: where have you seen this shape before?

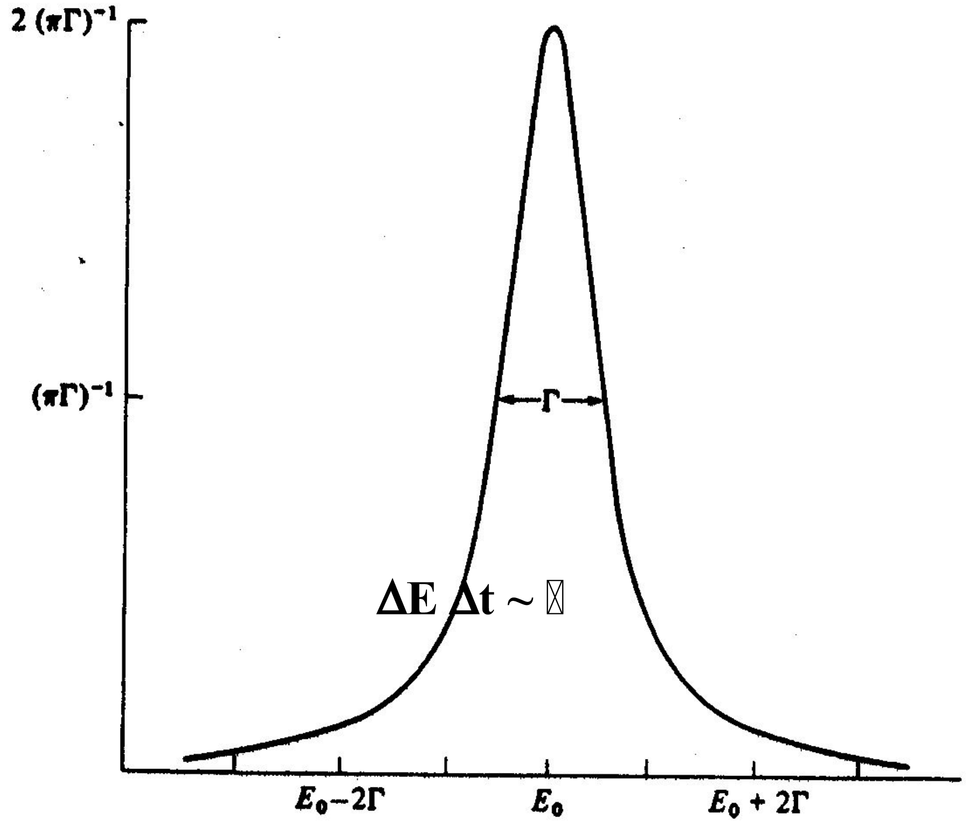
We will see this many times in NP and PP.

Breit-Wigner Resonance

- **Important in atomic, nuclear and particle physics.**
- **Uncertainty relationship**

$$\Delta E \, \Delta t \sim \hbar$$

- **Determine lifetimes of states from width.**
- $\tau \sim \hbar / \Gamma$, Γ**=FWHM;**



Fermi Golden Rule

- **Want to be able to calculate reaction rates in terms of matrix elements of H.**
- **Warning: We will use this many times to calculate σ but derivation not required for exams, given here for completeness.**

Discrete □ Continuum

- Decays to a channel i (range of states n). Density of states $n_i(E)$. Assume narrow resonance

$$P_i = \frac{2\pi}{\Gamma} \int |H_{i0}|^2 \, n_i(E) P(E - E_0) dE$$

$$P_i = \frac{2\pi}{\Gamma} |H_{i0}|^2 n_i(E_0)$$

$$P_i = \frac{\Gamma_i}{\Gamma} ; R_{Total} = \frac{\Gamma}{\hbar} ; R_i = P_i \, R_{Total}$$

$$R_i = \frac{\Gamma_i}{\hbar} = \frac{2\pi}{\hbar} |H_{i0}|^2 n_i(E_0)$$

Cross Section

- **Breit Wigner cross section.**
- **Definition of σ and flux F:**

$$R = F\,\sigma$$

$$\psi = V^{-1/2}\exp(i\vec{k}.\vec{r})$$

$$F = V^{-1}v$$

$$n(k) = \frac{V}{(2\pi)^3}4\pi k^2$$

$$E = \frac{(\hbar k)^2}{2m} \quad ; \quad \frac{dE}{dk} = \hbar v \qquad\qquad n(E) = \frac{V}{(2\pi)^3}\frac{4\pi k^2}{\hbar v}$$

Breit-Wigner Cross Section

$$R = |a_o(t)|^2 \frac{\Gamma_f}{\hbar} = \frac{|H_{01}|^2}{(E_1 - E_0)^2 + \Gamma^2/4} \frac{\Gamma_f}{\hbar}$$

$$\Gamma_i(E) = 2\pi |H_{10}|^2 n(E)$$

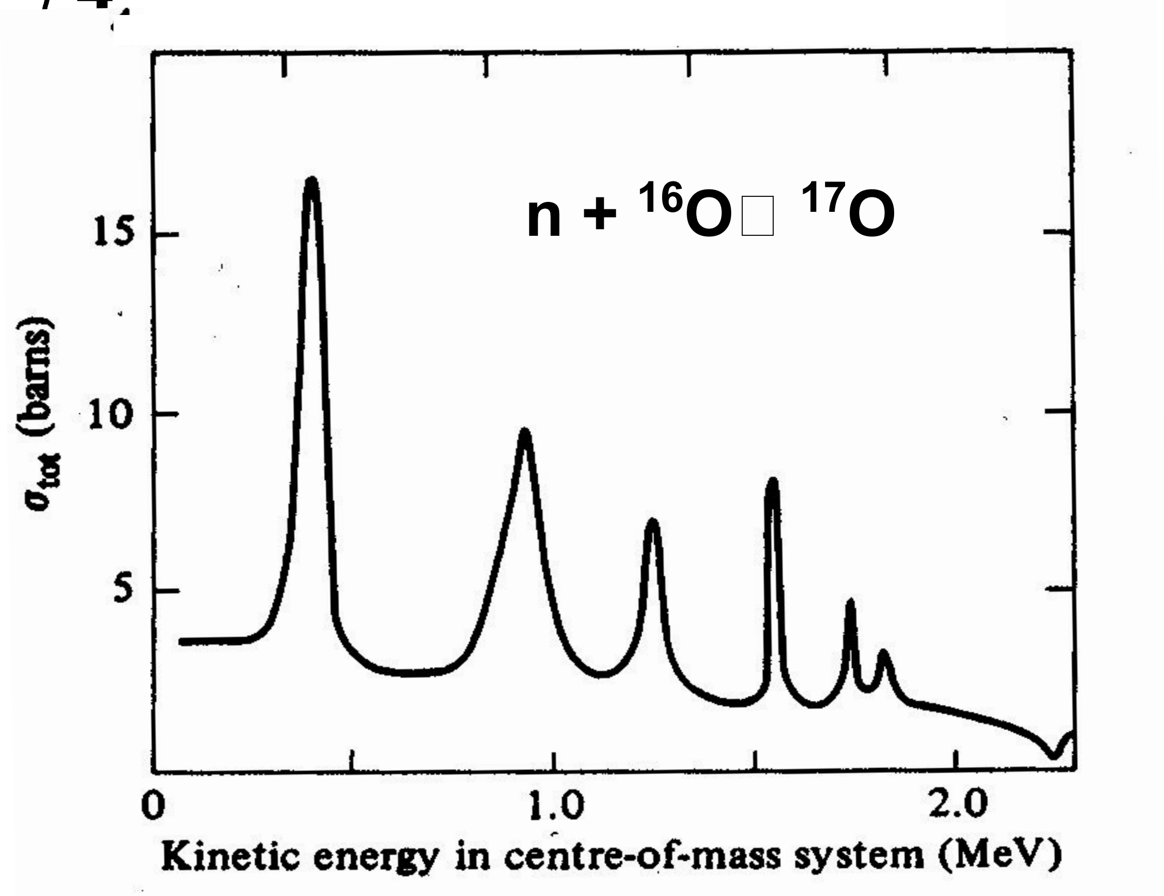
$$R = \frac{1}{2\pi\hbar} \frac{1}{n(E)} \frac{\Gamma_i(E_1)\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$

- Combine rate, flux & density states

$$\sigma = \left(\frac{V}{v}\right)\left(\frac{(2\pi)^3 \hbar v}{V 4\pi k^2}\right) \frac{1}{2\pi\hbar} \frac{\Gamma_i(E_1)\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$

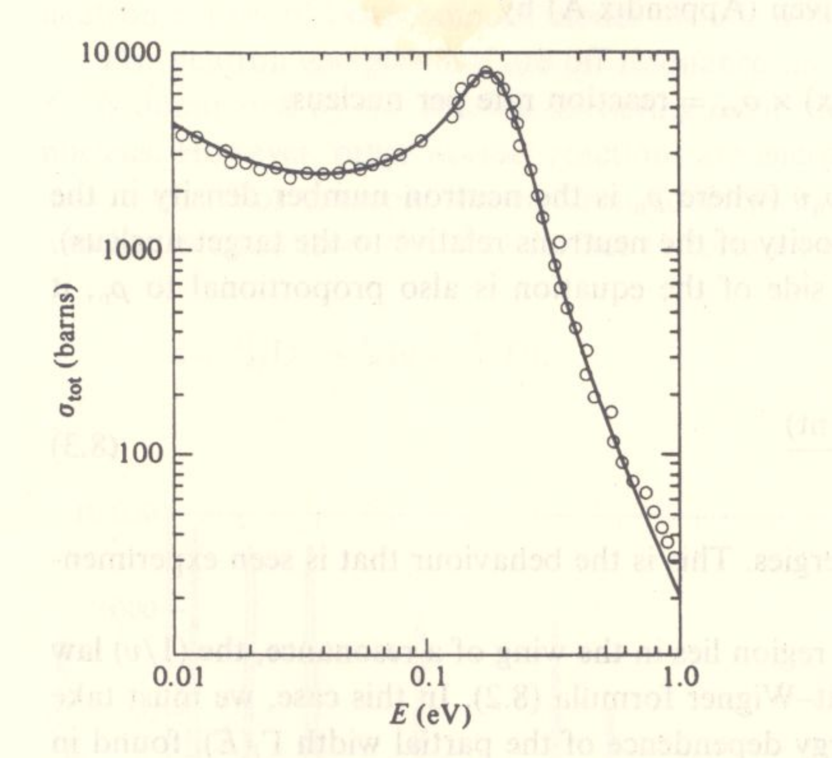
Breit-Wigner Cross Section

$$\sigma = \frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$



Low Energy Resonances

- n + Cd total cross section.
- Cross section scales $\sigma \sim 1/E^{1/2}$ at low E.
- B-W: $1/k^2$ and Γ~n(E)~k



Rutherford Scattering 1

$$V(r) = \frac{Z_1 Z_2 \alpha}{r} \quad ; \quad \alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \quad ; \hbar = c = 1$$

$$\psi_i = V^{-1/2} \exp(i\vec{k}_i.\vec{r}) \quad ; \quad \psi_f = V^{-1/2} \exp(i\vec{k}_f.\vec{r}) \qquad \vec{q} = \vec{k}_i - \vec{k}_f$$

$$H_{fi} = V^{-1} \int \exp(i\vec{k}_i.\vec{r}) \left(\frac{Z_1 Z_2 \alpha}{r} \right) \exp(-i\vec{k}_f.\vec{r}) d^3r$$

$$H_{fi} = V^{-1} Z_1 Z_2 \alpha \int \frac{\exp(i\vec{q}.\vec{r})}{r} d^3r$$

$$H_{fi} = V^{-1} Z_1 Z_2 2\pi\alpha \int \frac{\exp(iqr\cos\theta)}{r} r^2 dr\, d\cos\vartheta$$

Rutherford Scattering 2

$$\mathrm{H_{fi}} = \mathrm{V^{-1} Z_1 Z_2} 2\pi\alpha \int \frac{\exp(\mathrm{iqr}) - \exp(-\mathrm{iqr})}{\mathrm{iqr}^2} \mathrm{r^2 dr}$$

$$\mathrm{xV(r); \exp(-r/a)\ a \to \infty}$$

$$H_{fi} = V^{-1} \frac{Z_1 Z_2 2\pi\alpha}{iq} \int \exp(-1/a + iq)r - \exp(-1/a - iq)r\, dr$$

$$\mathrm{H_{fi}} = \mathrm{V}^{-1} \frac{\mathrm{Z_1 Z_2} 2\pi\alpha}{\mathrm{iq}} \left[\frac{1}{-1/\mathrm{a} + \mathrm{iq}} - \frac{1}{-1/\mathrm{a} - \mathrm{iq}} \right]$$

Rutherford Scattering 3

- **Fermi Golden Rule:** $$R = \frac{2\pi}{\hbar}\left|H_{fi}\right|^2 \frac{dn}{dE_f}$$

$$\frac{dn}{dp} = 4\pi p^2 \frac{V}{h^3}\frac{d\Omega}{4\pi} \quad ; \frac{dn}{dE} = \frac{dn}{dp}\frac{dp}{dE} \quad ; \frac{dp}{dE} = 1/v$$

$$n(E) = \frac{p^2 V}{v(2\pi)^3} d\Omega \quad ; (h = 2\pi\hbar = 2\pi)$$

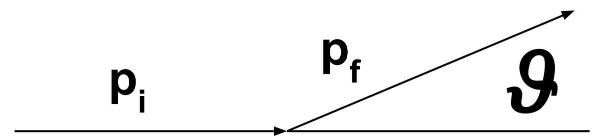
$$\sigma = R/F \quad ; \quad F = V^{-1}v$$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\left|\frac{Z_1 Z_2 4\pi\alpha}{Vq^2}\right|^2 \frac{p^2 V}{v(2\pi)^3}\frac{V}{v}$$

$$\frac{d\sigma}{d\Omega} = \frac{4p^2(Z_1 Z_2 \alpha)^2}{v^2 q^4}$$

Rutherford Scattering 4

$$q^2 = (\vec{p}_i - \vec{p}_f)^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2(\vartheta/2)$$



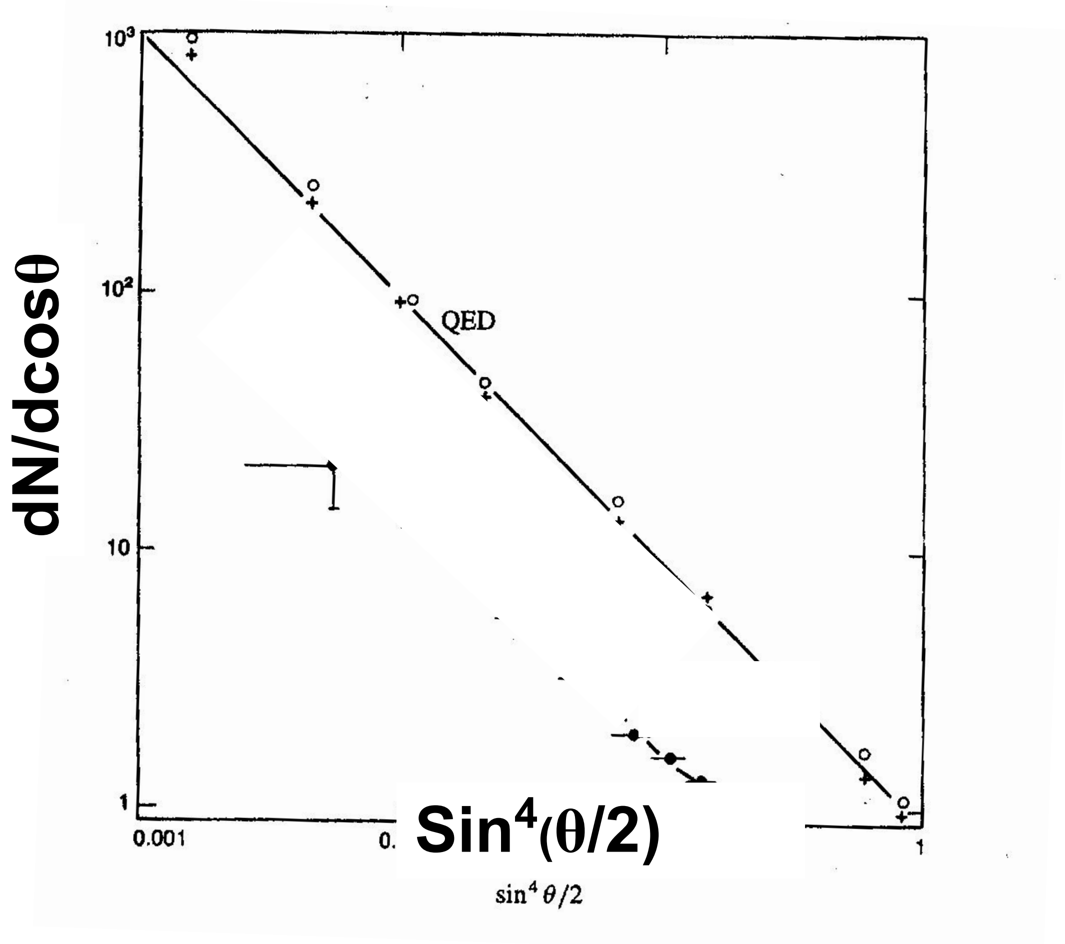
$$\frac{d\sigma}{d\Omega} = \frac{(Z_1 Z_2 \alpha)^2}{4p^2 v^2 \sin^4(\vartheta/2)}$$

Compare with experimental data at low energy

Q: what changes at high energy ?

Low Energy Experiment

- Scattering of α on Au & Ag agree with calculation assuming point nucleus



Higher Energy

- Deviation from Rutherford scattering at higher energy determine charge distribution in the nucleus.
- Form factors is F.T. of charge distribution.

Electron - Gold