

Cross Sections

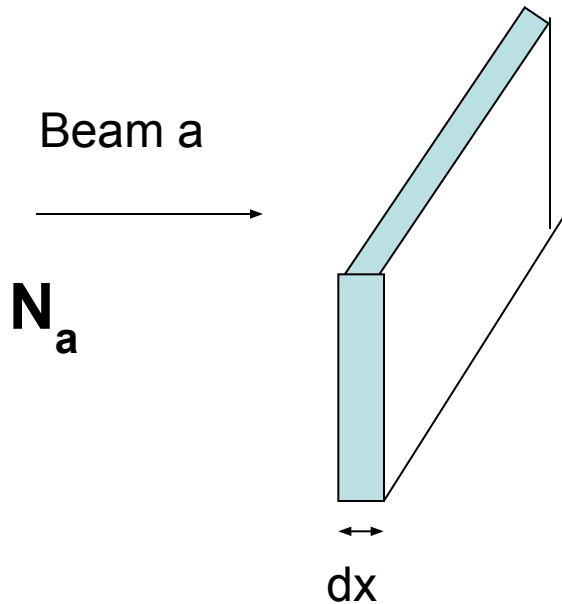
- **Definition of Cross Section**
 - Why its useful.
- **Breit-Wigner Resonances**
- **Rutherford Scattering**

Cross-Sections

- **Why concept is important**
 - Learn about dynamics of interaction and/or constituents (cf Feynman's watches).
 - Needed for practical calculations.
- **Experimental Definition**
- **How to calculate σ**
 - Fermi Golden Rule
 - Breit-Wigner Resonances
 - QM calculation of Rutherford Scattering

Definition of σ

- $a+b \rightarrow x$
- Effective area for reaction to occur is σ



$N_a(0)$ particles type a/unit time hit target b

N_b atoms b/unit volume

Number /unit area= $N_b dx$

Probability interaction = $\sigma N_b dx$

$dN_a = -N_a N_b dx \sigma$

$N_a(x) = N_a(0) \exp(-x/\lambda) ; \lambda = 1/(N_b \sigma)$

Reaction Rates

- N_a beam particles/unit volume, speed v
- Flux $F = N_a v$
- Rate/target b atom $R = F \sigma$
- Thin target $x \ll \lambda$: $R = (N_b T) F \sigma^{\text{Total}}$
- This is total cross section. Can also define differential cross sections, as a function of reaction product, energy, transverse momentum, angle etc.
- $dR(a+b \rightarrow c+d)/dE = (N_b T) F d\sigma(a+b \rightarrow c+d)/dE$

Breit-Wigner Line Shape

- Start with NR Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi \quad ; \quad \psi(t) = \sum_{n=0}^{\infty} a_n(t) \exp(-iE_n t / \hbar) \phi_n$$

$$i\hbar \dot{a}_n \exp(-iE_n t / \hbar) \phi_n + a_n E_n \exp(-iE_n t / \hbar) \phi_n = \sum_m a_m H \exp(-iE_m t / \hbar) \phi_n$$

X by ϕ_n^* and integrate $\int \phi_m^* \phi_n d^3 r = \delta_{nm}$; $H_{mn} = \int \phi_m^* H \phi_n d^3 r$

$$i\hbar \dot{a}_n \exp(-iE_n t / \hbar) + a_n E_n \exp(-iE_n t / \hbar) = \sum_m a_m H_{mn} \exp(-iE_m t / \hbar)$$

Start in state m \square exponential decay $a_m(t) = \exp(-\Gamma t / 2\hbar)$

$$|a_m(t)|^2 = \exp(-\Gamma t / \hbar)$$

Breit-Wigner Line Shape - 2

$$i\hbar \dot{a}_n = H_{mn} \exp\{-i(E_n - E_m) - \Gamma/2\}t/\hbar\}$$

$$i\hbar a_n(t) = \int_0^t dt H_{mn} \exp[-i(E_n - E_m) - \Gamma/2]t/\hbar$$

$$a_n(t) = \left[\frac{H_{mn} \exp[-i(E_n - E_m) - \Gamma/2]t/\hbar}{-i(E_n - E_m) - \Gamma/2} \right]_0^t$$

For $t \gg \hbar / \Gamma$

$$a_n(t) = \frac{H_{mn}}{i(E_n - E_m) + \Gamma/2}$$

Breit-Wigner Line Shape -3

$$|a_n(t)|^2 = \frac{|H_{mn}|^2}{(E_m - E_n)^2 + \Gamma^2 / 4}$$

$$|a_n(t)|^2 = \frac{2\pi}{\Gamma} |H_{mn}|^2 P(E_m - E_n)$$

Normalised Breit-Wigner line shape

$$P(E_m - E_n) = \frac{\Gamma}{2\pi} \frac{1}{(E_m - E_n)^2 + \Gamma^2 / 4}$$

Q: where have you seen this shape before?

We will see this many times in NP and PP.

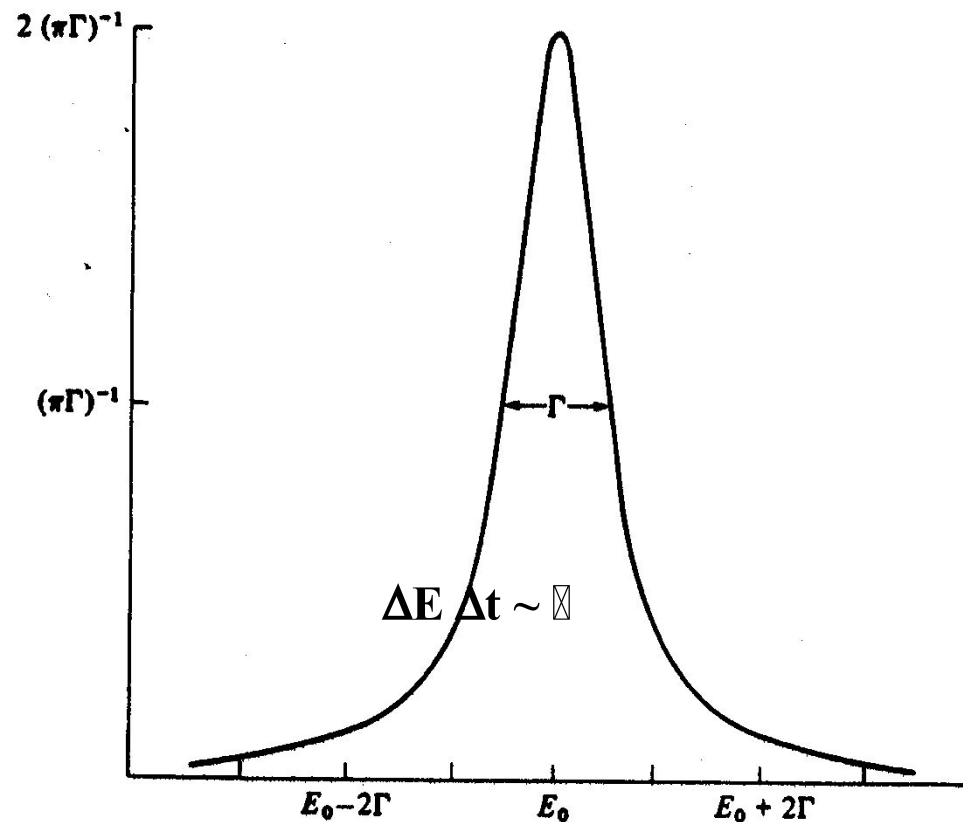
Breit-Wigner Resonance

- Important in atomic, nuclear and particle physics.
- Uncertainty relationship

$$\Delta E \Delta t \sim \hbar$$

- Determine lifetimes of states from width.

- $\tau \sim \hbar / \Gamma$, $\Gamma = \text{FWHM}$;



Fermi Golden Rule

- Want to be able to calculate reaction rates in terms of matrix elements of H .
- **Warning: We will use this many times to calculate σ but derivation not required for exams, given here for completeness.**

Discrete \square Continuum

- Decays to a channel i (range of states n). Density of states $n_i(E)$. Assume narrow resonance

$$P_i = \frac{2\pi}{\Gamma} \int |H_{i0}|^2 n_i(E) P(E - E_0) dE$$

$$P_i = \frac{2\pi}{\Gamma} |H_{i0}|^2 n_i(E_0)$$

$$P_i = \frac{\Gamma_i}{\Gamma} ; R_{\text{Total}} = \frac{\Gamma}{\hbar} ; R_i = P_i R_{\text{Total}}$$

$$R_i = \frac{\Gamma_i}{\hbar} = \frac{2\pi}{\hbar} |H_{i0}|^2 n_i(E_0)$$

Cross Section

- Breit Wigner cross section.
- Definition of σ and flux F :

$$R = F \sigma$$

$$\psi = V^{-1/2} \exp(i\mathbf{k} \cdot \mathbf{r})$$

$$\mathbf{F} = V^{-1} \mathbf{v}$$

$$n(\mathbf{k}) = \frac{V}{(2\pi)^3} 4\pi k^2$$

$$E = \frac{(\hbar k)^2}{2m} \quad ; \quad \frac{dE}{dk} = \hbar v$$

$$n(E) = \frac{V}{(2\pi)^3} \frac{4\pi k^2}{\hbar v}$$

Breit-Wigner Cross Section

$$R = |\mathbf{a}_0(\mathbf{t})|^2 \frac{\Gamma_f}{\hbar} = \frac{|H_{01}|^2}{(\mathbf{E}_1 - \mathbf{E}_0)^2 + \Gamma^2 / 4} \frac{\Gamma_f}{\hbar}$$

$$\Gamma_i(\mathbf{E}) = 2\pi |H_{10}|^2 n(\mathbf{E})$$

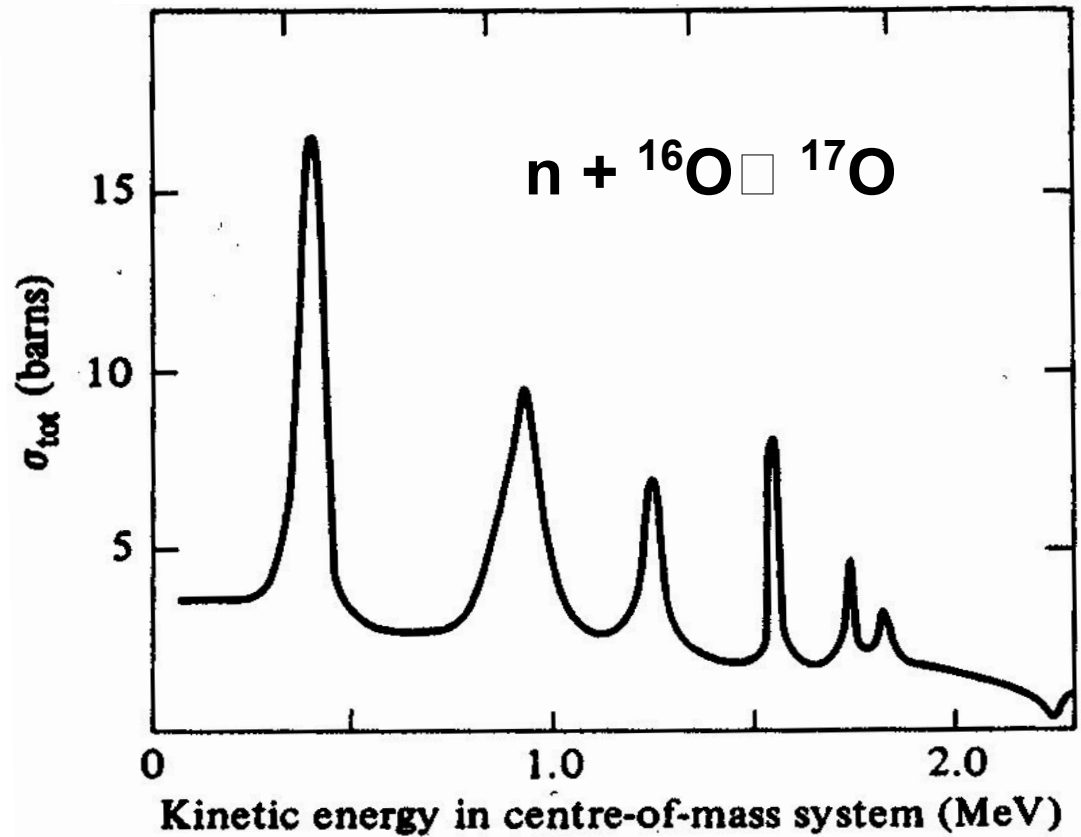
$$R = \frac{1}{2\pi\hbar} \frac{1}{n(\mathbf{E})} \frac{\Gamma_i(\mathbf{E}_1)\Gamma_f}{(\mathbf{E}_1 - \mathbf{E}_0)^2 + \Gamma^2 / 4}$$

- Combine rate, flux & density states \square

$$\sigma = \left(\frac{V}{v} \right) \left(\frac{(2\pi)^3 \hbar^3 v}{V 4\pi k^2} \right) \frac{1}{2\pi\hbar} \frac{\Gamma_i(\mathbf{E}_1)\Gamma_f}{(\mathbf{E}_1 - \mathbf{E}_0)^2 + \Gamma^2 / 4}$$

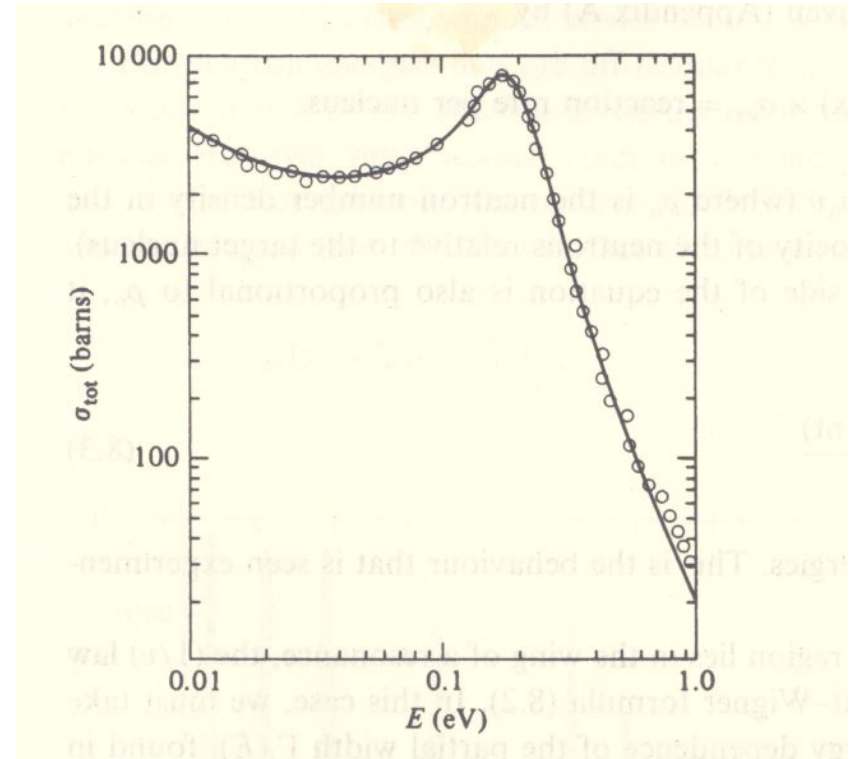
Breit-Wigner Cross Section

$$\sigma = \frac{\pi}{k^2} \frac{\Gamma_i \Gamma_f}{(E_1 - E_0)^2 + \Gamma^2 / 4}$$



Low Energy Resonances

- $n + \text{Cd}$ total cross section.
- Cross section scales $\sigma \sim 1/E^{1/2}$ at low E .
- **B-W:** $1/k^2$ and $\Gamma \sim n(E) \sim k$



Rutherford Scattering 1

$$V(\mathbf{r}) = \frac{Z_1 Z_2 \alpha}{r} \quad ; \quad \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \quad ; \quad \hbar = c = 1$$

$$\psi_i = V^{-1/2} \exp(i\mathbf{k}_i \cdot \mathbf{r}) \quad ; \quad \psi_f = V^{-1/2} \exp(i\mathbf{k}_f \cdot \mathbf{r}) \quad \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$$

$$H_{fi} = V^{-1} \int \exp(i\mathbf{k}_i \cdot \mathbf{r}) \left(\frac{Z_1 Z_2 \alpha}{r} \right) \exp(-i\mathbf{k}_f \cdot \mathbf{r}) d^3 r$$

$$H_{fi} = V^{-1} Z_1 Z_2 \alpha \int \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{r} d^3 r$$

$$H_{fi} = V^{-1} Z_1 Z_2 2\pi\alpha \int \frac{\exp(iqr \cos\theta)}{r} r^2 dr d\cos\theta$$

Rutherford Scattering 2

$$H_{fi} = V^{-1} Z_1 Z_2 2\pi\alpha \int \frac{\exp(iqr) - \exp(-iqr)}{iqr^2} r^2 dr$$

$$x V(r); \exp(-r/a) \quad a \rightarrow \infty$$

$$H_{fi} = V^{-1} \frac{Z_1 Z_2 2\pi\alpha}{iq} \int \exp(-1/a + iq)r - \exp(-1/a - iq)r dr$$

$$H_{fi} = V^{-1} \frac{Z_1 Z_2 2\pi\alpha}{iq} \left[\frac{1}{-1/a + iq} - \frac{1}{-1/a - iq} \right]$$

Rutherford Scattering 3

- Fermi Golden Rule:

$$R = \frac{2\pi}{\hbar} |\mathbf{H}_{fi}|^2 \frac{dn}{dE_f}$$

$$\frac{dn}{dp} = 4\pi p^2 \frac{V}{h^3} \frac{d\Omega}{4\pi} \quad ; \quad \frac{dn}{dE} = \frac{dn}{dp} \frac{dp}{dE} \quad ; \quad \frac{dp}{dE} = 1/v$$

$$n(E) = \frac{p^2 V}{v(2\pi\hbar)^3} d\Omega \quad ; \quad (\hbar = 2\pi\hbar = 2\pi)$$

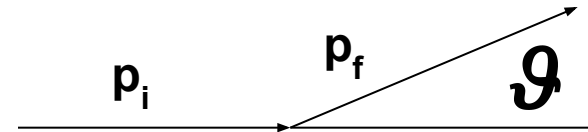
$$\sigma = R / F \quad ; \quad F = V^{-1} v$$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} \left| \frac{Z_1 Z_2 4\pi\alpha}{Vq^2} \right|^2 \frac{p^2 V}{v(2\pi\hbar)^3} \frac{v}{v}$$

$$\frac{d\sigma}{d\Omega} = \frac{4p^2 (Z_1 Z_2 \alpha)^2}{v^2 q^4}$$

Rutherford Scattering 4

$$q^2 = (\vec{p}_i - \vec{p}_f)^2 = 2p^2(1 - \cos\theta) = 4p^2 \sin^2(\theta/2)$$



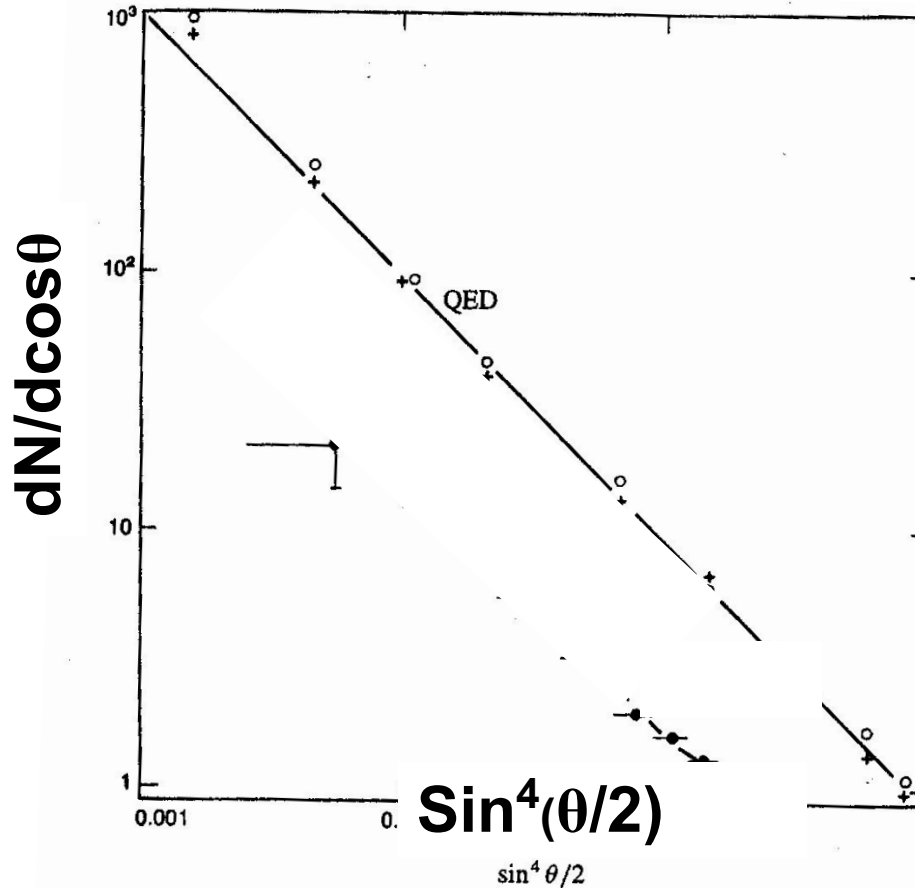
$$\frac{d\sigma}{d\Omega} = \frac{(Z_1 Z_2 \alpha)^2}{4p^2 v^2 \sin^4(\theta/2)}$$

Compare with experimental data at low energy

Q: what changes at high energy ?

Low Energy Experiment

- Scattering of α on Au & Ag agree with calculation assuming point nucleus



Higher Energy

- Deviation from Rutherford scattering at higher energy \square determine charge distribution in the nucleus.
- Form factors is F.T. of charge distribution.

Electron - Gold

