Cross Sections

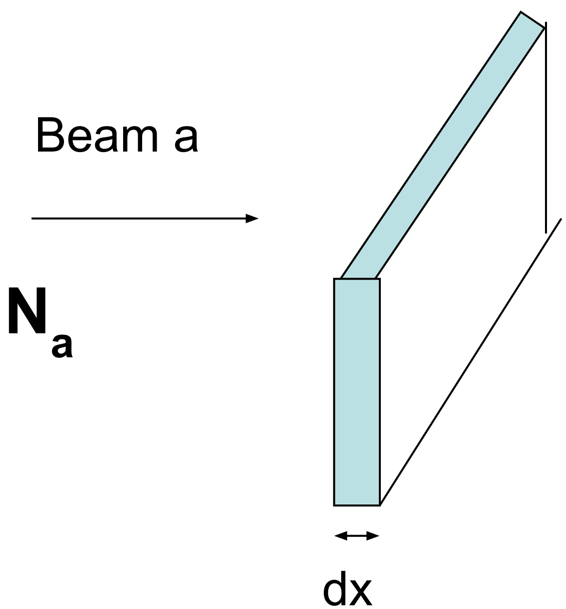
- Definition of Cross Section
  - Why its useful.
- Breit-Wigner Resonances
- Rutherford Scattering

Cross-Sections

- Why concept is important
  - Learn about dynamics of interaction and/or constituents (cf Feynman's watches).
  - Needed for practical calculations.
- Experimental Definition
- How to calculate σ
  - Fermi Golden Rule
  - Breit-Wigner Resonances
  - QM calculation of Rutherford Scattering

Definition of σ

- a+b□x
- Effective area for reaction to occur is σ



$N_a(0)$ particles type a/unit time hit target b

N_b atoms b/unit volume

Number /unit area= N_b dx

Probability interaction = $\sigma N_b dx$

$dN_a = -N_a N_b \, dx \, \sigma$

$N_a(x) = N_a(0) \exp(-x/\lambda)$; $\lambda = 1/(N_b \sigma)$

Reaction Rates

- N_a beam particles/unit volume, speed v
- Flux $F = N_a v$
- Rate/target b atom $R = F\sigma$
- Thin target $x \ll \lambda$: $R = (N_b^T)\, F\, \sigma^{Total}$
- This is total cross section. Can also define differential cross sections, as a function of reaction product, energy, transverse momentum, angle etc.
- **dR(a+b□c+d)/dE=(N_b^T) F dσ(a+b□c+d) /dE**

Breit-Wigner Line Shape

- **Start with NR Schrödinger equation:**

$$i\hbar\frac{\partial\psi}{dt} = H\psi \quad ;\psi(t) = \sum_{n=0}^{\infty} a_n(t)\exp(-iE_n t/\hbar)\varphi_n$$

$$i\hbar\dot{a}_n \exp(-iE_n t/\hbar)\,\varphi_n + a_n E_n \exp(-iE_n t/\hbar)\,\varphi_n = \sum_m a_m H \exp(-iE_m t/\hbar)\,\varphi_n$$

X by φ^*_n and integrate $\int \phi_m^{\,*}\phi_n d^3r = \delta_{nm} \quad ; \quad H_{mn} = \int \phi_m^{\,*} H\phi_n d^3r$

$$i\hbar\dot{a}_n \exp(-iE_n t/\hbar) + a_n E_n \exp(-iE_n t/\hbar) = \sum_m a_m H_{mn} \exp(-iE_m t/\hbar)$$

Start in state m □ exponential decay $\quad a_m(t) = \exp(-\Gamma t/2\hbar)$

$$|a_m(t)|^2 = \exp(-\Gamma t/\hbar)$$

Breit-Wigner Line Shape - 2

$$i\hbar\dot{a}_n = H_{mn}\exp\{[-i(E_n - E_m) - \Gamma/2]t/\hbar\}$$

$$i\hbar a_n(t) = \int_0^t dt\, H_{mn}\exp[-i(E_n - E_m) - \Gamma/2]t/\hbar$$

$$a_n(t) = \left[\frac{H_{mn}\exp[-i(E_n - E_m) - \Gamma/2]t/\hbar}{-i(E_n - E_m) - \Gamma/2}\right]_0^t$$

For $t \gg \hbar/\Gamma$

$$a_n(t) = \frac{H_{mn}}{i(E_n - E_m) + \Gamma/2}$$

Breit-Wigner Line Shape -3

$$|a_n(t)|^2 = \frac{|H_{mn}|^2}{(E_m - E_n)^2 + \Gamma^2/4}$$

$$|a_n(t)|^2 = \frac{2\pi}{\Gamma}|H_{mn}|^2 P(E_m - E_n)$$

Normalised Breit-Wigner line shape

$$P(E_m - E_n) = \frac{\Gamma}{2\pi}\frac{1}{(E_m - E_n)^2 + \Gamma^2/4}$$

Q: where have you seen this shape before?

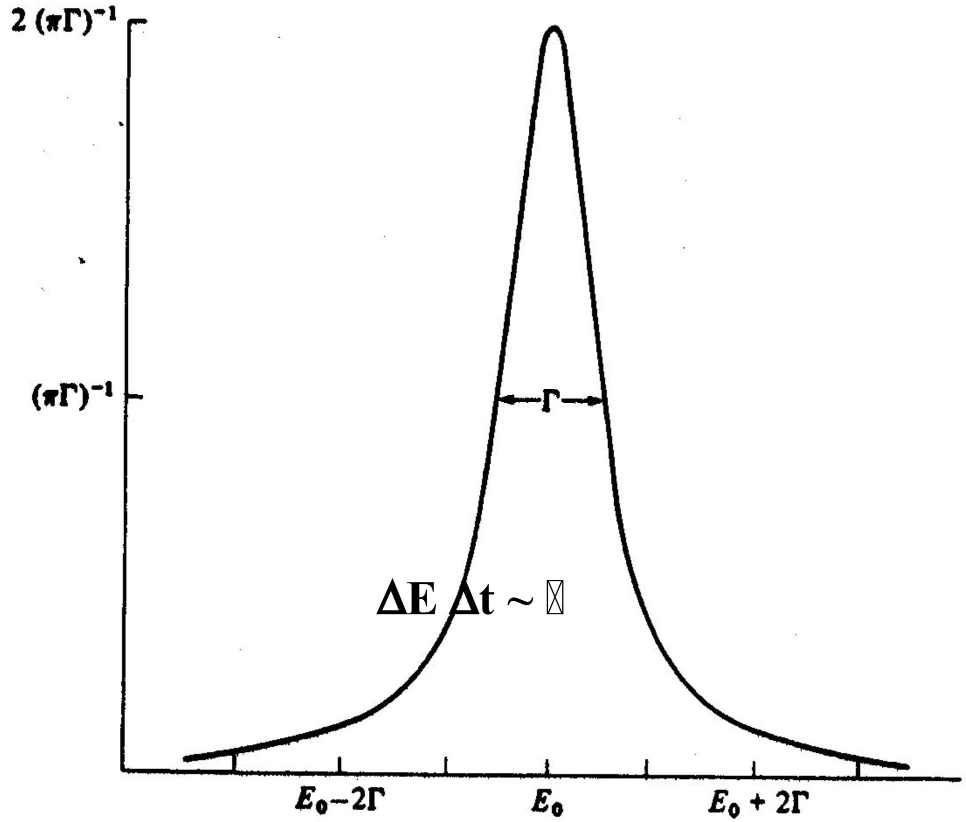
We will see this many times in NP and PP.

Breit-Wigner Resonance

- **Important in atomic, nuclear and particle physics.**
- **Uncertainty relationship**

$$\Delta E \, \Delta t \sim \hbar$$

- **Determine lifetimes of states from width.**
- $\tau \sim \hbar / \Gamma$, Γ=**FWHM;**



Fermi Golden Rule

- Want to be able to calculate reaction rates in terms of matrix elements of H.
- Warning: We will use this many times to calculate σ but derivation not required for exams, given here for completeness.

Discrete → Continuum

- Decays to a channel i (range of states n). Density of states $n_i(E)$. Assume narrow resonance

$$P_i = \frac{2\pi}{\Gamma}\int |H_{i0}|^2\, n_i(E) P(E - E_0)\, dE$$

$$P_i = \frac{2\pi}{\Gamma}|H_{i0}|^2 n_i(E_0)$$

$$P_i = \frac{\Gamma_i}{\Gamma}\,;\ R_{Total} = \frac{\Gamma}{\hbar}\,;\ R_i = P_i\, R_{Total}$$

$$R_i = \frac{\Gamma_i}{\hbar} = \frac{2\pi}{\hbar}|H_{i0}|^2 n_i(E_0)$$

Cross Section

- **Breit Wigner cross section.**
- **Definition of σ and flux F:**

$$R = F\,\sigma$$

$$\psi = V^{-1/2}\exp(i\vec{k}.\vec{r})$$

$$F = V^{-1}v$$

$$n(k) = \frac{V}{(2\pi)^3}4\pi k^2$$

$$E = \frac{(\hbar k)^2}{2m} \quad ; \quad \frac{dE}{dk} = \hbar v \qquad n(E) = \frac{V}{(2\pi)^3}\frac{4\pi k^2}{\hbar v}$$

Breit-Wigner Cross Section

$$R = |a_o(t)|^2 \frac{\Gamma_f}{\hbar} = \frac{|H_{01}|^2}{(E_1 - E_0)^2 + \Gamma^2/4} \frac{\Gamma_f}{\hbar}$$

$$\Gamma_i(E) = 2\pi |H_{10}|^2 n(E)$$

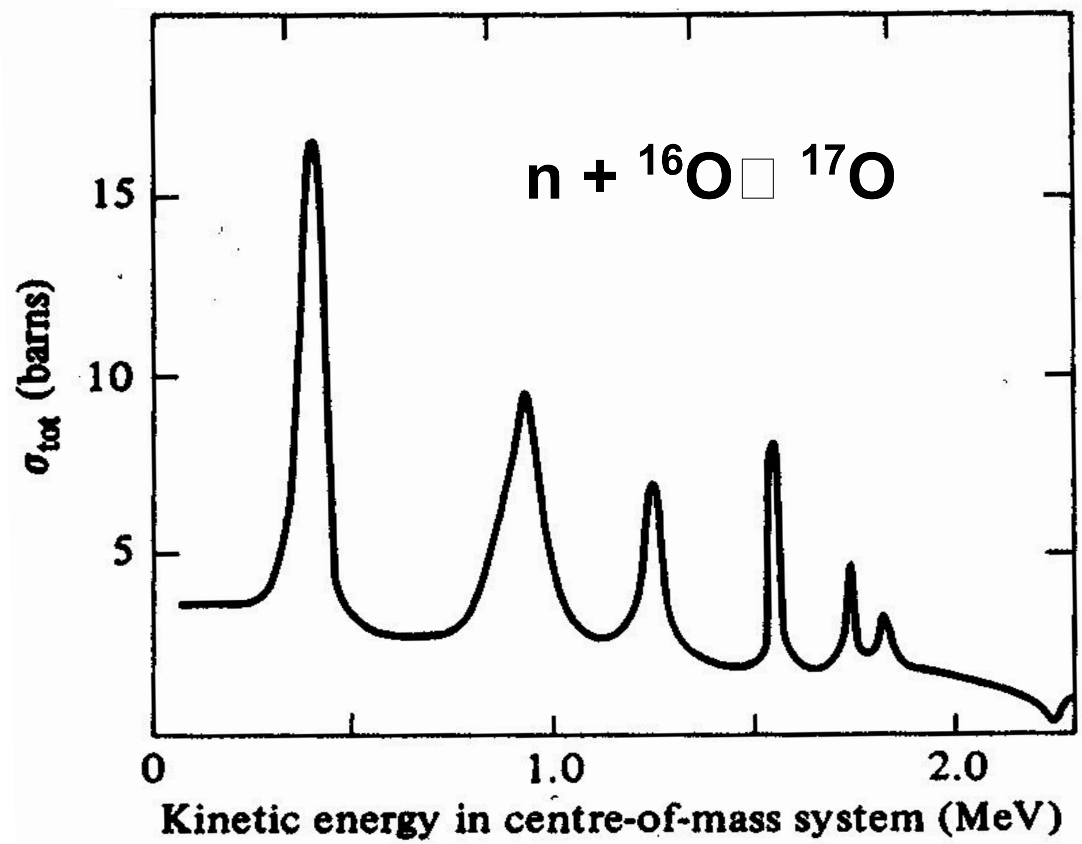
$$R = \frac{1}{2\pi\hbar} \frac{1}{n(E)} \frac{\Gamma_i(E_1)\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$

- Combine rate, flux & density states □

$$\sigma = \left(\frac{V}{v}\right)\left(\frac{(2\pi)^3 \hbar v}{V 4\pi k^2}\right)\frac{1}{2\pi\hbar} \frac{\Gamma_i(E_1)\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$

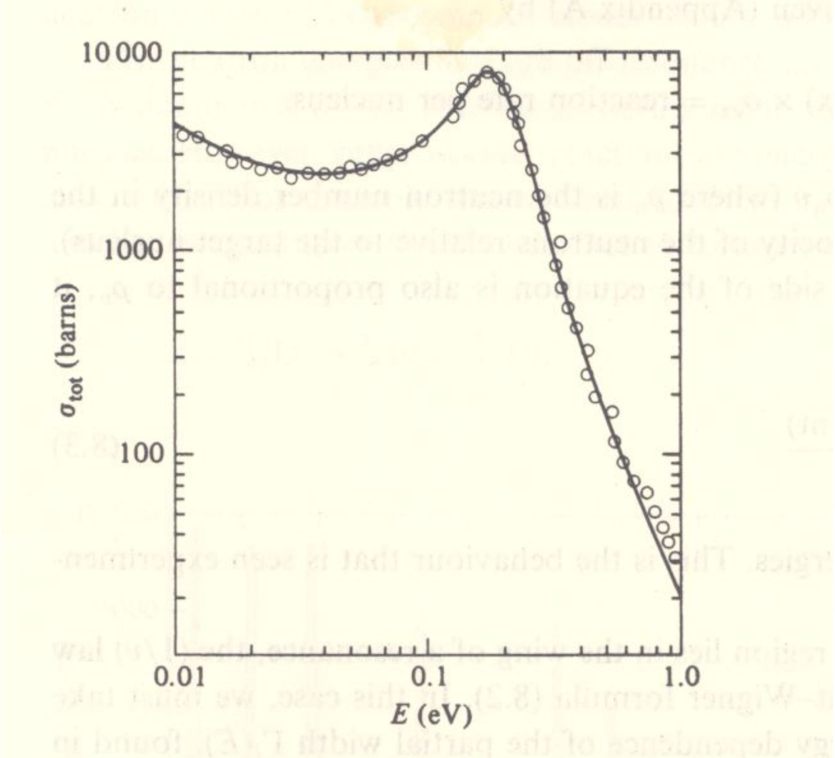
Breit-Wigner Cross Section

$$\sigma = \frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}$$



Low Energy Resonances

- n + Cd total cross section.
- Cross section scales $\sigma \sim 1/E^{1/2}$ at low E.
- B-W: $1/k^2$ and $\Gamma \sim n(E) \sim k$



Rutherford Scattering 1

$$V(r) = \frac{Z_1 Z_2 \alpha}{r} \quad ; \quad \alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \quad ; \hbar = c = 1$$

$$\psi_i = V^{-1/2} \exp(i\vec{k}_i . \vec{r}) \quad ; \quad \psi_f = V^{-1/2} \exp(i\vec{k}_f . \vec{r}) \qquad \vec{q} = \vec{k}_i - \vec{k}_f$$

$$H_{fi} = V^{-1} \int \exp(i\vec{k}_i . \vec{r}) \left(\frac{Z_1 Z_2 \alpha}{r} \right) \exp(-i\vec{k}_f . \vec{r}) d^3 r$$

$$H_{fi} = V^{-1} Z_1 Z_2 \alpha \int \frac{\exp(i\vec{q} . \vec{r})}{r} d^3 r$$

$$H_{fi} = V^{-1} Z_1 Z_2 2\pi\alpha \int \frac{\exp(iqr\cos\theta)}{r} r^2 dr \, d\cos\vartheta$$

Rutherford Scattering 2

$$\mathbf{H_{fi}} = \mathbf{V^{-1}Z_1Z_2}2\pi\alpha\int\frac{\mathbf{exp(iqr) - exp(-iqr)}}{\mathbf{iqr^2}}\mathbf{r^2dr}$$

$$\mathbf{xV(r); exp(-r/a)\ a \to \infty}$$

$$H_{fi} = V^{-1}\frac{Z_1Z_2 2\pi\alpha}{iq}\int \exp(-1/a + iq)r - \exp(-1/a - iq)r\,dr$$

$$\mathbf{H_{fi}} = \mathbf{V^{-1}}\frac{\mathbf{Z_1Z_2}2\pi\alpha}{\mathbf{iq}}\left[\frac{1}{-1/a+iq} - \frac{1}{-1/a-iq}\right]$$

Rutherford Scattering 3

- **Fermi Golden Rule:** $$\mathrm{R} = \frac{2\pi}{\hbar}\left|\mathrm{H_{fi}}\right|^2 \frac{\mathrm{dn}}{\mathrm{dE_f}}$$

$$\frac{\mathrm{dn}}{\mathrm{dp}} = 4\pi\mathrm{p}^2 \frac{\mathrm{V}}{\mathrm{h}^3}\frac{\mathrm{d}\Omega}{4\pi} \quad ; \frac{\mathrm{dn}}{\mathrm{dE}} = \frac{\mathrm{dn}}{\mathrm{dp}}\frac{\mathrm{dp}}{\mathrm{dE}} \quad ; \frac{\mathrm{dp}}{\mathrm{dE}} = 1/\mathrm{v}$$

$$n(E) = \frac{p^2 V}{v(2\pi)^3} d\Omega \quad ;(h = 2\pi\hbar = 2\pi)$$

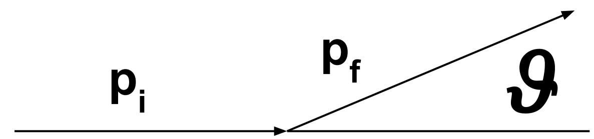
$$\sigma = \mathrm{R}/\mathrm{F} \quad ; \quad \mathrm{F} = \mathrm{V}^{-1}\mathrm{v}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi}{\hbar}\left|\frac{\mathrm{Z_1 Z_2} 4\pi\alpha}{\mathrm{Vq}^2}\right|^2 \frac{\mathrm{p^2 V}}{\mathrm{v}(2\pi)^3}\frac{\mathrm{V}}{\mathrm{v}}$$

$$\frac{d\sigma}{d\Omega} = \frac{4p^2(Z_1 Z_2 \alpha)^2}{v^2 q^4}$$

Rutherford Scattering 4

$$q^2 = (\vec{p}_i - \vec{p}_f)^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2(\vartheta/2)$$



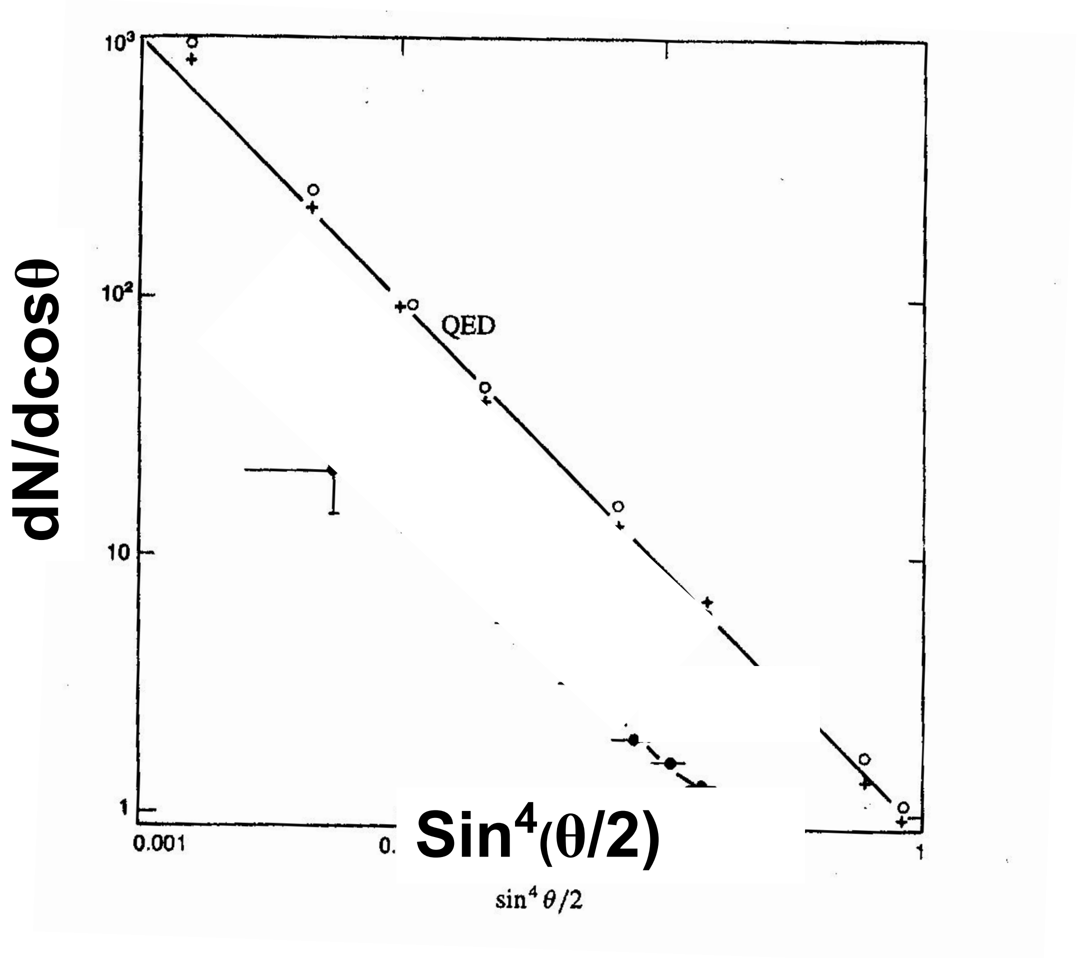
$$\frac{d\sigma}{d\Omega} = \frac{(Z_1 Z_2 \alpha)^2}{4p^2 v^2 \sin^4(\vartheta/2)}$$

Compare with experimental data at low energy

Q: what changes at high energy ?

Low Energy Experiment

- Scattering of α on Au & Ag agree with calculation assuming point nucleus



Higher Energy

- Deviation from Rutherford scattering at higher energy determine charge distribution in the nucleus.
- Form factors is F.T. of charge distribution.

Electron - Gold