Probability Theory And Statistics

The Course of Lectures.

Seminar 1.

Event Algebra.

Basic Concepts.

Random Experiment

- Suppose that a process that could lead to two or more different outcomes is to be observed and there is uncertainty beforehand as to which outcome will occur.
- Some examples are the following:
  - A coin is thrown.
  - A die is rolled.
  - A consumer is asked which of two products he or she prefers.

Random Experiment

- Some examples are the following:
  - An item from a set of accounts is examined by an auditor.
  - The daily change in an index of stock market prices is observed.
  - A batch of a chemical produced by a particular process is tested to determine whether it contains more than an allowable percentage of impurity.
- Each of these examples involves a **random experiment.**

Definition: a random experiment.

- **A random experiment** is a process leading to at least two possible outcomes with uncertainty as to which will occur

Random Experiment

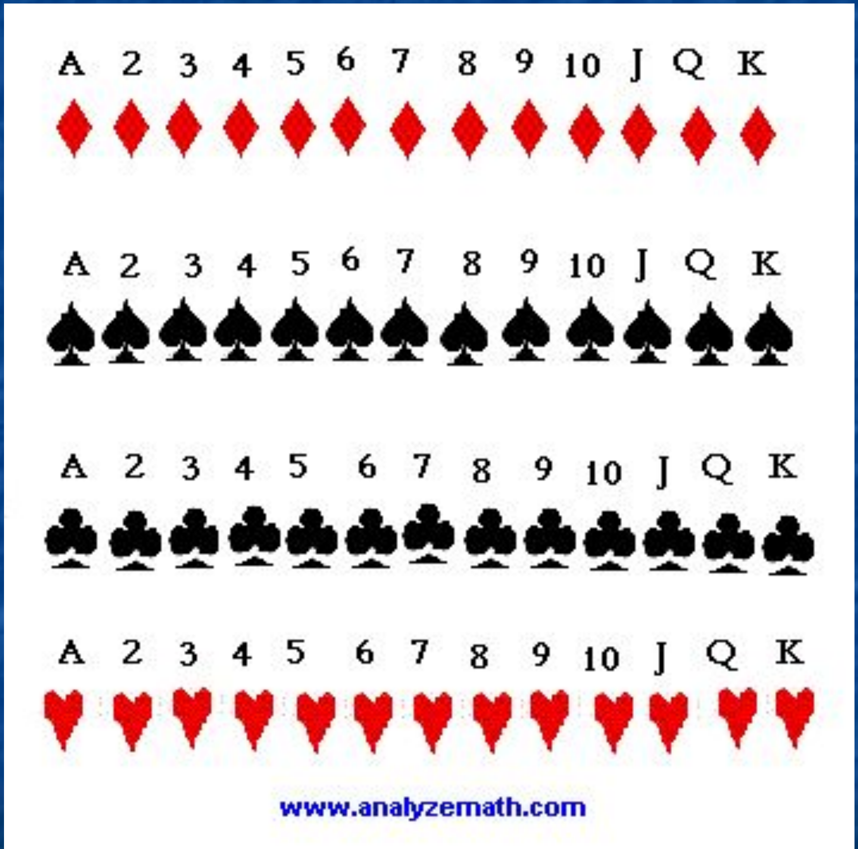
- In each of the first three experiments listed, it is possible to specify what outcomes might arise.
  - If a coin is thrown, the result will be either "head" or "tail."
  - If a die is rolled, the result will be one of the numbers 1, 2, 3,4, 5, or 6.
  - A consumer might indicate a preference for one of the products or no preference.

Definition: Outcomes

- The possible outcomes of a random experiment are called the **basic outcomes**,

  and the set of all basic outcomes is called the **sample space**

  Example : pack of playing cards



Example

- A die is rolled.
- The basic outcomes are the numbers 1, 2, 3, 4, 5, 6.
- Thus, the sample space is

  S = [1, 2, 3, 4, 5, 6]

- Here we see that there are six basic outcomes.
- No two can occur together, and one of them must occur

Definition: Events

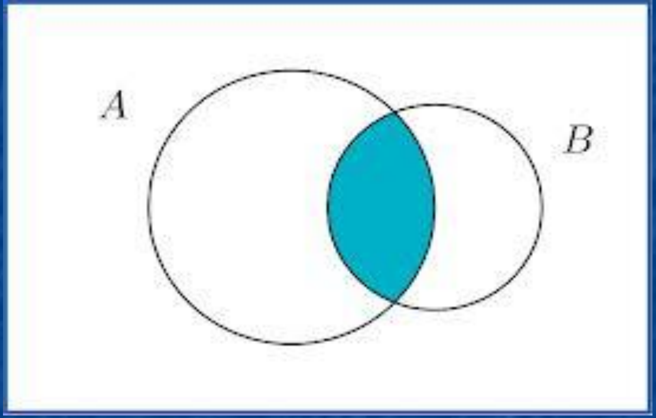
- **An event** is a set of basic outcomes from the sample space, and it is said to **occur** if the random experiment gives rise to one of its constituent basic outcomes

Definition:
The intersection of events

- Let A and B be two events in the sample space S.
- Their **intersection**, denoted $A \cap B$ is the set of all basic outcomes in S that belong to both A and B.
- Hence, the intersection $A \cap B$ occurs if and only if both A and B occur.

Truth table for intersection of events. Venn diagram

A	B	$A \cap B$
1	1	1
1	0	0
0	1	0
0	0	0



Clearly, a basic outcome will be in $A \cap B$ if and only if it is in both *A* and *B*

Intersection of events: example

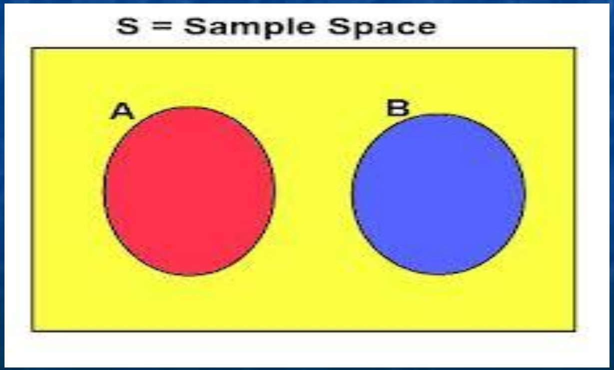
- In rolling a die, the outcomes 4 and 6 both belong to the two events

  A = "Even number results"

  B = "Number at least 4 results”.

No intersection

- It is possible that events *A* and *B* have no common basic outcomes, in which case the figures will not intersect.
- Such events are said to be **mutually exclusive**



A and B

are mutually exclusive

Definition: mutually exclusive events

- If the events *A* and *B* have no common basic outcomes, they are called **mutually exclusive** and their intersection $A \cap B$ is said to be the **empty set.**
- It follows, then, that $A \cap B$ cannot occur

The **union** of events

- When considering jointly several events, another possibility of interest is that at least one of them will occur.
- This will happen if the basic outcome of the random experiment belongs to at least one of the events.
- The set of basic outcomes belonging to at least one of the events is called their **union.**

The union of events: example

- In the die throw experiment,

  the outcomes 2,4, 5, and 6 all belong to at least one of the events

  A= "Even number results"

  or

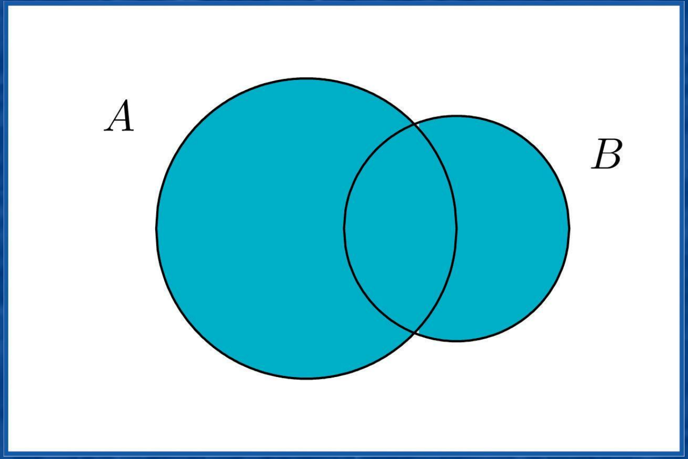
  B= "Number at least 4 results."

Definition: the union of events

- Let *A* and *B* be two events in the sample space *S.*
- Their **union**, denoted $A \cup B$,
  is the set of all basic outcomes in *S*
  that belong to at least one of these two events.
  Hence, the union $A \cup B$ occurs
  if and only if either *A* or *B* (or both) occurs.

Truth table for union of events. Venn diagram

A	B	$A \cup B$
1	1	1
1	0	1
0	1	1
0	0	0



It is clear that a basic outcome will be in $A \cup B$ if and only if it is in either *A* or *B* (or both).

Collectively exhaustive events

- A case of special interest concerns a collection of several events whose union is the whole sample space *S.*
- Since every basic outcome is always contained in *S,* it follows that every outcome of the random experiment will be in at least one of this collection of events.
- These events are then said to be **collectively exhaustive**

Collectively exhaustive events: example

- If a die is thrown, the events
  A="Result is at least 3"
  and
  B="Result is at most 5"
  are together collectively exhaustive —

at least one of these two events must occur.

The complement of the event

- Next, let *A* be an event, and suppose our interest is that *A* not occur.
- This will happen if the basic outcome of the random experiment lies in *S* (as it must)

  but *not* in *A*.
- The set of basic outcomes belonging to the sample space but not to a particular event is called the **complement** of that event and is denoted $\overline{A}$

The complement of the event

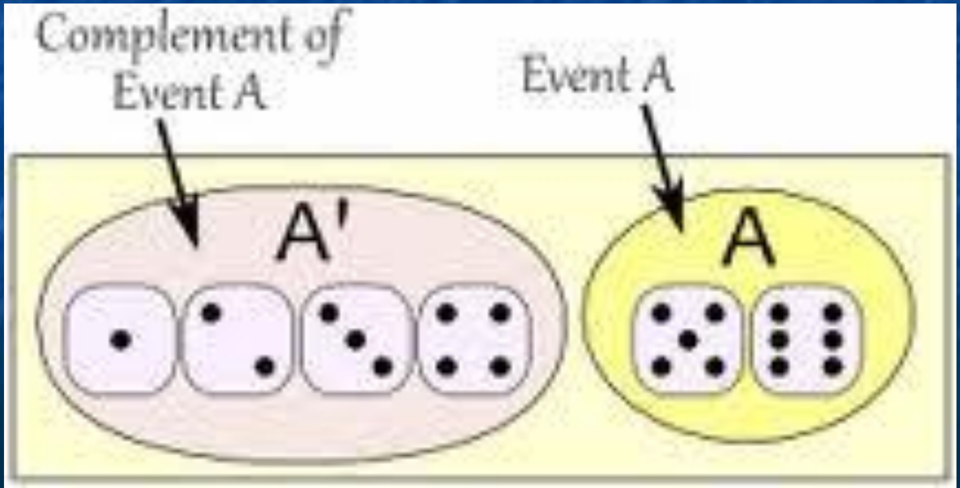
- Clearly, the events A and $\overline{A}$ are mutually exclusive (no basic outcome can belong to both) and collectively exhaustive (every basic outcome must belong to one or the other).

Definition: the complement of A

- Let A be an event in the sample space *S.*
- The set of basic outcomes of a random experiment belonging to *S* but not to *A* is called the **complement** of A and is denoted $\bar{A}$

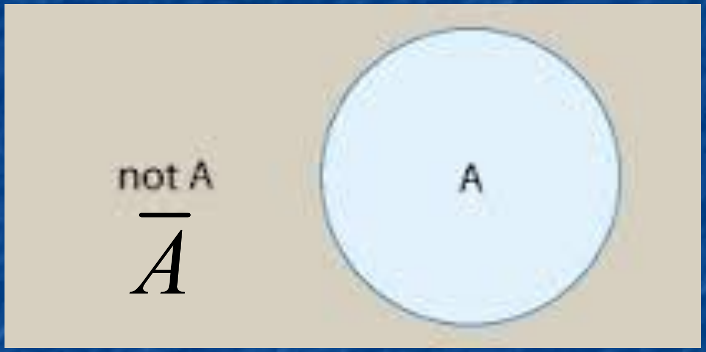
The complement of A: example

- If a die is thrown, the complement of event A="Result is at least 5" is event $\overline{A}$ ="Result is at most 4"



Truth table for compliment of event. Venn diagram

A	$\overline{A}$
1	0
0	1



Clearly, an event $\overline{A}$ occurs
if and only if an event A does not occur

Question 1

- Prove the statement

$$A \cup \overline{B} = A \cup \left(\overline{A} \cap \overline{B} \right)$$

- with the help of truth tables

Answer 1

A	B	$\bar{A}$	$\bar{B}$	$A \cup \bar{B}$	$\bar{A} \cap \bar{B}$	$A \cup \left(\bar{A} \cap \bar{B}\right)$
1	1	0	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	0	0	0
0	0	1	1	1	1	1

Question 2

- A problem often faced in sociological research is that some of the questions we would like to ask are so sensitive that many subjects will either refuse to reply or will give a dishonest answer.
- One way of attacking this problem is through the method of *randomized response.*

Question 2 (continued)

- This technique involves pairing the sensitive question with a nonsensitive question.
- For instance, we should want to obtain an information concerning dodging taxes.
- So we might create the following pair:
  - (a) Have you purposely evaded taxes in the last 12 months? (sensitive question)
  - (b) Have you obtained a "head" in the trial of coin tossing? (nonsensitive question)

Question 2 (continued)

- Subjects are asked to flip a coin and then to answer question
  - (a) if the result is "head"
  - and flip a coin once again and answer (b) otherwise.
- Since the investigator cannot ***know*** which question is answered, it is hoped that honest responses will be obtained in this way.

Question 2 (continued)

- The nonsensitive question is one for which the investigator already has information.
- Thus, in our example, the investigator knows what proportion of “tails” is $1/2$.

Question 2 (continued)

- Now, we define the following events:
  - A : Subject answers "yes."
  - E_1 : Subject answers sensitive question.
  - E_2 : Subject answers nonsensitive question.
- Clearly, the events E_1 and E_2 are mutually exclusive and collectively exhaustive.

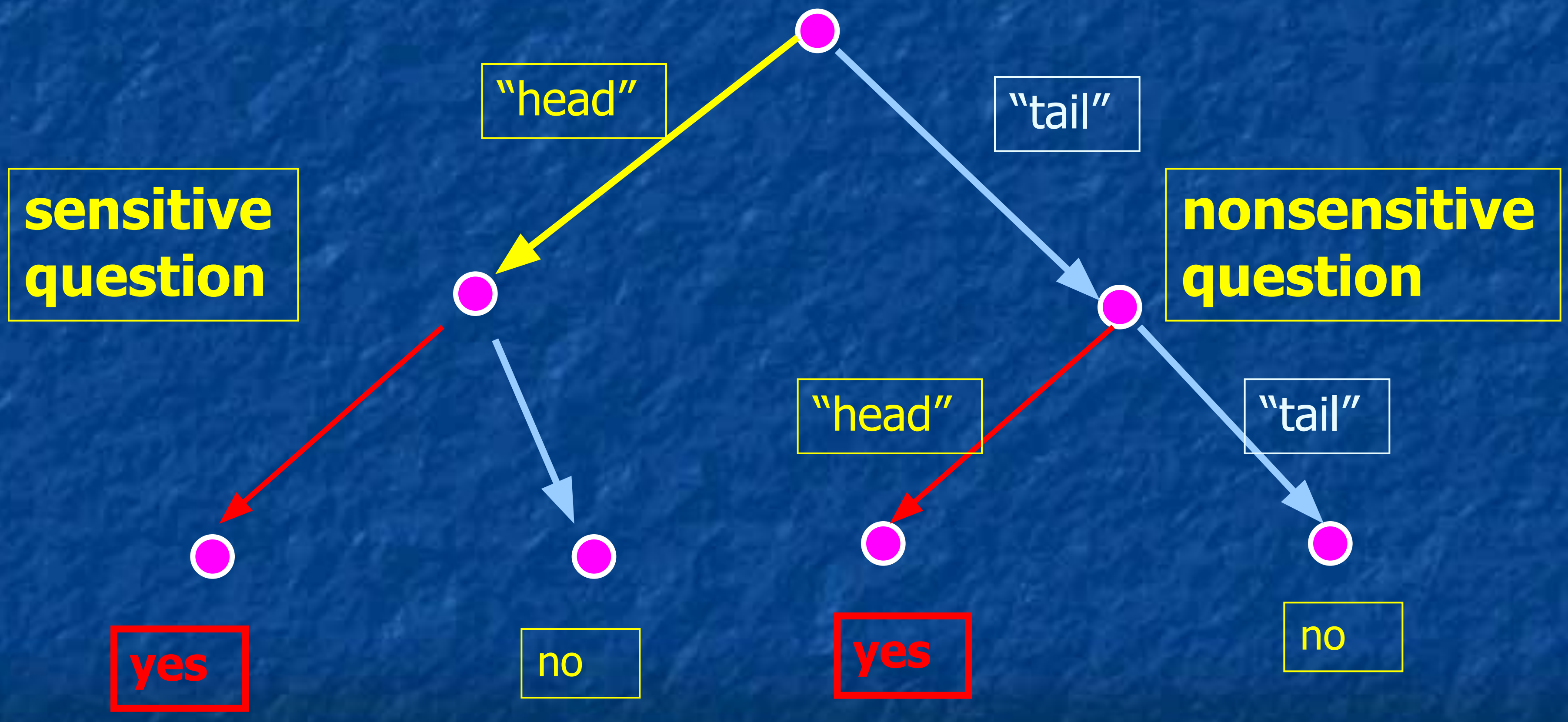
Question 2 (continued)

- Thus, the conditions of result (**3**) are satisfied, and it follows that the events
  - $A \cap E_1$ Subject both responds "yes" and has answered the sensitive question
  - $A \cap E_2$: Subject both responds "yes" and has answered the nonsensitive question

  are mutually exclusive.
- Furthermore, their union must be the event *A;* that is

$$A = \left(A \cap E_1\right) \cup \left(A \cap E_2\right)$$

Question 2 (continued)



Question 2 (continued)

- Let the proportion of population evaded taxes be 20%. What is the proportion of "yes" answers in our survey?
- In real survey 30% of people answer "yes". What is the proportion of people dodging taxes?

What is Probability?

- Suppose that a random experiment is to be carried out and we are interested in the chance of a particular event's occurring.
- The concept of probability is intended to provide a numerical measure for the likelihood of an event's occurrence

What is Probability?

- Probability is measured on a scale from 0 to 1.
- At the extremes of this range, a probability of 0 implies that the event is impossible (it is certain not to occur),
- whereas a probability of 1 implies that the event is certain to occur.
- For uncertain events, we want to attach a probability between 0 and 1 such that the more likely the event is to occur, the higher the probability

What is Probability?

- In practice, such ideas are frequently met.
- It is known that rain is more likely under certain meteorological conditions than others.
- An experienced manager may judge that one product is more likely to achieve substantial market penetration than another.

What is Probability?

- To take a very simple example, suppose a coin is thrown.
- The statement "The probability that a head results is M" may be viewed through two distinct ideas — *relative frequency* and *subjective probability*

Relative Frequency

- Suppose that a random experiment can be replicated in such a way that,
- after each trial, it is possible to return to the initial state and repeat the experiment so that the resulting outcome is unaffected by previous outcomes.
- For example, a coin or die can be thrown repeatedly in this way.

Relative Frequency

- If some number *N* of experiments is conducted and the event *A* occurs in N_A of them (N_A clearly depending on *N*), we have

$$\text{Proportion of occurrences of } A \text{ in } N \text{ trials } = \frac{N_A}{N}$$

Relative Frequency

- Now, if N is very large, we would not expect much variation in the proportion N_A / N as *N* increases;
- that is, the proportion of occurrences of *A* will remain approximately constant.
- This notion underlies the **relative frequency** concept of probability

Definition: Relative Frequency

- Let N_A be the number of occurrences of event A in N repeated trials.
- Then, under the **relative frequency** concept of probability, the probability that *A* occurs is the limit of the ratio N_A/N as the number of trials *N* becomes infinitely large

Relative Frequency

- Under this definition, if we say

  "The probability of a head resulting from a single throw of a coin is "$1/2$
- we mean that if the coin is thrown repeatedly, the proportion of heads resulting will get very close to $1/2$ as the number of trials gets very large
- The relative frequency notion provides a convenient framework for thinking ab$1/2$out probability, but it does involve conceptual difficulties

Subjective Probability

- An alternative view, which does not depend on the notion of repeatable experiments, regards probability as a personal subjective concept, expressing an individual's degree of belief about the chance that an event will occur.
- One way to understand this idea is in terms of *fair bets*

Subjective Probability

- For example, if I assert that the probability of a head resulting from the throw of a coin is $1/2$, what I have in mind is that the coin appears to be perfectly fair and that the throw is just as likely to produce a head as a tail.
- In assessing this subjective probability, I am not necessarily thinking in terms of repeated experimentation but am concerned with only a single throw of the coin.

Subjective Probability

- My subjective probability assessment implies that I would view as fair a bet in which I had to pay $1 if the result was tail and would receive $1 if the result was head.
- If I were to receive more than $1 if the throw yielded a head, I would regard the bet as in my favor.

Subjective Probability

- Similarly, if I believe that the probability of a horse's winning a particular race is .4, I am asserting the personal view that there is a 40-60 chance of its winning.
- Given this belief, I would regard as fair a bet in which I lost $2 if the horse did not win and gained $3 if it did

Subjective Probability

- It should be emphasized that subjective probabilities are personal;
- there is no requirement that different individuals considering the same event should arrive at the same probabilities.
- In the coin-throwing example, most people will conclude that the appropriate probability for a head is $1/2$

Subjective Probability

- However, an individual with more information about the coin in question might believe otherwise.
- In the example of the horse race, it is likely that two bettors will reach different subjective probabilities.
- They may not, for example, have the same information, and even if they do, they might not interpret it in the same way.

Subjective Probability

- It is certainly clear that individual investors do not all hold the same views on the likely future behavior of the stock market!
- Their subjective probabilities might be thought of as depending on the knowledge they have and the way they interpret it

Thank you
for your attention!