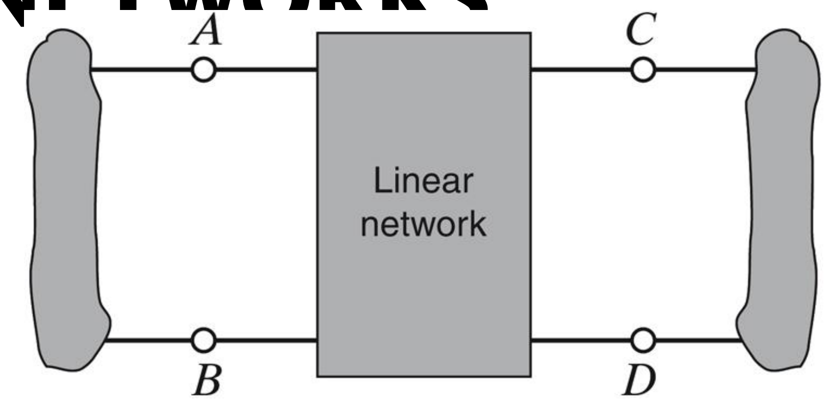
TWO-PORT NETWORKS



In many situations one is not interested in the internal organization of a network. A description relating input and output variables may be sufficient

A two-port model is a description of a network that relates voltages and currents
at two pairs of terminals

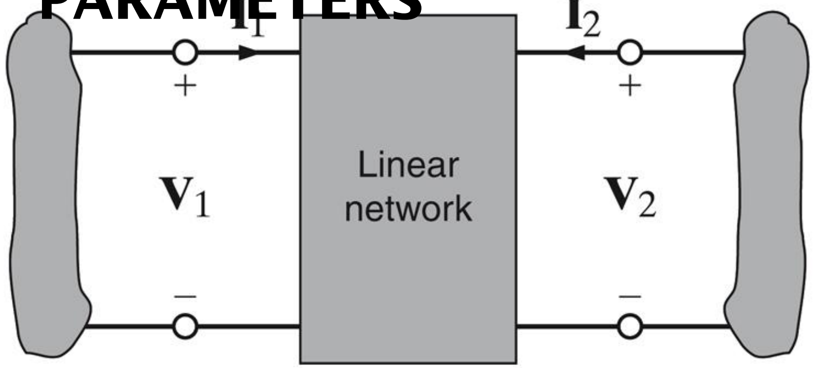
LEARNING GOALS

Study the basic types of two-port models

- Admittance parameters
- Impedance parameters
- Hybrid parameters
- Transmission parameters

Understand how to convert one model into another

ADMITTANCE PARAMETERS



The network contains NO independent sources

The admittance parameters describe the currents in terms of the voltages

y_{21} determines the current flowing into port 2 when the port is short-circuited and a voltage is applied to port 1

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

The first subindex identifies the output port. The second the input port.

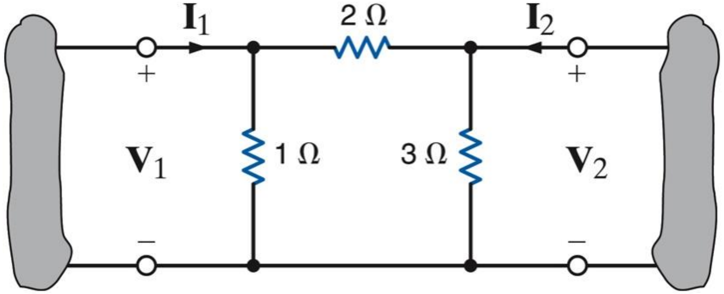
The computation of the parameters follows directly from the definition

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

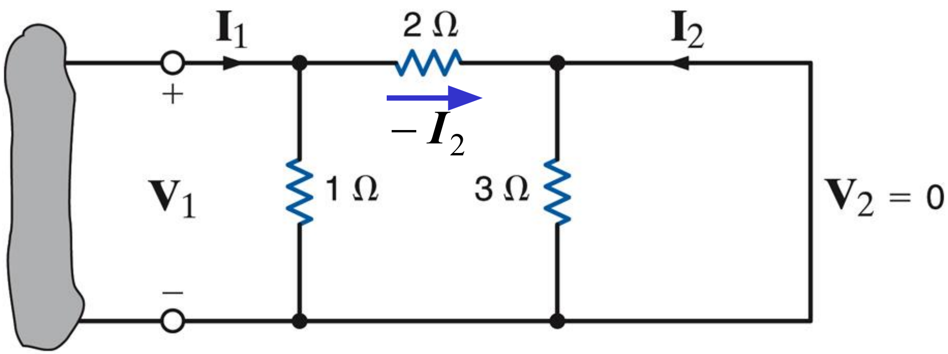
LEARNING EXAMPLE

Find the admittance parameters for the network



$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

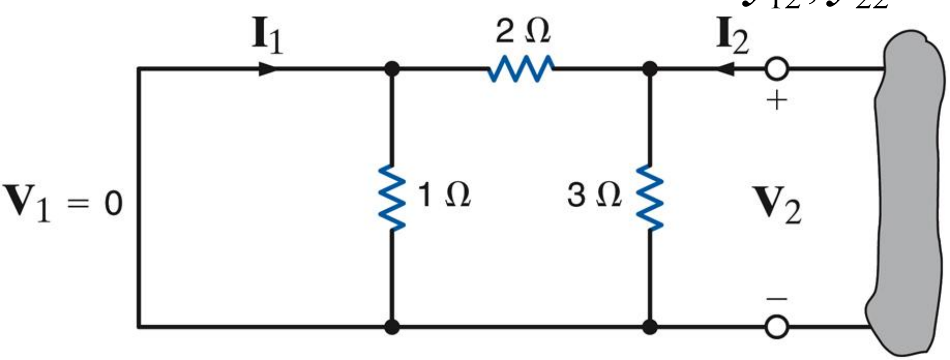
Circuit used to determine y_{11}, y_{21}



$$I_1 = (1+\frac{1}{2})V_1 \Rightarrow y_{11} = \frac{3}{2}[S]$$

$$-I_2 = \frac{1}{1+2}I_1 \Rightarrow I_2 = -\frac{1}{2}V_1 \Rightarrow \quad y_{21} = -\frac{1}{2}[S]$$

Circuit used to determine y_{12}, y_{22}



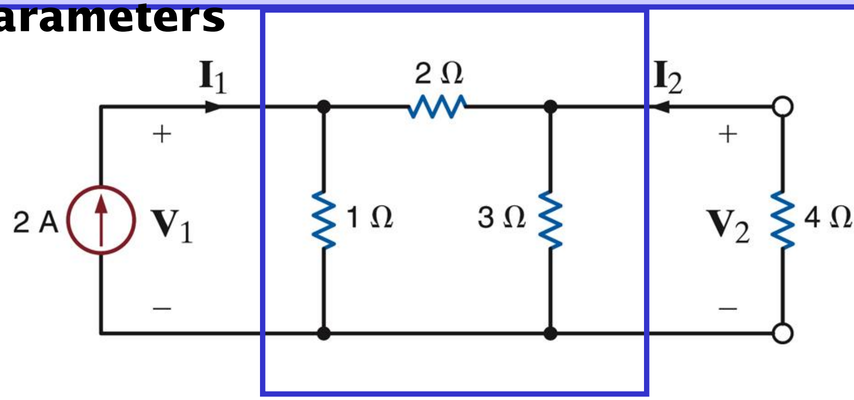
$$I_2 = \left(\frac{1}{2}+\frac{1}{3}\right)V_2 \Rightarrow y_{22} = \frac{5}{6}[S]$$

$$-I_1 = \frac{3}{2+3}I_2 = \frac{3\times 5}{5\times 6}V_2 \Rightarrow y_{12} = \frac{1}{2}[S]$$

Next we show one use of this model

An application of the admittance parameters

Determine the current through the 4 Ohm resistor



$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$I_1 = \frac{3}{2}V_1 - \frac{1}{2}V_2$$

$$I_2 = -\frac{1}{2}V_1 + \frac{5}{6}V_2$$

$$I_1 = 2A, \quad V_2 = -4I_2 \qquad I_2 = -\frac{1}{4}V_2$$

The model plus the conditions at the ports are sufficient to determine the other variables.

$$2 = \frac{3}{2}V_1 - \frac{1}{2}V_2$$

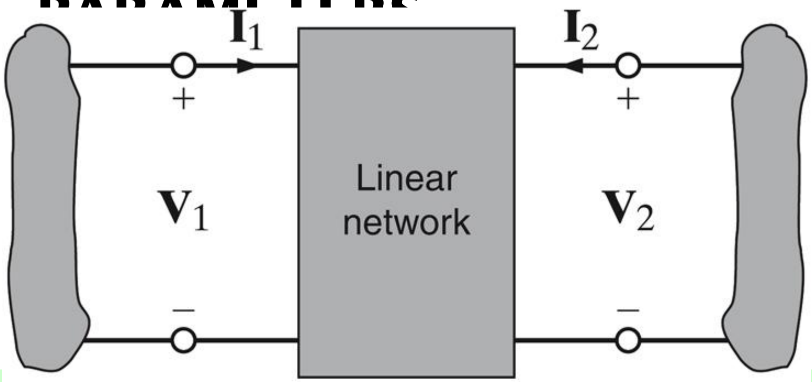
$$0 = -\frac{1}{2}V_1 + \left(\frac{5}{6} + \frac{1}{4}\right)V_2$$

$$V_1 = \frac{13}{6}V_2$$

$$V_2 = \frac{8}{11}[V]$$

$$I_2 = -\frac{2}{11}[A]$$

IMPEDANCE PARAMETERS



The network contains NO independent sources

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

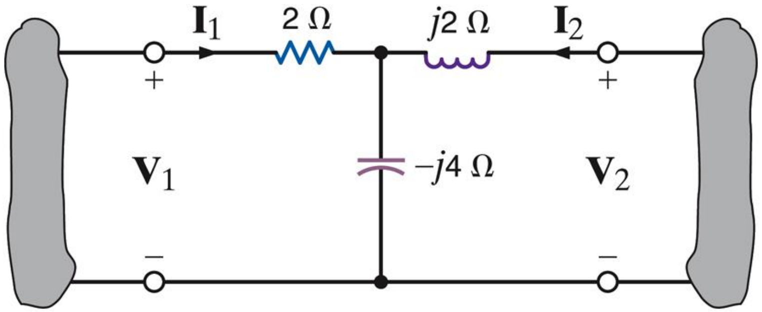
The '*z parameters*' can be derived in a manner similar to the Y parameters

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

LEARNING EXAMPLE

Find the Z parameters



$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

Write the loop equations

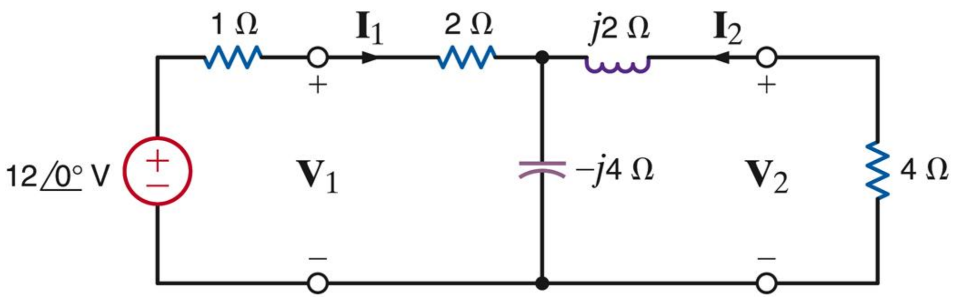
$$V_1 = 2I_1 - j4(I_1 + I_2)$$
$$V_2 = j2I_2 - j4(I_2 + I_1)$$

rearranging

$$V_1 = (2 - j4)I_1 - j4I_2 \qquad \Rightarrow z_{11} = 2 - j4\Omega \qquad z_{12} = -j4\Omega$$
$$V_2 = -j4I_1 - j2I_2 \qquad z_{21} = -j4\Omega \qquad z_{22} = -j2\Omega$$

LEARNING EXAMPLE

Use the Z parameters to find the current through the 4 Ohm resistor



$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

Output port constraint

$$V_2 = -4I_2$$

Input port constraint

$$V_1 = 12\angle 0° - (1)I_1$$

$$V_1 = (2 - j4)I_1 - j4I_2$$

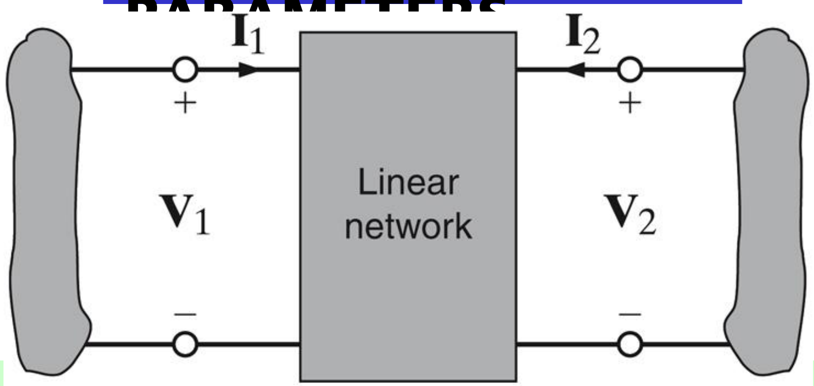
$$V_2 = -j4I_1 - j2I_2$$

$$0 = -j4I_1 + (4 - j2)I_2 \quad \times(3 - j4)$$

$$12 = (3 - j4)I_1 - j4I_2 \quad \times j4$$

$$48j = (16 + (4 - j2)(3 - j4))I_2 \quad \Rightarrow I_2 = 1.61\angle 137.73°$$

HYBRID PARAMETERS



The network contains NO independent sources

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

h_{11} = short-circuit input impedance

h_{12} = open-circuit reverse voltage gain

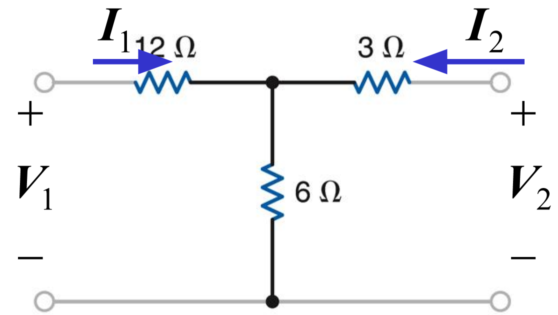
h_{21} = short-circuit forward current gain

h_{22} = open-circuit output admittance

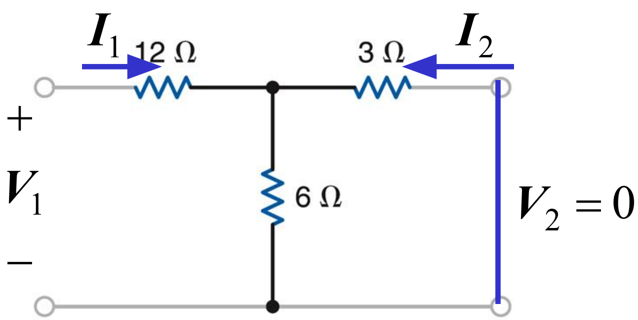
These parameters are very common in modeling transistors

LEARNING EXAMPLE

Find the hybrid parameters for the network

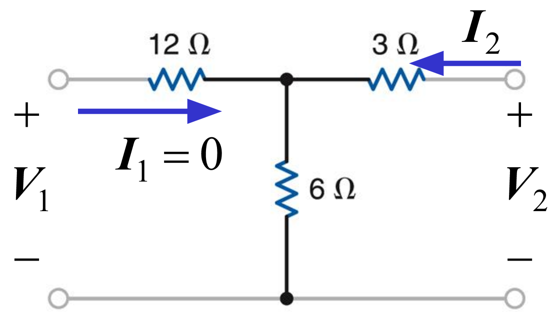


$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$



$$V_1 = (12 + (6 \| 3)) I_1 \Rightarrow h_{11} = 14\Omega$$

$$I_2 = -\frac{6}{3+6} I_1 \Rightarrow h_{21} = -\frac{2}{3}$$

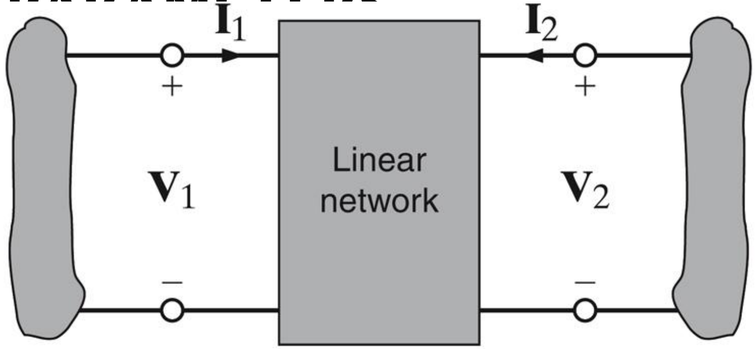


$$V_1 = \frac{6}{3+6} V_2 \Rightarrow h_{12} = \frac{2}{3}$$

$$I_2 = \frac{V_2}{9} \Rightarrow h_{22} = \frac{1}{9} [S]$$

TRANSMISSION PARAMETERS

ABCD parameters



The network contains NO independent sources

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \qquad D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

A = open circuit voltage ratio

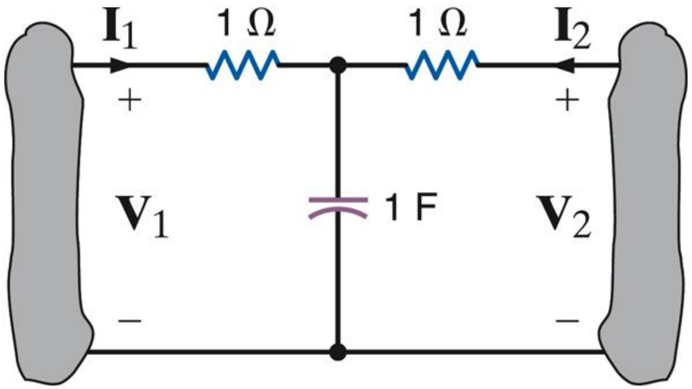
B = negative short-circuit transfer impedance

C = open-circuit transfer admittance

D = negative short-circuit current ratio

LEARNING EXAMPLE

Determine the transmission parameters



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$\boxed{A = \left.\frac{V_1}{V_2}\right|_{I_2=0}} \quad \boxed{C = \left.\frac{I_1}{V_2}\right|_{I_2=0}}$$

$$\boxed{B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}} \quad \boxed{D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}}$$

when $I_2 = 0$

$$V_2 = \frac{\frac{1}{j\omega}}{1+\frac{1}{j\omega}} V_1 \Rightarrow A = 1 + j\omega$$

$$V_2 = \frac{1}{j\omega} I_1 \Rightarrow \frac{I_1}{V_2} = j\omega$$

when $V_2 = 0$

$$I_2 = -\frac{\frac{1}{j\omega}}{1+\frac{1}{j\omega}} I_1 = -\frac{1}{1+j\omega} I_1 \qquad \Rightarrow D = 1 + j\omega$$

$$V_1 = \left[1 + (1 \,||\, \frac{1}{j\omega})\right] I_1 = \left[\frac{2+j\omega}{1+j\omega}\right] [-(1+j\omega)] I_2$$

$$B = 2 + j\omega$$

PARAMETER CONVERSIONS

If all parameters exist, they can be related by conventional algebraic manipulations.
As an example consider the relationship between Z and Y parameters

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta_Z} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

with $\Delta_Z = z_{11}z_{22} - z_{21}z_{12}$

In the following conversion table, the symbol Δ stands for the determinant of the corresponding matrix

$$\Delta_Z = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \Delta_Y = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}, \Delta_H = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \Delta_T = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

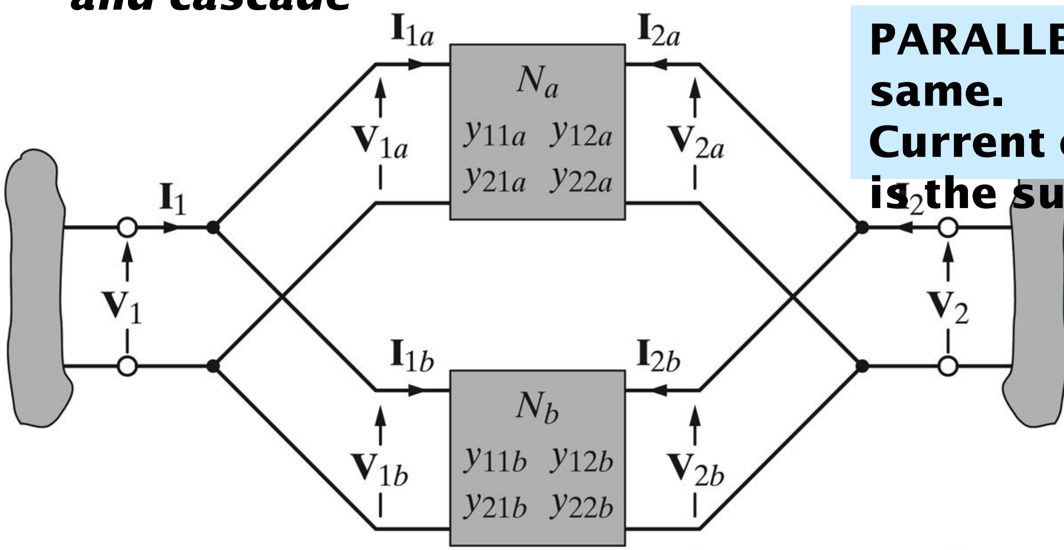
TABLE 16.1 Two-port parameter conversion formulas

$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y} \\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta_H}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{22}}{\Delta_Z} & \frac{-z_{12}}{\Delta_Z} \\ \frac{-z_{21}}{\Delta_Z} & \frac{z_{11}}{\Delta_Z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_H}{h_{11}} \end{bmatrix}$
$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_Z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_Y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta_H}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$
$\begin{bmatrix} \frac{\Delta_Z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_Y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

INTERCONNECTION OF TWO PORTS

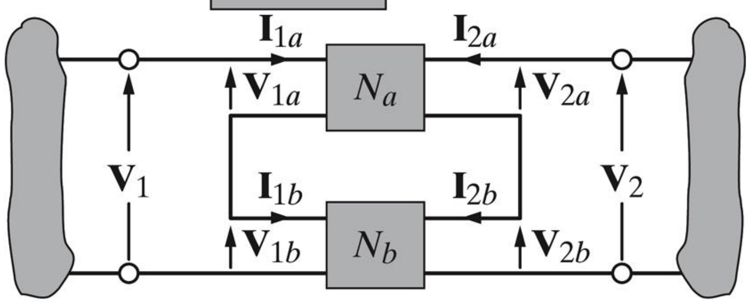
Interconnections permit the description of complex systems in terms of simpler components or subsystems

The basic interconnections to be considered are: *parallel, series and cascade*

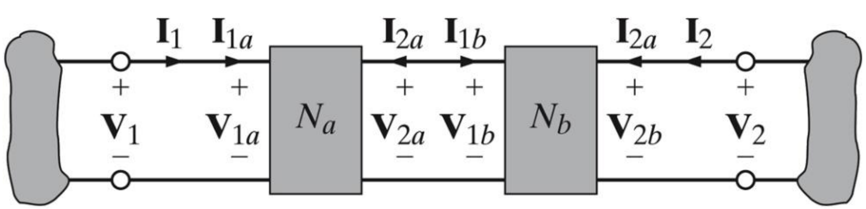


PARALLEL: Voltages are the same.
Current of interconnection is the sum of currents

The rules used to derive models for interconnection assume that each subsystem behaves in the same manner before and after the interconnection

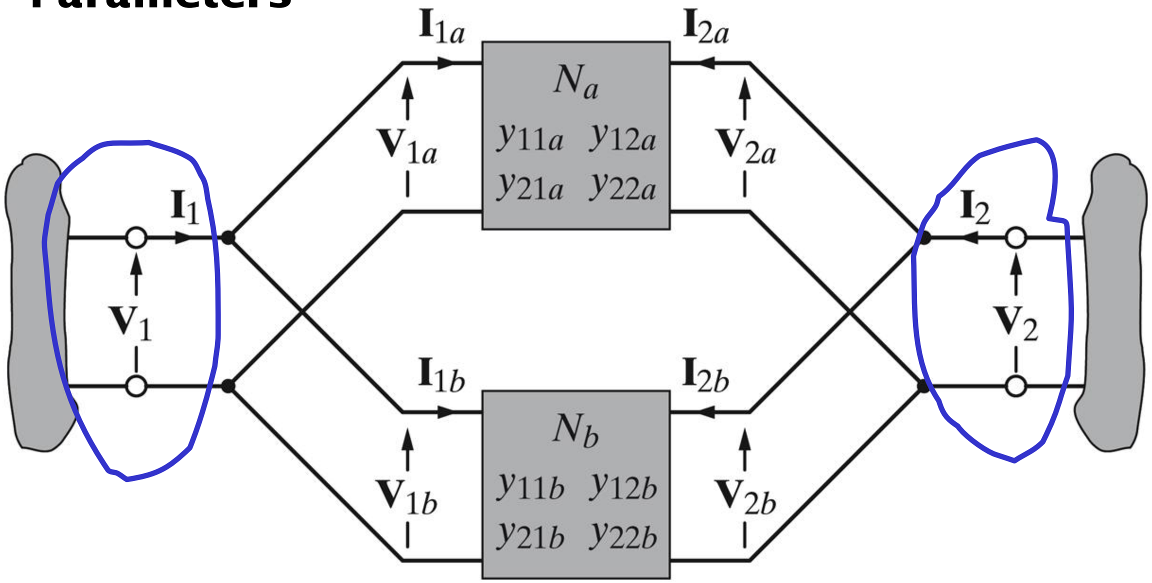


SERIES: Currents are the same.
Voltage of interconnection is the sum of voltages



CASCADE:
Output of first subsystem acts as input for the second

Parallel Interconnection: Description Using Y Parameters



Interconnection description

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I = YV$$

$$I_a = \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix}, V_a = \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix}, Y_a = \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22b} \end{bmatrix} \Rightarrow I_a = Y_a V_a$$

In a similar manner

$$I_b = Y_b V_b$$

Interconne ction constraint s:

$$I_1 = I_{1a} + I_{1b}, \; I_2 = I_{2a} + I_{2b}$$

$$V_1 = V_{1a} = V_{1b}, \; V_2 = V_{2a} = V_{2b}$$

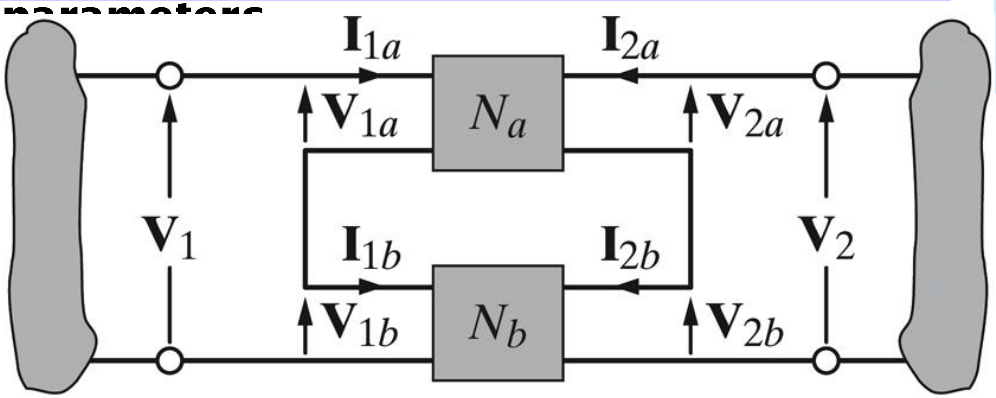
$$\Rightarrow \begin{cases} I = I_a + I_b \\ V = V_a = V_b \end{cases}$$

$$\Rightarrow I = Y_a V_a + Y_b V_b = (Y_a + Y_b) V$$

$$Y = Y_a + Y_b$$

Series interconnection using Z parameters

SERIES: Currents are the same. Voltage of interconnection is the sum of voltages



Description of each subsystem

$$V_a = Z_a I_a, \quad V_b = Z_b I_b$$

Interconnection constraints

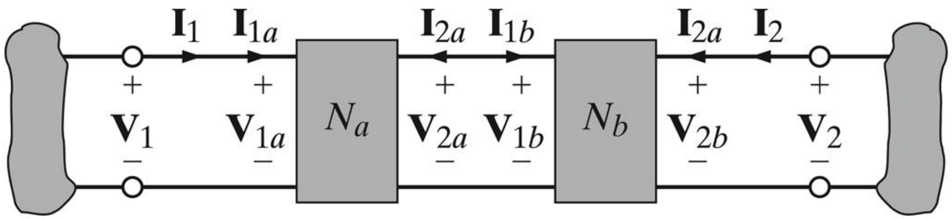
$$I_a = I_b = I$$

$$V = V_a + V_b$$

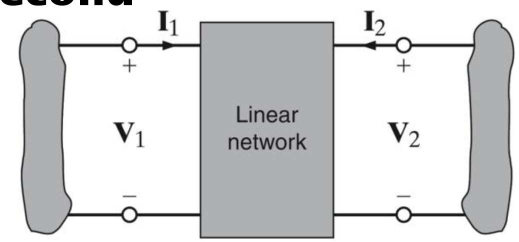
$$\Rightarrow V = Z_a I + Z_b I = (Z_a + Z_b) I$$

$$Z = Z_a + Z_b$$

Cascade connection using transmission parameters



CASCADE:
Output of first subsystem acts as input for the second



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Interconnection constraints:

$$I_{2a} = -I_{1b} \qquad V_{2a} = V_{1b}$$
$$V_1 = V_{1a} \qquad V_2 = V_{2b}$$
$$I_1 = I_{1a} \qquad I_2 = I_{2b}$$

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix}$$

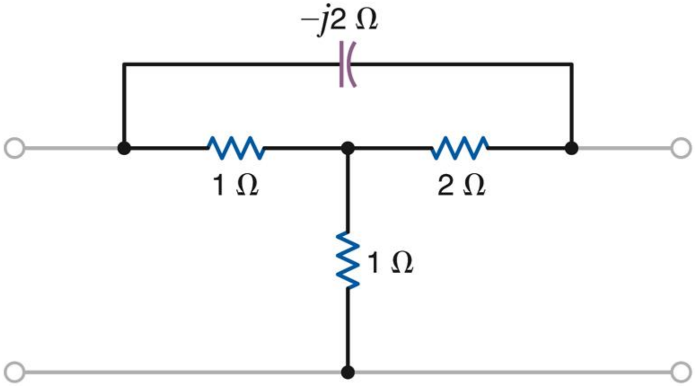
$$\begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

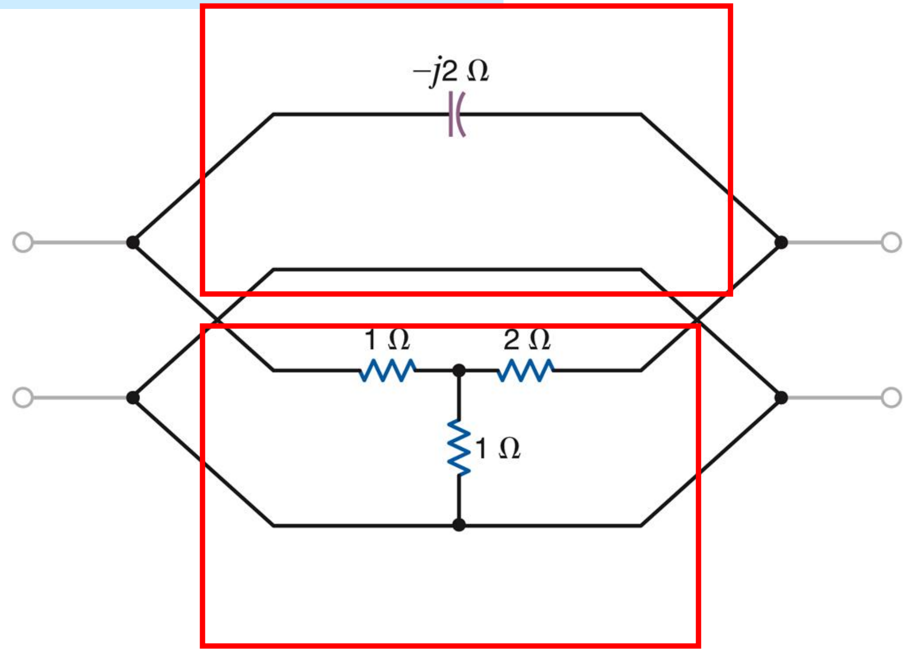
Matrix multiplication does not commute.
Order of the interconnection is important

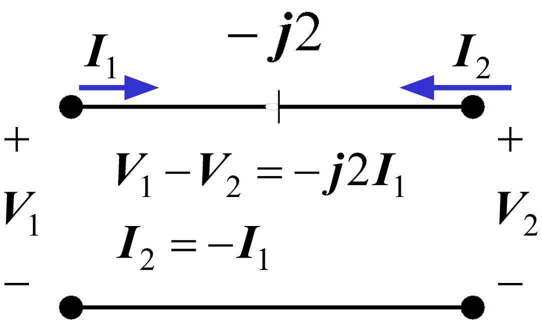
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

LEARNING EXAMPLE

Find the Y parameters for the network



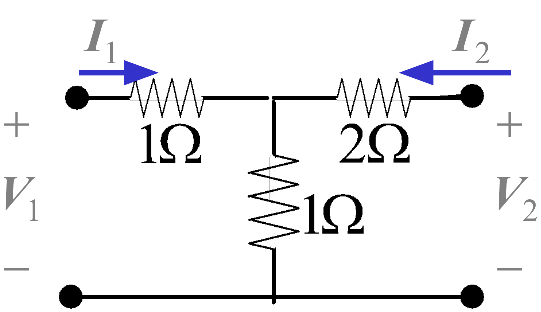




$$V_1 - V_2 = -j2I_1$$
$$I_2 = -I_1$$

$$y_{11a} = j\frac{1}{2},\ y_{12a} = -j\frac{1}{2}$$
$$y_{21a} = -j\frac{1}{2},\ \ y_{22a} = j\frac{1}{2}$$

$$\boldsymbol{Y} = \begin{bmatrix} \frac{3}{5} + \boldsymbol{j}\frac{1}{2} & -\left(\frac{1}{5} + \boldsymbol{j}\frac{1}{2}\right) \\ -\left(\frac{1}{5} + \boldsymbol{j}\frac{1}{2}\right) & \frac{2}{5} + \boldsymbol{j}\frac{1}{2} \end{bmatrix} [\boldsymbol{S}]$$

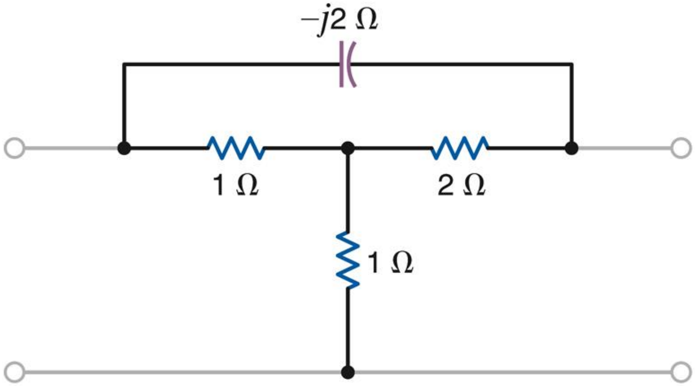


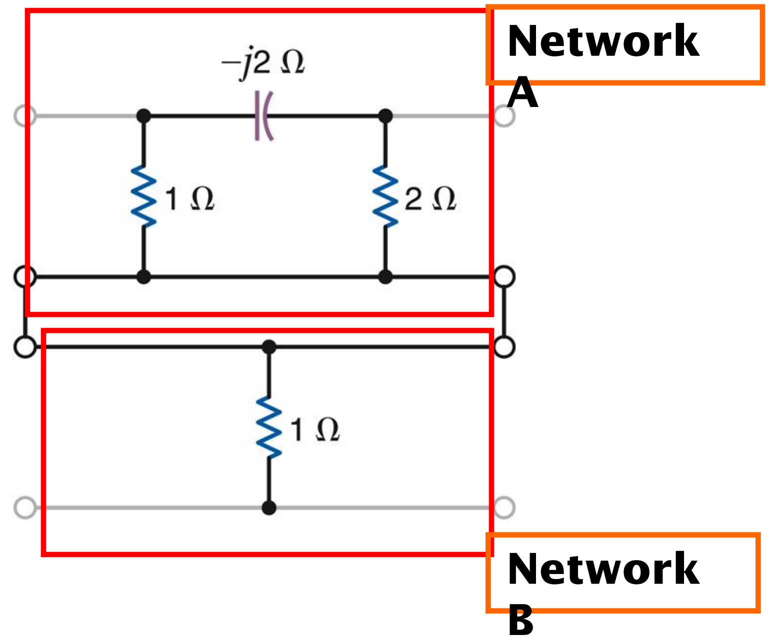
$$V_1 = 2I_1 + I_2$$
$$V_2 = I_1 + 3I_2$$

$$Y_b = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

LEARNING EXAMPLE

Find the Z parameters of the network





Use direct method,
or given the Y parameters transform to Z
… or decompose the network in a series connection of simpler networks

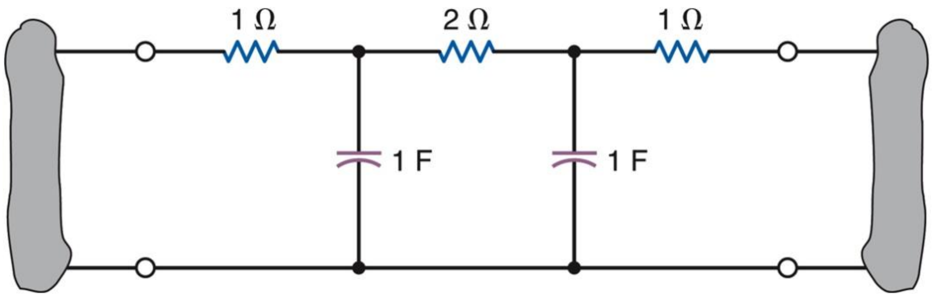
$$Z_a = \begin{bmatrix} \dfrac{2-2j}{3-2j} & \dfrac{2}{3-2j} \\ \dfrac{2}{3-2j} & \dfrac{2-4j}{3-2j} \end{bmatrix}$$

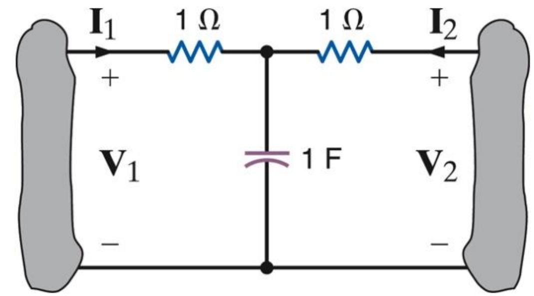
$$Z_b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Z = Z_a + Z_b = \begin{bmatrix} \dfrac{5-4j}{3-2j} & \dfrac{5-2j}{3-2j} \\ \dfrac{5-2j}{3-2j} & \dfrac{5-6j}{3-2j} \end{bmatrix}$$

LEARNING EXAMPLE

Find the transmission parameters





$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+j\omega & 2+j\omega \\ j\omega & 1+j\omega \end{bmatrix}$$

By splitting the 2-Ohm resistor, the network can be viewed as the cascade connection of two identical networks

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+j\omega & 2+j\omega \\ j\omega & 1+j\omega \end{bmatrix} \begin{bmatrix} 1+j\omega & 2+j\omega \\ j\omega & 1+j\omega \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (1+j\omega)^2 + (2+j\omega)j\omega & (1+j\omega)(2+j\omega)+(2+j\omega)(1+j\omega) \\ j\omega(1+j\omega)+(1+j\omega)(j\omega) & j\omega(2+j\omega)+(1+j\omega)^2 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+4j\omega-2\omega^2 & 4+6j\omega-2\omega^2 \\ 2j\omega-2\omega^2 & 1+4j\omega-2\omega^2 \end{bmatrix}$$