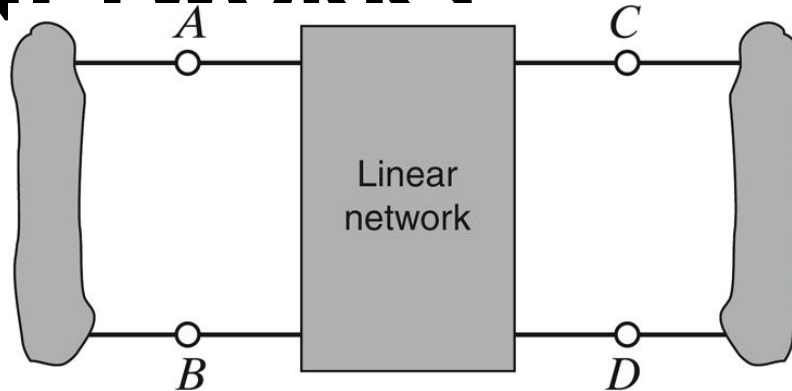


# TWO-PORT NETWORKS



In many situations one is not interested in the internal organization of a network. A description relating input and output variables may be sufficient

A two-port model is a description of a network that relates voltages and currents at two pairs of terminals

## LEARNING GOALS

Study the basic types of two-port models

Admittance parameters

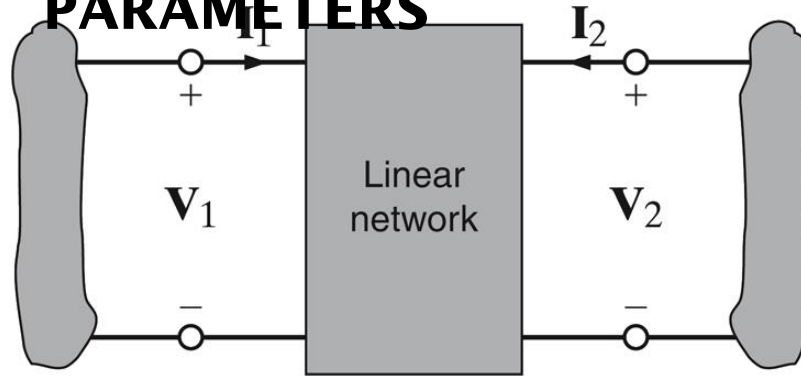
Impedance parameters

Hybrid parameters

Transmission parameters

Understand how to convert one model into another

# ADMITTANCE PARAMETERS



The network contains **NO independent sources**

The admittance parameters describe the currents in terms of the voltages

$y_{21}$  determines the current flowing into port 2 when the port is short-circuited and a voltage is applied to port 1

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

The first subindex identifies the output port. The second the input port.

The computation of the parameters follows directly from the definition

$$y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0}$$

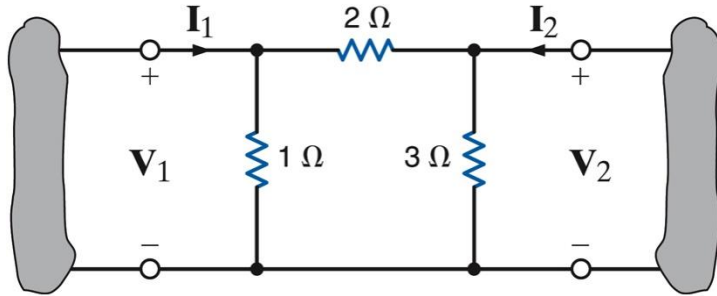
$$y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0}$$

$$y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0}$$

$$y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0}$$

**LEARNING  
EXAMPLE**

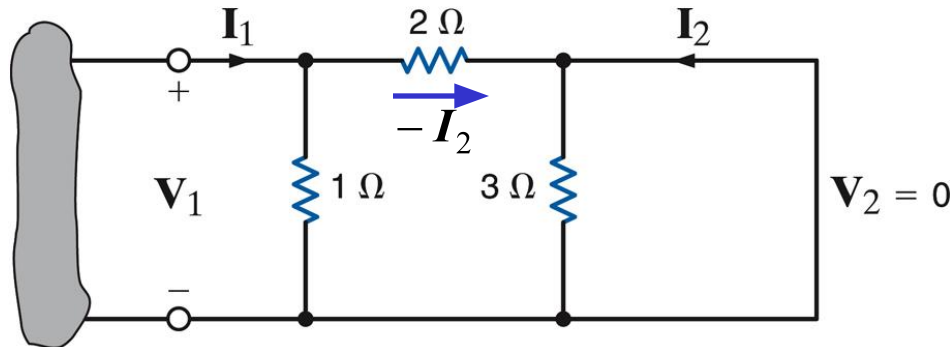
**Find the admittance parameters for the  
network**



$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

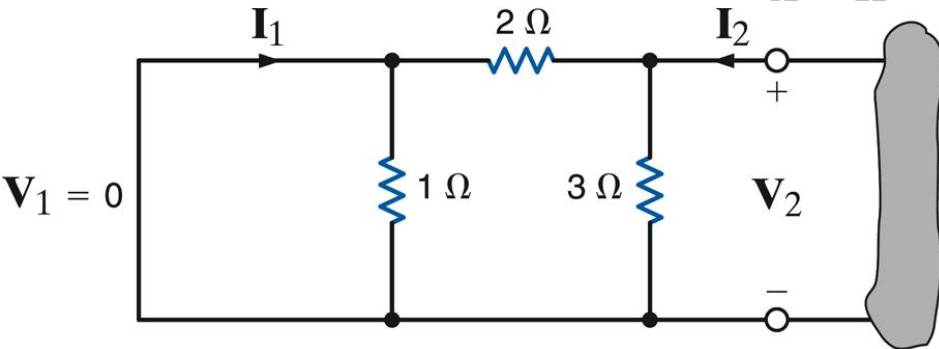
Circuit used to determine  $y_{11}, y_{21}$



$$I_1 = \left(1 + \frac{1}{2}\right)V_1 \Rightarrow y_{11} = \frac{3}{2}[\mathcal{S}]$$

$$-I_2 = \frac{1}{1+2}I_1 \Rightarrow I_2 = -\frac{1}{2}V_1 \Rightarrow y_{21} = -\frac{1}{2}[\mathcal{S}]$$

Circuit used to determine  $y_{12}, y_{22}$



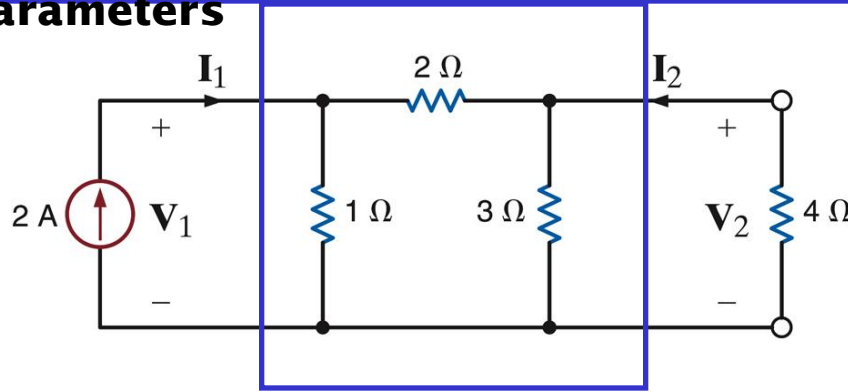
$$I_2 = \left(\frac{1}{2} + \frac{1}{3}\right)V_2 \Rightarrow y_{22} = \frac{5}{6}[\mathcal{S}]$$

$$-I_1 = \frac{3}{2+3}I_2 = \frac{3 \times 5}{5 \times 6}V_2 \Rightarrow y_{12} = \frac{1}{2}[\mathcal{S}]$$

**Next we show one use of this  
model**

**An application of the admittance parameters**

**Determine the current through the 4 Ohm resistor**



$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$I_1 = \frac{3}{2}V_1 - \frac{1}{2}V_2$$

$$I_2 = -\frac{1}{2}V_1 + \frac{5}{6}V_2$$

$$I_1 = 2A, \quad V_2 = -4I_2 \quad I_2 = -\frac{1}{4}V_2$$

**The model plus the conditions at the ports are sufficient to determine the other variables**

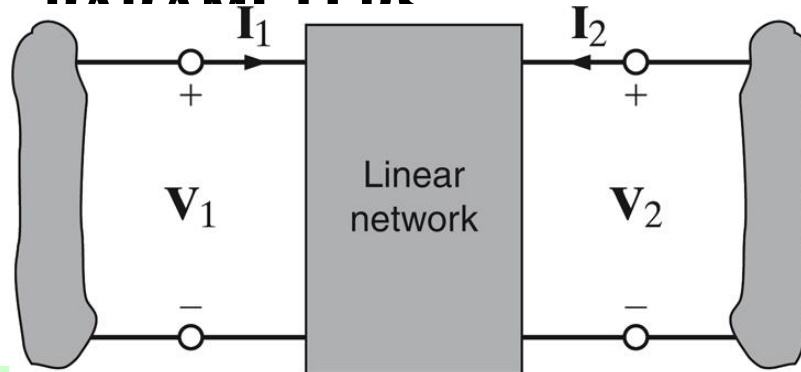
$$0 = -\frac{1}{2}V_1 + \left(\frac{5}{6} + \frac{1}{4}\right)V_2$$

$$V_1 = \frac{13}{6}V_2$$

$$V_2 = \frac{8}{11}[V]$$

$$I_2 = -\frac{2}{11}[A]$$

# IMPEDANCE



The network contains **NO** independent sources

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

The '*z* parameters' can be derived in a manner similar to the *Y* parameters

$$z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0}$$

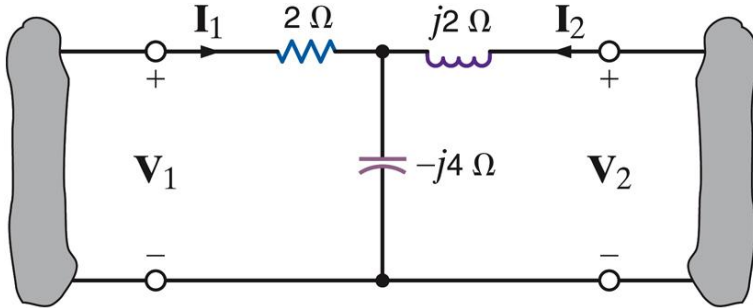
$$z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0}$$

$$z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0}$$

$$z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0}$$

## LEARNING EXAMPLE

## Find the Z parameters



$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0}$$

$$z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0}$$

$$z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0}$$

$$z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0}$$

## Write the loop equations

$$V_1 = 2I_1 - j4(I_1 + I_2)$$

$$V_2 = j2I_2 - j4(I_2 + I_1)$$

## rearrangin

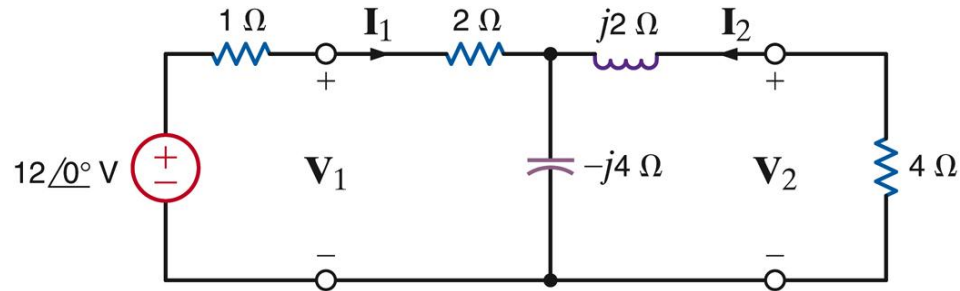
g

$$V_1 = (2 - j4)I_1 - j4I_2 \quad \Rightarrow \quad z_{11} = 2 - j4\Omega \quad z_{12} = -j4\Omega$$

$$V_2 = -j4I_1 - j2I_2 \quad z_{21} = -j4\Omega \quad z_{22} = -j2\Omega$$

## LEARNING EXAMPLE

Use the Z parameters to find the current through the 4 Ohm resistor



$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

Output port  
constraint

$$V_2 = -4I_2$$

Input port  
constraint

$$V_1 = 12\angle 0^\circ - (1)I_1$$

$$V_1 = (2 - j4)I_1 - j4I_2$$

$$V_2 = -j4I_1 - j2I_2$$

$$0 = -j4I_1 + (4 - j2)I_2 \quad \times (3 - j4)$$

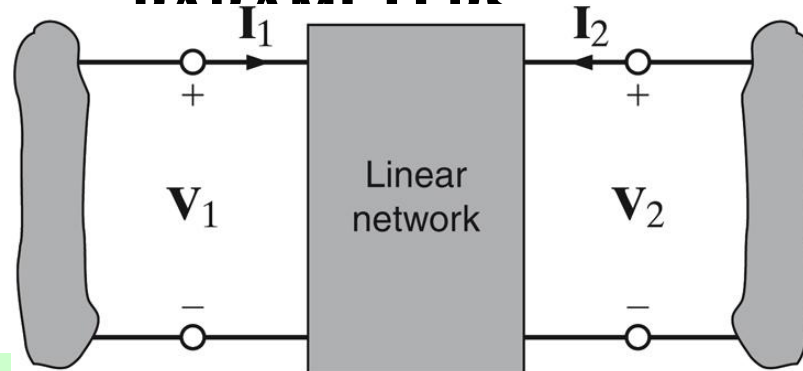
$$12 = (3 - j4)I_1 - j4I_2 \quad \times j4$$

---

$$48j = (16 + (4 - j2)(3 - j4))I_2 \quad \Rightarrow I_2 = 1.61\angle 137.73^\circ$$

# HYBRID

## PARAMETERS



The network contains **NO** independent sources

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0}$$

$$h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0}$$

$h_{11}$  = short - circuit input impedance

$h_{12}$  = open - circuit reverse voltage gain

$$h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0}$$

$$h_{22} = \left. \frac{I_2}{V_2} \right|_{I_1=0}$$

$h_{21}$  = short - circuit forward current gain

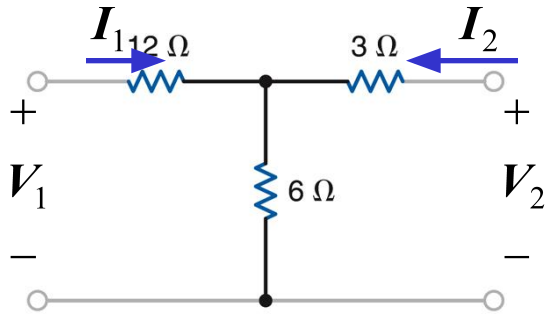
$h_{22}$  = open - circuit output admittance

**These parameters are very common in modeling transistors**



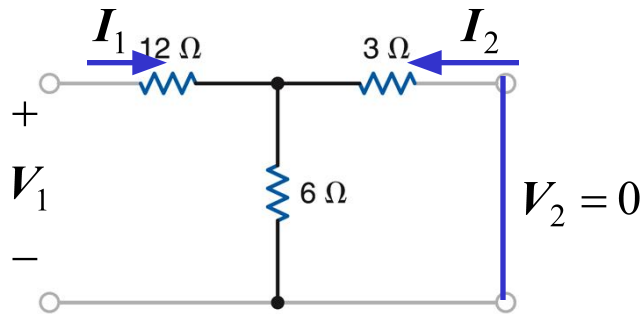
**LEARNING  
EXAMPLE**

**Find the hybrid parameters for the  
network**



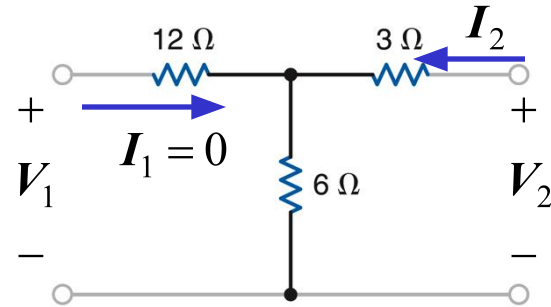
$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$



$$V_1 = (12 + (6 \parallel 3))I_1 \Rightarrow h_{11} = 14\Omega$$

$$I_2 = -\frac{6}{3+6}I_1 \Rightarrow h_{21} = -\frac{2}{3}$$



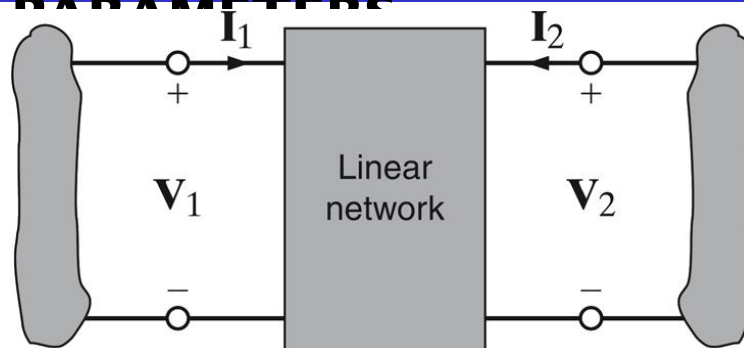
$$V_1 = \frac{6}{3+6}V_2 \Rightarrow h_{12} = \frac{2}{3}$$

$$I_2 = \frac{V_2}{9} \Rightarrow h_{22} = \frac{1}{9}[S]$$

# TRANSMISSION

PARAMETERS

# ABCD parameters



The network contains **NO independent sources**

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

$$A = \left. \frac{V_1}{V_2} \right|_{I_2=0}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$B = - \left. \frac{V_1}{I_2} \right|_{V_2=0}$$

$$D = - \left. \frac{I_1}{I_2} \right|_{V_2=0}$$

A = open circuit voltage ratio

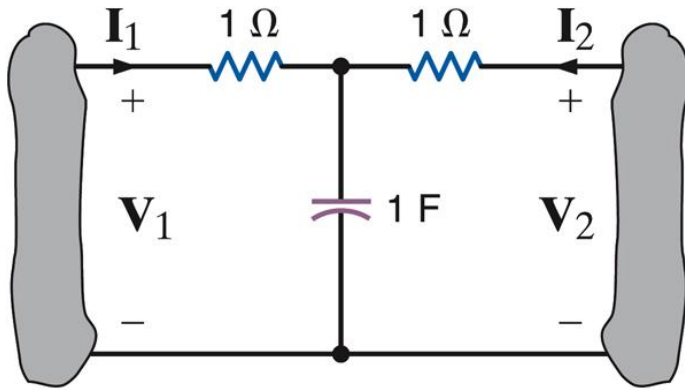
B = negative short -circuit transfer impedance

C = open -circuit transfer admittance

D = negative short -circuit current ratio

# LEARNING EXAMPLE

# Determine the transmission parameters



$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

$$A = \left. \frac{V_1}{V_2} \right|_{I_2=0}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$B = - \left. \frac{V_1}{I_2} \right|_{V_2=0}$$

$$D = - \left. \frac{I_1}{I_2} \right|_{V_2=0}$$

when  $I_2 = 0$

$$V_2 = \frac{1}{1 + \frac{1}{j\omega}} V_1 \Rightarrow A = 1 + j\omega$$

$$V_2 = \frac{1}{j\omega} I_1 \Rightarrow \frac{I_1}{V_2} = j\omega$$

when  $V_2 = 0$

$$I_2 = - \frac{\frac{1}{j\omega}}{1 + \frac{1}{j\omega}} I_1 = - \frac{1}{1 + j\omega} I_1 \Rightarrow D = 1 + j\omega$$

$$V_1 = \left[ 1 + \left( 1 \parallel \frac{1}{j\omega} \right) \right] I_1 = \left[ \frac{2 + j\omega}{1 + j\omega} \right] [-(1 + j\omega)] I_2$$

$$B = 2 + j\omega$$

# PARAMETER CONVERSIONS

If all parameters exist, they can be related by conventional algebraic manipulations.

As an example consider the relationship between Z and Y parameters

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta_Z} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

$$\text{with } \Delta_Z = z_{11}z_{22} - z_{21}z_{12}$$

In the following conversion table, the symbol  $\Delta$  stands for the determinant of the corresponding matrix

$$\Delta_Z = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \Delta_Y = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}, \Delta_H = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \Delta_T = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

**TABLE 16.1** Two-port parameter conversion formulas

$$\begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{z}_{22}}{\Delta_Z} & \frac{-\mathbf{z}_{12}}{\Delta_Z} \\ \frac{-\mathbf{z}_{21}}{\Delta_Z} & \frac{\mathbf{z}_{11}}{\Delta_Z} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{z}_{11}}{\mathbf{z}_{21}} & \frac{\Delta_Z}{\mathbf{z}_{21}} \\ 1 & \frac{\mathbf{z}_{22}}{\mathbf{z}_{21}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\Delta_Z}{\mathbf{z}_{22}} & \frac{\mathbf{z}_{12}}{\mathbf{z}_{22}} \\ -\frac{\mathbf{z}_{21}}{\mathbf{z}_{22}} & \frac{1}{\mathbf{z}_{22}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{y}_{22}}{\Delta_Y} & \frac{-\mathbf{y}_{12}}{\Delta_Y} \\ \frac{-\mathbf{y}_{21}}{\Delta_Y} & \frac{\mathbf{y}_{11}}{\Delta_Y} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-\mathbf{y}_{22}}{\mathbf{y}_{21}} & \frac{-1}{\mathbf{y}_{21}} \\ \mathbf{y}_{21} & \frac{\mathbf{y}_{11}}{\mathbf{y}_{21}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\mathbf{y}_{11}} & \frac{-\mathbf{y}_{12}}{\mathbf{y}_{11}} \\ \frac{\mathbf{y}_{21}}{\mathbf{y}_{11}} & \frac{\Delta_Y}{\mathbf{y}_{11}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{A}}{\mathbf{C}} & \frac{\Delta_T}{\mathbf{C}} \\ 1 & \frac{\mathbf{D}}{\mathbf{C}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{D}}{\mathbf{B}} & \frac{-\Delta_T}{\mathbf{B}} \\ -\frac{1}{\mathbf{B}} & \frac{\mathbf{A}}{\mathbf{B}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathbf{B}}{\mathbf{D}} & \frac{\Delta_T}{\mathbf{D}} \\ -\frac{1}{\mathbf{D}} & \frac{\mathbf{C}}{\mathbf{D}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\Delta_H}{\mathbf{h}_{22}} & \frac{\mathbf{h}_{12}}{\mathbf{h}_{22}} \\ -\frac{\mathbf{h}_{21}}{\mathbf{h}_{22}} & \frac{1}{\mathbf{h}_{22}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\mathbf{h}_{11}} & \frac{-\mathbf{h}_{12}}{\mathbf{h}_{11}} \\ \frac{\mathbf{h}_{21}}{\mathbf{h}_{11}} & \frac{\Delta_H}{\mathbf{h}_{11}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-\Delta_H}{\mathbf{h}_{21}} & \frac{-\mathbf{h}_{11}}{\mathbf{h}_{21}} \\ \frac{-\mathbf{h}_{22}}{\mathbf{h}_{21}} & \frac{-1}{\mathbf{h}_{21}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix}$$

# INTERCONNECTION OF

Interconnections permit the description of complex systems in terms of simpler

The basic interconnections to be considered are: *parallel*, *series* and *cascade*

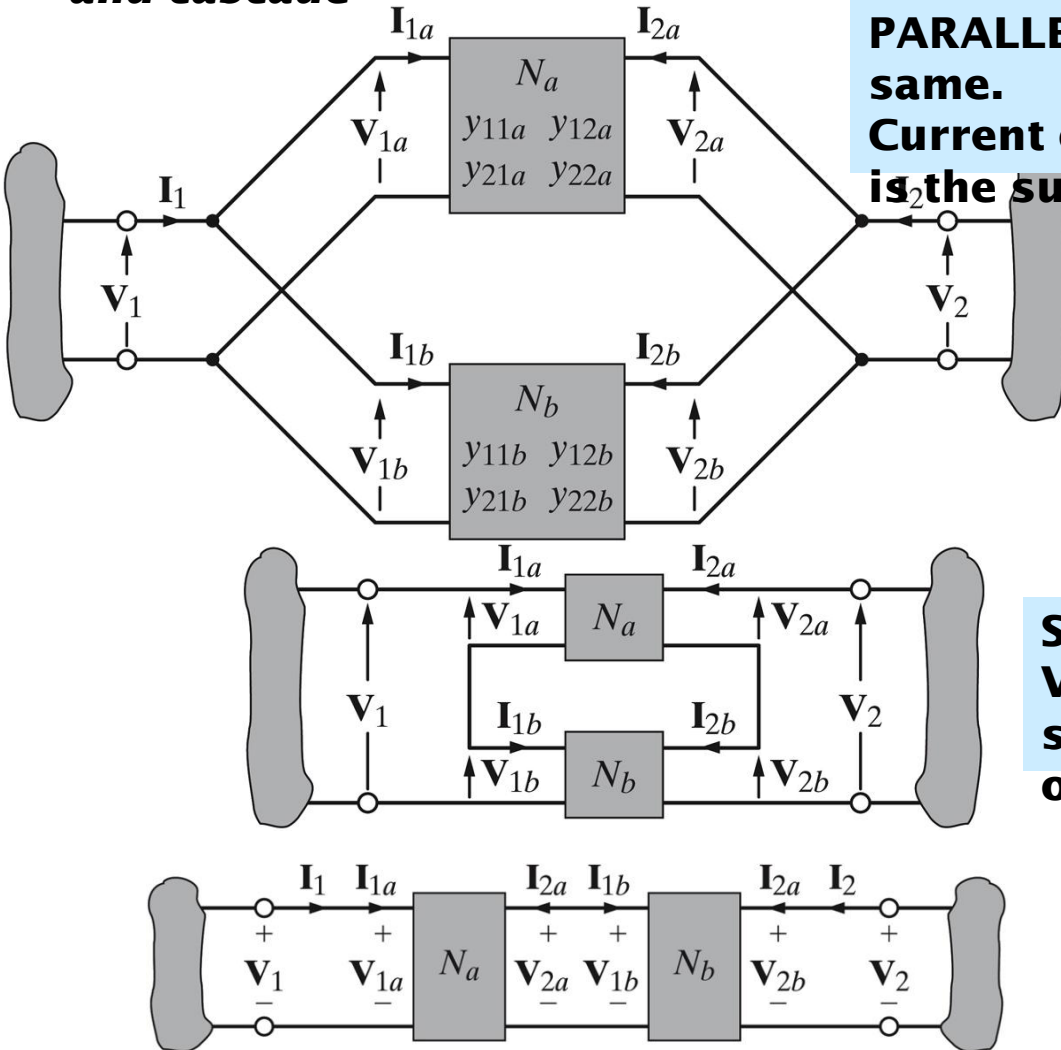
**PARALLEL:** Voltages are the same.  
Current of interconnection is the sum of currents

The rules used to derive models for interconnection assume that each subsystem behaves in the

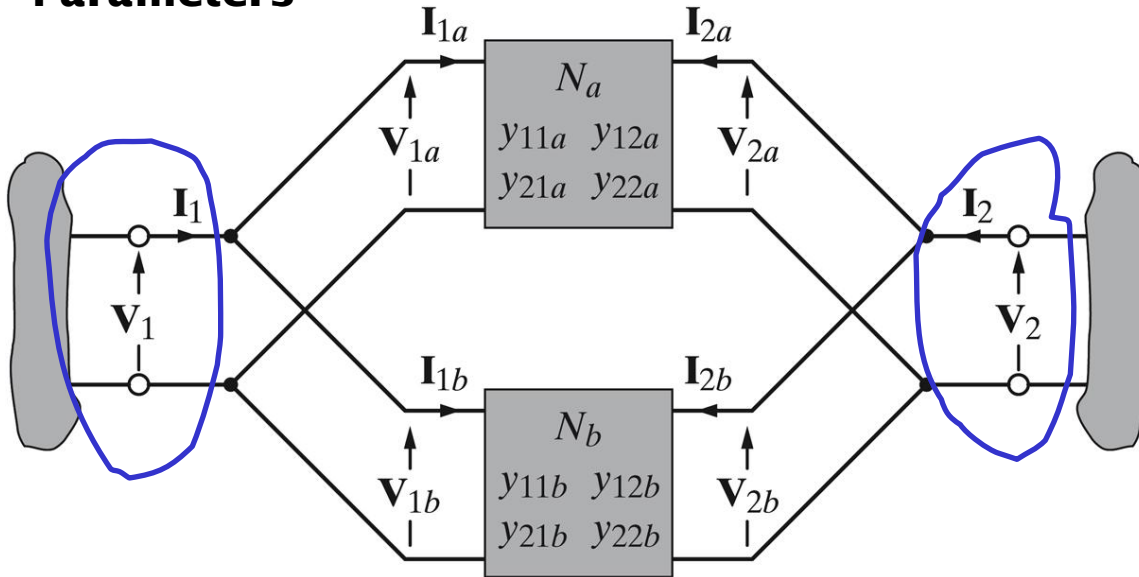
same manner before and after the interconnection

**SERIES:** Currents are the same.  
Voltage of interconnection is the sum of voltages

**CASCADE:**  
Output of first subsystem acts as input for the



## Parallel Interconnection: Description Using Y Parameters



Interconnection description

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I = YV$$

$$I_a = \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix}, V_a = \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix}, Y_a = \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} \Rightarrow I_a = Y_a V_a$$

In a similar manner

$$I_b = Y_b V_b$$

Interconnection constraints:

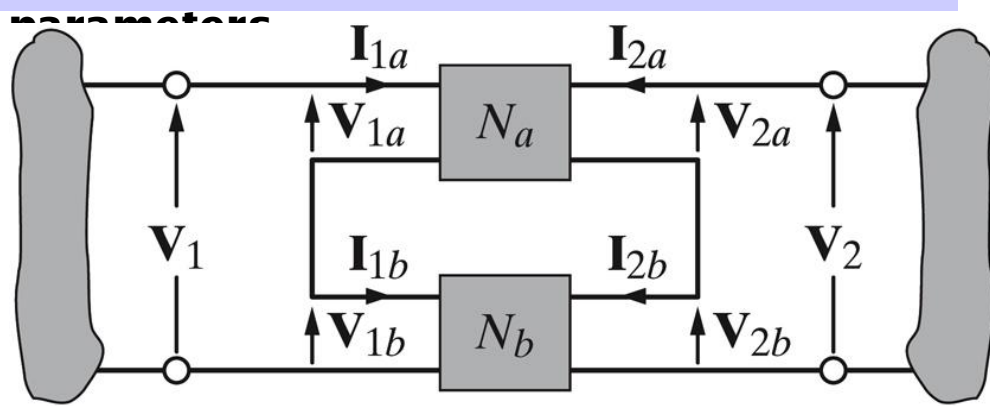
$$I_1 = I_{1a} + I_{1b}, \quad I_2 = I_{2a} + I_{2b}$$

$$V_1 = V_{1a} = V_{1b}, \quad V_2 = V_{2a} = V_{2b}$$

$$\Rightarrow \begin{cases} I = I_a + I_b \\ V = V_a = V_b \end{cases} \Rightarrow I = Y_a V_a + Y_b V_b = (Y_a + Y_b) V$$

$$Y = Y_a + Y_b$$

## Series interconnection using Z



Interconnection constraints

$$I_a = I_b = I$$

$$V = V_a + V_b$$

$$\Rightarrow V = Z_a I + Z_b I = (Z_a + Z_b) I$$

**SERIES: Currents are the same. Voltage of interconnection is the sum of voltages**

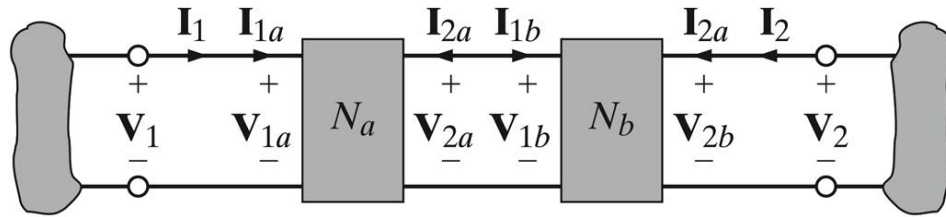
Description of each subsystem

$$V_a = Z_a I_a, \quad V_b = Z_b I_b$$

$$\mathbf{Z} = \mathbf{Z}_a + \mathbf{Z}_b$$



# Cascade connection using transmission parameters



Interconnection constraints:

$$I_{2a} = -I_{1b} \quad V_{2a} = V_{1b}$$

$$V_1 = V_{1a} \quad V_2 = V_{2b}$$

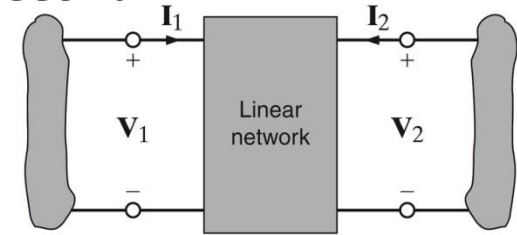
$$I_1 = I_{1a} \quad I_2 = I_{2b}$$

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix}$$

$$\begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

**CASCADE:**  
Output of first subsystem acts as input for the second



$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

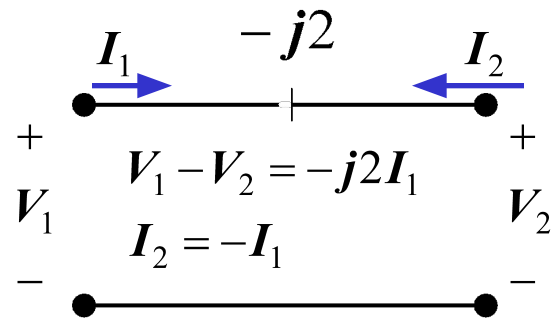
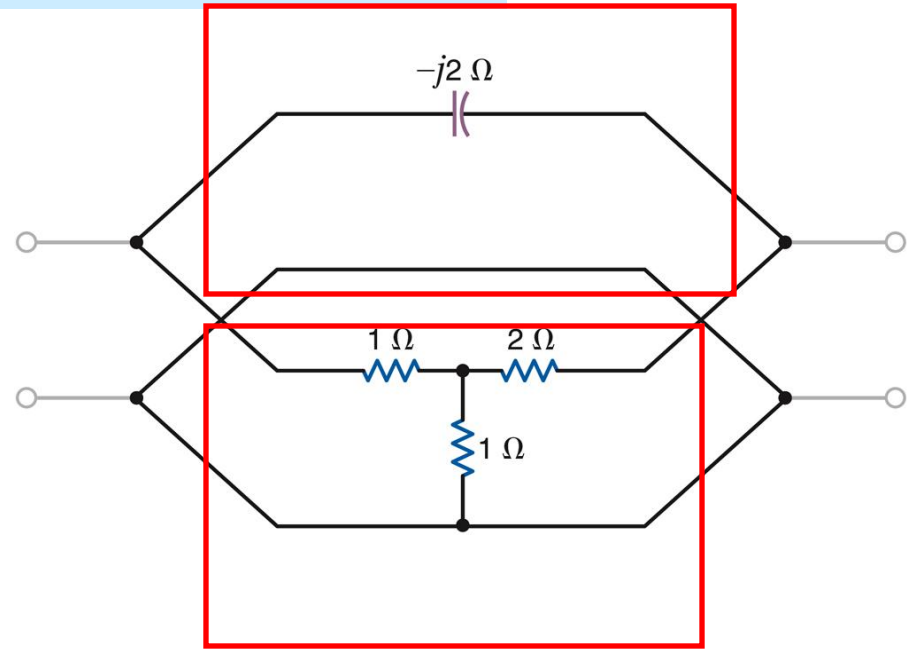
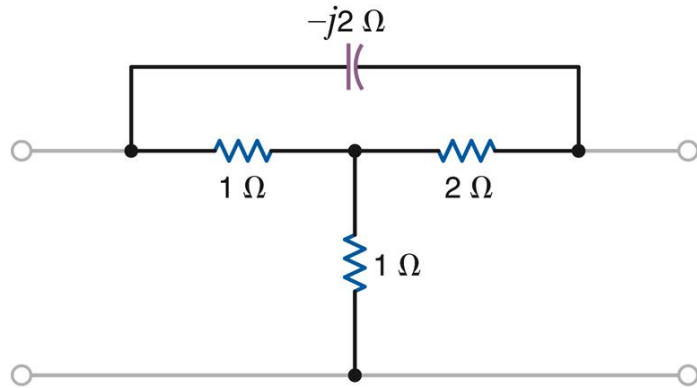
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

**Matrix multiplication does not commute.**

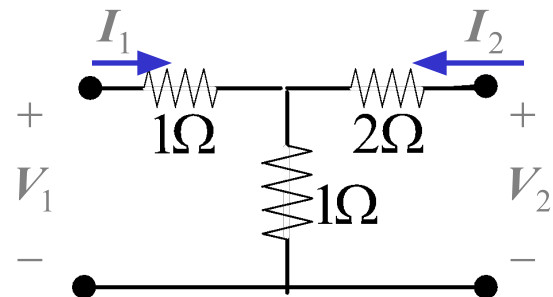
**Order of the interconnection is important**

# LEARNING EXAMPLE

# Find the Y parameters for the network



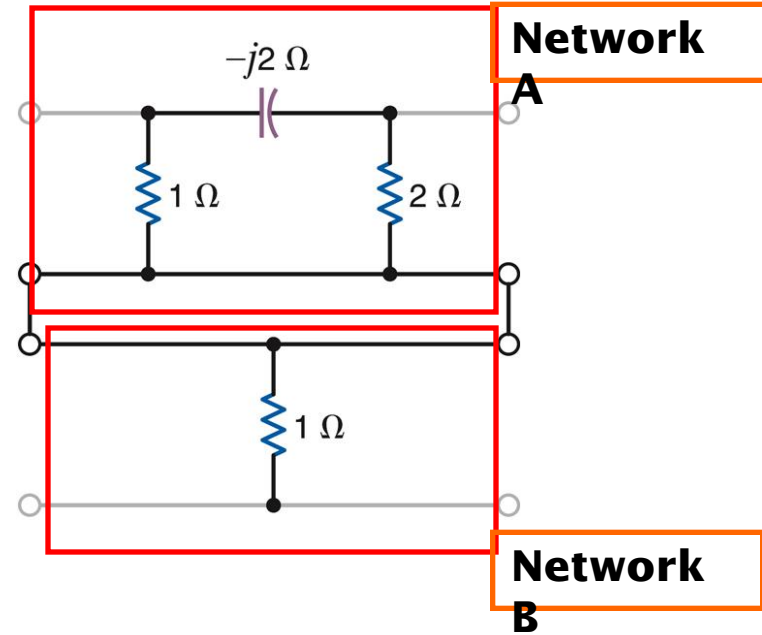
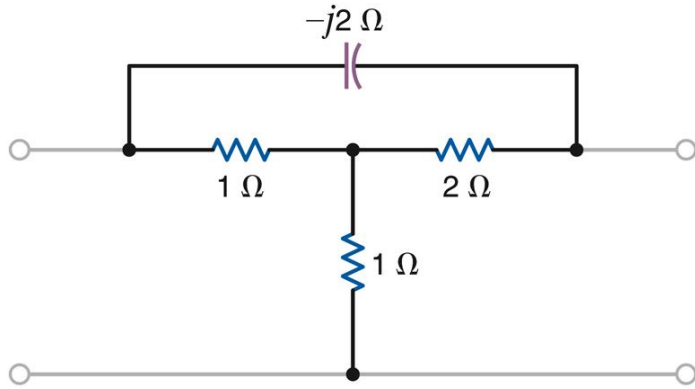
$$\begin{aligned}
 y_{11a} &= j\frac{1}{2}, & y_{12a} &= -j\frac{1}{2} \\
 y_{21a} &= -j\frac{1}{2}, & y_{22a} &= j\frac{1}{2}
 \end{aligned}
 \quad
 \mathbf{Y} = \begin{bmatrix} \frac{3}{5} + j\frac{1}{2} & -\left(\frac{1}{5} + j\frac{1}{2}\right) \\ -\left(\frac{1}{5} + j\frac{1}{2}\right) & \frac{2}{5} + j\frac{1}{2} \end{bmatrix} [\mathbf{S}]$$



$$\begin{aligned}
 V_1 &= 2I_1 + I_2 \\
 V_2 &= I_1 + 3I_2
 \end{aligned}
 \quad
 \mathbf{Y}_b = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

**LEARNING  
EXAMPLE**

**Find the Z parameters of the  
network**



**Use direct method,  
or given the Y parameters transform  
to Z**

**... or decompose the network in a  
series connection of simpler networks**

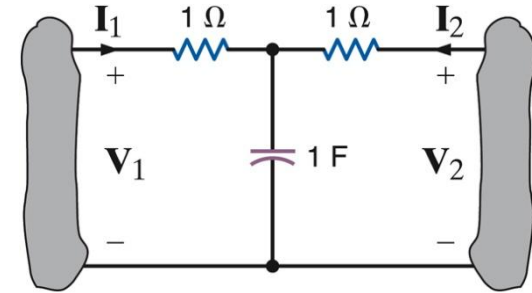
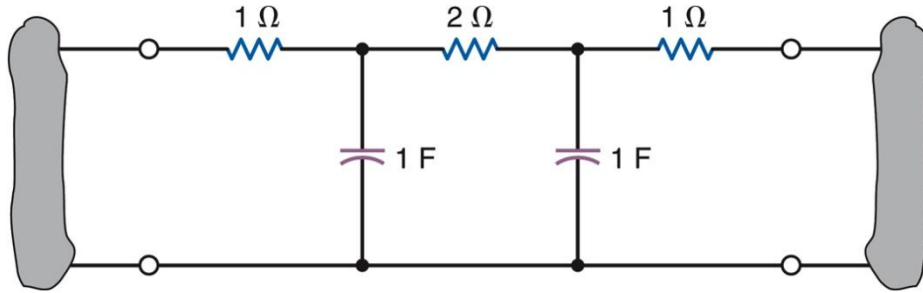
$$\mathbf{Z}_a = \begin{bmatrix} 2-2j & 2 \\ 3-2j & 3-2j \end{bmatrix}$$

$$\mathbf{Z}_b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{Z} = \mathbf{Z}_a + \mathbf{Z}_b = \begin{bmatrix} 5-4j & 5-2j \\ 3-2j & 3-2j \\ 5-2j & 5-6j \\ 3-2j & 3-2j \end{bmatrix}$$

**LEARNING  
EXAMPLE**

**Find the transmission  
parameters**



**By splitting the 2-Ohm resistor,  
the network can be viewed as the  
cascade connection of two  
identical**

**networks**

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + j\omega & 2 + j\omega \\ j\omega & 1 + j\omega \end{bmatrix} \begin{bmatrix} 1 + j\omega & 2 + j\omega \\ j\omega & 1 + j\omega \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + j\omega & 2 + j\omega \\ j\omega & 1 + j\omega \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (1 + j\omega)^2 + (2 + j\omega)j\omega & (1 + j\omega)(2 + j\omega) + (2 + j\omega)(1 + j\omega) \\ j\omega(1 + j\omega) + (1 + j\omega)(j\omega) & j\omega(2 + j\omega) + (1 + j\omega)^2 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + 4j\omega - 2\omega^2 & 4 + 6j\omega - 2\omega^2 \\ 2j\omega - 2\omega^2 & 1 + 4j\omega - 2\omega^2 \end{bmatrix}$$