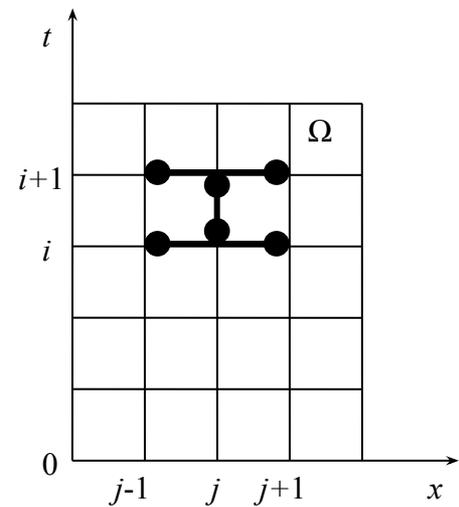


УРАВНЕНИЯ ПАРАБОЛИЧЕСКОГО ТИПА



$$\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2} + f(t, x)$$

$$u(0, x) = U(x)$$

с краевыми условиями

$$u(t, 0) = U_0(t), \quad u(t, L) = U_L(t)$$

**Схема с
«весами»**

$$\frac{1}{\tau} (u_j^{\boxtimes} - u_j) = \frac{\eta \sigma}{h^2} (u_{j+1}^{\boxtimes} - 2u_j^{\boxtimes} + u_{j-1}^{\boxtimes}) + \frac{\eta(1-\sigma)}{h^2} (u_{j+1} - 2u_j + u_{j-1}) + \tilde{f}$$

$$0 \leq \sigma \leq 1, \quad \tilde{f} = f(t_{i+1/2}, x_j)$$

При $\sigma = 0$ схема чисто явная,
при $\sigma = 1$ схема чисто неявная

1. Погрешность аппроксимации

$$u(t_{i+1}, x_j) = u(t_i, x_j) + u'(t_i, x_j)\tau + \frac{u''(t_i, x_j)}{2}\tau^2 + \frac{u'''(t_i, x_j)}{3!}\tau^3 + O(\tau^4)$$

$$u(t_i, x_{j\pm 1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} +$$

$$+ u^{iv}(t_i, x_j)\frac{h^4}{24} \pm u^v(t_i, x_j)\frac{h^5}{120} + O(h^6)$$

$$f(t_{i+1/2}, x_j) = f(t_i, x_j) + \frac{\partial f}{\partial t}(t_i, x_j) \frac{\tau}{2} + \frac{\partial^2 f}{\partial t^2}(t_i, x_j) \frac{\tau^2}{8} + O(\tau^3)$$

$$\begin{aligned} u(t_{i+1}, x_{j\pm 1}) &= u(t_i, x_j) + \cancel{u}(t_i, x_j)\tau \pm \cancel{u}'(t_i, x_j)h + \\ &+ \frac{1}{2} \left[\cancel{u}(t_i, x_j)\tau^2 \pm 2\cancel{u}(t_i, x_j)\tau h + u''(t_i, x_j)h^2 \right] + \\ &+ \frac{1}{3!} \left[\cancel{u}(t_i, x_j)\tau^3 \pm 3\cancel{u}(t_i, x_j)\tau^2 h + \cancel{u}''(t_i, x_j)\tau h^2 \pm u'''(t_i, x_j)h^3 \right] + \\ &+ \frac{1}{4!} \left[\cancel{u}(t_i, x_j)\tau^4 \pm 4\cancel{u}(t_i, x_j)\tau^3 h + 6\cancel{u}''(t_i, x_j)\tau^2 h^2 \pm 4u'''(t_i, x_j)h^3 \tau + u^{iv}(t_i, x_j)h^4 \right] + \dots \end{aligned}$$

$$\begin{aligned} \psi_j^i &= \left[\cancel{u}(t_i, x_j) + \cancel{u}(t_i, x_j) \frac{\tau}{2} + \cancel{u}(t_i, x_j) \frac{\tau^2}{3!} + O(\tau^3) \right] - \\ &- \eta \sigma \left[u''(t_i, x_j) + \cancel{u}''(t_i, x_j)\tau + \cancel{u}''(t_i, x_j) \frac{\tau^2}{2} + u^{iv}(t_i, x_j) \frac{h^2}{12} \right] - \\ &- \eta(1-\sigma) \left[u''(t_i, x_j) + u^{iv}(t_i, x_j) \frac{h^2}{12} \right] - \end{aligned}$$

$$\left[f(t_i, x_j) + \frac{\partial f}{\partial t}(t_i, x_j) \frac{\tau}{2} + \frac{\partial^2 f}{\partial t^2}(t_i, x_j) \frac{\tau^2}{8} \right] + O(\tau^3, h^3)$$

$$\psi_j^i = \left[\bar{u}(t_i, x_j) - \eta u''(t_i, x_j) - f(t_i, x_j) \right] + \frac{\tau}{2} \left[\bar{u}(t_i, x_j) - 2\eta \sigma \bar{u}''(t_i, x_j) - \bar{f}(t_i, x_j) \right] +$$

$$+ \frac{\tau^2}{24} \left[4\bar{u}(t_i, x_j) - 12\eta \sigma \bar{u}''(t_i, x_j) - 3\bar{f}(t_i, x_j) \right] - \eta \frac{h^2}{12} u^{iv}(t_i, x_j) + O(\tau^3, h^3)$$

$$\psi_j^i = \frac{\tau}{2} \left[\bar{u}(t_i, x_j) - 2\eta \sigma \bar{u}''(t_i, x_j) - \bar{f}(t_i, x_j) \right] +$$

$$+ \frac{\tau^2}{24} \left[4\bar{u}(t_i, x_j) - 12\eta \sigma \bar{u}''(t_i, x_j) - 3\bar{f}(t_i, x_j) \right] - \eta \frac{h^2}{12} u^{iv}(t_i, x_j) + O(\tau^3, h^3) = O(\tau, h^2)$$

Дополнительно $\sigma = 1/2$ и

$$\bar{u} = \frac{d}{dt}(u + f) = \eta u'' + \bar{f}$$

тогда

$$\psi_j^i = \frac{\tau^2}{24} \left[4\bar{u}(t_i, x_j) - 12\eta \sigma \bar{u}''(t_i, x_j) - 3\bar{f}(t_i, x_j) \right] - \eta \frac{h^2}{12} u^{iv}(t_i, x_j) + O(\tau^3, h^3) = O(\tau^2, h^2)$$

При $\sigma = 1/2$ схема называется схемой Крэнка–Николсона

2. Устойчивость

$$\begin{aligned} & \boxed{u}_{j-1} \left(-\frac{\eta\sigma}{h^2} \right) + \boxed{u}_j \left(\frac{1}{\tau} + \frac{2\eta\sigma}{h^2} \right) + \boxed{u}_{j+1} \left(-\frac{\eta\sigma}{h^2} \right) = \\ & = u_{j-1} \frac{\eta(1-\sigma)}{h^2} + u_j \left(\frac{1}{\tau} - \frac{2\eta(1-\sigma)}{h^2} \right) + u_{j+1} \frac{\eta(1-\sigma)}{h^2} + \tilde{f} \end{aligned}$$

$$\alpha_{\max} = \frac{1}{\tau} + \frac{2\eta\sigma}{h^2}$$

тогда по принципу максимума

$$\frac{1}{\tau} + \frac{2\eta\sigma}{h^2} \geq \frac{2\eta\sigma}{h^2} + \frac{2\eta(1-\sigma)}{h^2} + \left| \frac{1}{\tau} - \frac{2\eta(1-\sigma)}{h^2} \right|$$

$$\frac{1}{\tau} - \frac{2\eta(1-\sigma)}{h^2} \geq \left| \frac{1}{\tau} - \frac{2\eta(1-\sigma)}{h^2} \right|$$

$$\frac{1}{\tau} - \frac{2\eta(1-\sigma)}{h^2} \geq 0$$

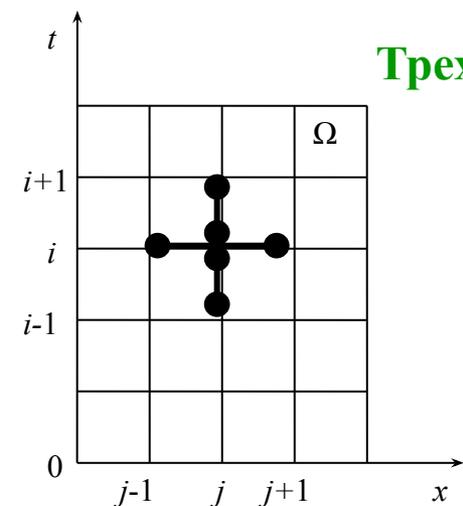
$$\tau \leq \frac{h^2}{2\eta(1-\sigma)}$$

Условие устойчивости по правой части

$$\frac{1}{\tau} + \frac{2\eta\sigma}{h^2} - \frac{2\eta\sigma}{h^2} \geq \frac{\omega}{\tau}, \quad 0 < \omega < 1$$

Т.о. схема условно устойчива

Трехслойная схема Ричардсона



$$\frac{\bar{u}_j - \underline{u}_j}{2\tau} = \eta \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + f(t_i, x_j)$$

Устойчивость схемы
(методом Неймана)

$$(\delta u_j^{m-1})^{(k)} = (\rho_k)^{m-1} a_k e^{ikx_j}, \quad (\delta u_j^m)^{(k)} = (\rho_k)^m a_k e^{ikx_j}, \quad (\delta u_j^{m+1})^{(k)} = (\rho_k)^{m+1} a_k e^{ikx_j}$$

$$\frac{1}{2\tau} \left((\rho_k)^{m+1} - (\rho_k)^{m-1} \right) e^{ikx_j} = (\rho_k)^m \frac{\eta}{h^2} \left(e^{ik(x_j+h)} - 2e^{ikx_j} + e^{ik(x_j-h)} \right) \quad \Bigg| \quad \cdot \frac{(\rho_k)^{m-1} e^{ikx_j}}{2\tau}$$

$$\rho_k^2 - \rho_k \frac{2\tau\eta}{h^2} (e^{ikh} - 2 + e^{-ikh}) - 1 = 0$$

$$\rho_k^2 + \rho_k \frac{8\tau\eta}{h^2} \sin^2\left(\frac{kh}{2}\right) - 1 = 0$$

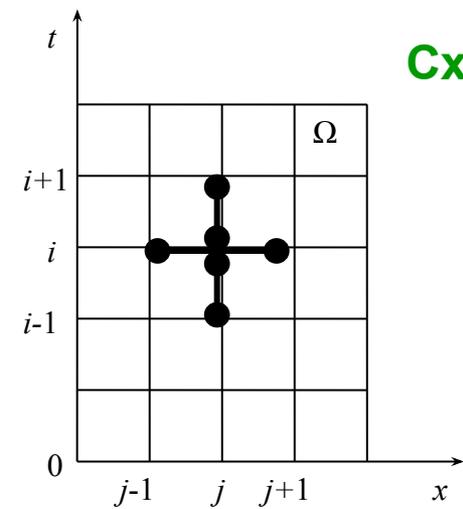
Уравнение имеет только действительные корни (т.к. дискриминант положительный)

$$\rho_k = \frac{4\tau\eta}{h^2} \sin^2\left(\frac{kh}{2}\right) \pm \sqrt{\frac{16\tau^2\eta^2}{h^4} \sin^4\left(\frac{kh}{2}\right) + 1}$$

$$|(\rho_k)_1 \cdot (\rho_k)_2| = 1$$

Схема абсолютно неустойчива !!!

Схема Дюфорты и Франкела



$$\frac{\overset{\boxtimes}{u}_j - \overset{\boxtimes}{u}_j}{2\tau} = \frac{\eta}{h^2} \left[u_{j+1} - 2 \left\{ \frac{\overset{\boxtimes}{u}_j + \overset{\boxtimes}{u}_j}{2} \right\} + u_{j-1} \right] + f(t_i, x_j)$$

1. Погрешность аппроксимации

$$u(t_{i\pm 1}, x_j) = u(t_i, x_j) \pm \overset{\boxtimes}{u}(t_i, x_j)\tau + \overset{\boxtimes}{u}(t_i, x_j)\frac{\tau^2}{2} \pm \overset{\boxtimes}{u}(t_i, x_j)\frac{\tau^3}{3!} + O(\tau^4)$$

$$u(t_i, x_{j\pm 1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} +$$

$$+ u^{iv}(t_i, x_j)\frac{h^4}{24} \pm u^v(t_i, x_j)\frac{h^5}{120} + O(h^6)$$

$$\psi_j^i = \frac{u(t_{i+1}, x_j) - u(t_{i-1}, x_j)}{2\tau} - \frac{\eta}{h^2} \left[\underline{u(t_i, x_{j+1})} - 2 \left\{ \frac{u(t_{i+1}, x_j) + u(t_{i-1}, x_j)}{2} \right\} + \underline{u(t_i, x_{j-1})} \right] - f(t_i, x_j) =$$

$$= \left[\overset{\boxtimes}{u}(t_i, x_j) + \overset{\boxtimes}{u}(t_i, x_j)\frac{\tau^2}{6} + O(\tau^3) \right] -$$

$$\eta \left[\frac{2u(t_i, x_j)}{h^2} + u''(t_i, x_j) + u^{iv}(t_i, x_j) \frac{h^2}{12} + O(h^4) - \left\{ \frac{2u(t_i, x_j)}{h^2} + u(t_i, x_j) \frac{\tau^2}{h^2} + O\left(\frac{\tau^4}{h^2}\right) \right\} \right] - f(t_i, x_j) =$$

$$= [u(t_i, x_j) - \eta u''(t_i, x_j) - f(t_i, x_j)] + \left[u(t_i, x_j) \frac{\tau^2}{6} - \eta u^{iv}(t_i, x_j) \frac{h^2}{12} - \eta u(t_i, x_j) \frac{\tau^2}{h^2} \right] + \dots$$

$$= O\left(\tau^2, h^2, \frac{\tau^2}{h^2}\right)$$

Погрешность аппроксимации зависит от соотношения шагов по времени и координате и мала, если

$$\lim_{\tau, h \rightarrow 0} \frac{\tau}{h} = 0$$

2. Устойчивость схемы (методом Неймана)

$$\frac{1}{2\tau} \left(\underbrace{(\rho_k)^{m+1}}_{\text{red}} - \underbrace{(\rho_k)^{m-1}}_{\text{green}} \right) e^{ikx_j} = \frac{\eta}{h^2} \left(\underbrace{(\rho_k)^m e^{ik(x_j+h)}}_{\text{blue}} - \left\{ \underbrace{(\rho_k)^{m+1} e^{ikx_j}}_{\text{red}} + \underbrace{(\rho_k)^{m-1} e^{ikx_j}}_{\text{green}} \right\} + \underbrace{(\rho_k)^m e^{ik(x_j-h)}}_{\text{blue}} \right)$$

$$\left(\frac{1}{2\tau} + \frac{\eta}{h^2} \right) \rho_k^2 - \rho_k \frac{\eta}{h^2} (e^{ikh} + e^{-ikh}) - \left(\frac{1}{2\tau} - \frac{\eta}{h^2} \right) = 0$$

$$\rho_k^2 - \rho_k \frac{4\tau\eta \cos kh}{2\tau\eta + h^2} - \frac{2\tau\eta - h^2}{2\tau\eta + h^2} = 0$$

Схема бегущего счета

Для четных слоев

$$\frac{u_j - \overset{\Delta}{u}_j}{2\tau} = \frac{\eta}{h^2} [u_{j-1} - u_j - \overset{\boxtimes}{u}_j + \overset{\boxtimes}{u}_{j+1}] + f(t_{i-1/2}, x_j)$$

$$u_j \left(\frac{1}{\tau} + \frac{\eta}{h^2} \right) = u_{j-1} \frac{\eta}{h^2} + \overset{\boxtimes}{u}_j \left(\frac{1}{\tau} - \frac{\eta}{h^2} \right) + \overset{\boxtimes}{u}_{j+1} \frac{\eta}{h^2} + f(t_{i-1/2}, x_j), \quad j = \overline{1, N-1}$$

Для нечетных слоев

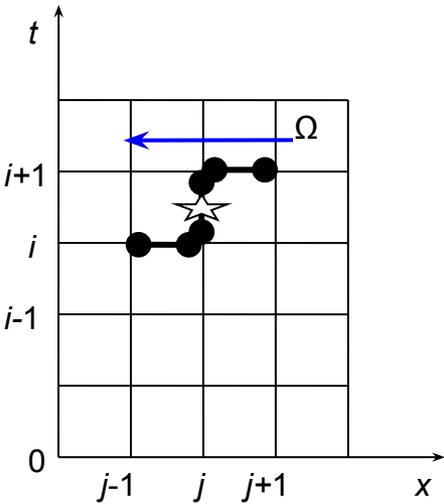
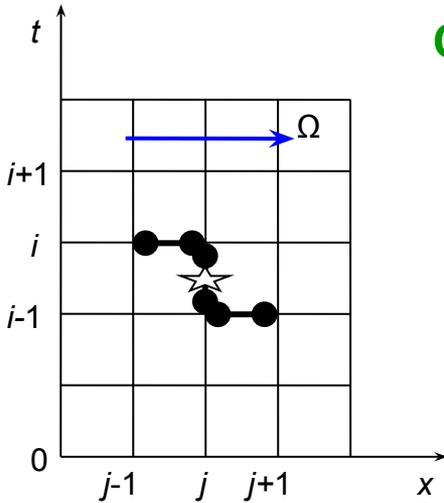
$$\frac{\overset{\Delta}{u}_j - u_j}{2\tau} = \frac{\eta}{h^2} [u_{j-1} - u_j - \overset{\boxtimes}{u}_j + \overset{\boxtimes}{u}_{j+1}] + f(t_{i+1/2}, x_j)$$

$$\overset{\boxtimes}{u}_j \left(\frac{1}{\tau} + \frac{\eta}{h^2} \right) = \overset{\boxtimes}{u}_{j+1} \frac{\eta}{h^2} + u_j \left(\frac{1}{\tau} - \frac{\eta}{h^2} \right) + u_{j+1} \frac{\eta}{h^2} + f(t_{i+1/2}, x_j), \quad j = \overline{N-1, 1}$$

Для каждого слоя погрешность аппроксимации: $\Psi_j^i = O\left(\tau, h, \frac{\tau}{h}, \tau h\right)$

Для двух слоев: $\Psi_j^i = O\left(\tau^2, h^2, \frac{\tau^2}{h^2}\right)$

Условие устойчивости схемы по начальным данным $\tau \leq \frac{h^2}{\eta}$



УРАВНЕНИЯ ГИПЕРБОЛИЧЕСКОГО ТИПА

$$\frac{\partial^2 u}{\partial t^2} = \lambda^2 \frac{\partial^2 u}{\partial x^2} + f(t, x), \quad \lambda^2 = \frac{F}{\rho}$$

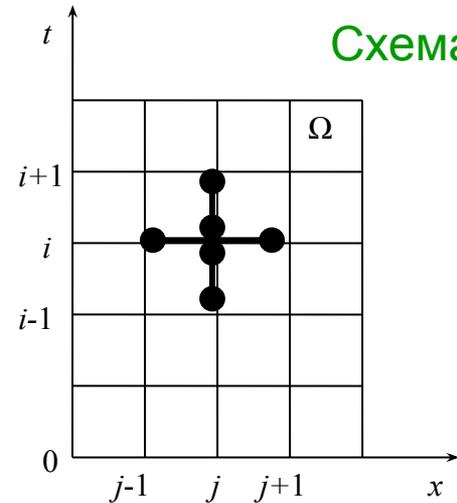
начальные условия

$$u(0, x) = U(x), \quad \frac{\partial u(0, x)}{\partial t} = V(x)$$

и граничные условия

$$u(t, 0) = U_0(t), \quad u(t, L) = U_L(t)$$

Схема «крест»



$$\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\tau^2} = \lambda^2 \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{h^2} + f(t_i, x_j)$$

1. Погрешность аппроксимации

$$u(t_{i\pm 1}, x_j) = u(t_i, x_j) \pm u'(t_i, x_j)\tau + u''(t_i, x_j)\frac{\tau^2}{2} \pm u'''(t_i, x_j)\frac{\tau^3}{3!} + u^{iv}(t_i, x_j)\frac{\tau^4}{6!} + O(\tau^5)$$

$$u(t_i, x_{j\pm 1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} + u^{iv}(t_i, x_j)\frac{h^4}{24} + O(h^5)$$

$$\begin{aligned}
\Psi_j^i &= \frac{u(t_{i+1}, x_j) - 2u(t_i, x_j) + u(t_{i-1}, x_j))}{2\tau} - \frac{\lambda^2}{h^2} [u(t_i, x_{j+1}) - 2u(t_i, x_j) + u(t_i, x_{j-1})] - f(t_i, x_j) = \\
&= \left[\cancel{u}(t_i, x_j) + \cancel{u}(t_i, x_j) \frac{\tau^2}{12} + O(\tau^4) \right] - \lambda^2 \left[u''(t_i, x_j) + u^{iv}(t_i, x_j) \frac{h^2}{12} + O(h^4) \right] - f(t_i, x_j) = \\
&= \left[\cancel{u}(t_i, x_j) - \lambda^2 u''(t_i, x_j) - f(t_i, x_j) \right] + \left[\cancel{u}(t_i, x_j) \frac{\tau^2}{12} - \lambda^2 u^{iv}(t_i, x_j) \frac{h^2}{12} + O(\tau^4, h^4) \right] = \\
&O(\tau^2, h^2)
\end{aligned}$$

Полученная схема *явная, трехслойная*. Значения для нулевого временного слоя – из начальных условий на саму функцию. Значения для первого временного слоя – аппроксимируются из начальных условий на скорость:

$$u(\tau, x_j) = u(0, x_j) + \cancel{u}(0, x_j)\tau + \cancel{u}(0, x_j) \frac{\tau^2}{2} + O(\tau^3) =$$

С учетом уравнения

$$= U(x_j) + V(x_j)\tau + \left[\lambda^2 U''(x_j) + f(0, x_j) \right] \frac{\tau^2}{2} + O(\tau^3)$$

тогда для первого временного слоя

$$u_j = U(x_j) + V(x_j)\tau + \left[\lambda^2 U''(x_j) + f(0, x_j) \right] \frac{\tau^2}{2}, \quad j = \overline{1, N-1}$$

2. Устойчивость схемы по начальным данным (методом Неймана)

$$\frac{1}{\tau^2}(\delta u_j^{\boxtimes} - 2\delta u_j + \delta u_j^{\boxtimes}) - \frac{\lambda^2}{h^2}(\delta u_{j+1} - 2\delta u_j + \delta u_{j-1}) = 0$$

$$\delta u_j^{\boxtimes} = a_k e^{ikx_j}, \quad \delta u_j = \rho_k a_k e^{ikx_j}, \quad \delta u_j^{\boxtimes} = \rho_k^2 \delta u_j^{\boxtimes} = \rho_k^2 a_k e^{ikx_j}$$

$$\frac{1}{\tau^2}(\rho_k^2 - 2\rho_k + 1)e^{ikx_j} - \frac{\lambda^2}{h^2}(\rho_k e^{ik(x_j+1)} - 2\rho_k e^{ikx_j} + \rho_k e^{ik(x_j-1)}) = 0$$

$$\left[(\rho_k^2 - 2\rho_k + 1) - \rho_k \frac{\lambda^2 \tau^2}{h^2} (e^{ikh} - 2 + e^{-ikh}) \right] e^{ikx_j} = 0$$

$$\rho_k^2 - 2\rho_k \left[1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \right] + 1 = 0$$

$$\rho_{k1} = \left(1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \right) + \sqrt{\left(1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \right)^2 - 1}$$

$$\rho_{k2} = \left(1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \right) - \sqrt{\left(1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \right)^2 - 1}$$

По теореме Виета $|\rho_{k1}| \cdot |\rho_{k2}| = 1$

Схема будет устойчива, если $|\rho_{k1}| = 1$ $|\rho_{k2}| = 1$

то есть корни будут *комплексные* (дискриминант отрицательный)

$$\left(1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2}\right)^2 \leq 1$$

$$\left|1 - 2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2}\right| \leq 1$$

$$-2 \leq -2 \frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \leq 0$$

$$\frac{\lambda^2 \tau^2}{h^2} \sin^2 \frac{kh}{2} \leq 1$$

$$\frac{\lambda^2 \tau^2}{h^2} \leq 1$$

Таким образом, схема условно устойчива при $\tau \leq h/\lambda$

