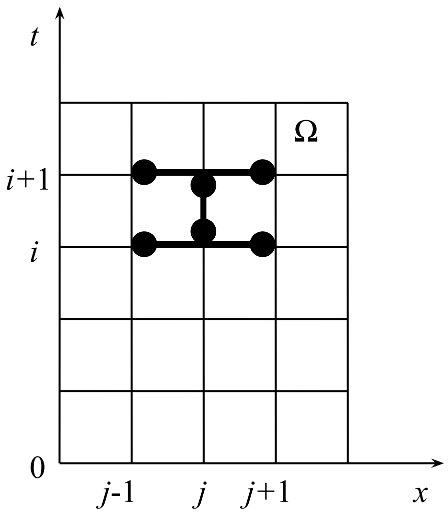


УРАВНЕНИЯ ПАРАБОЛИЧЕСКОГО ТИПА

$$\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2} + f(t, x)$$

с краевыми условиями

$$u(0, x) = U(x)$$

$$u(t,0) = U_0(t), \quad u(t, L) = U_L(t)$$

Схема с «весами»

$$\frac{1}{\tau}(\hat{u}_j - u_j) = \frac{\eta\sigma}{h^2}(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}) + \frac{\eta(1-\sigma)}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \tilde{f}$$

$$0 \le \sigma \le 1, \quad \tilde{f} = f(t_{i+1/2}, x_j)$$

При $\sigma = 0$ схема чисто явная,
при $\sigma = 1$ схема чисто неявная

1. Погрешность аппроксимации

$$u(t_{i+1}, x_j) = u(t_i, x_j) + \dot{u}(t_i, x_j)\tau + \ddot{u}(t_i, x_j)\frac{\tau^2}{2} + \dddot{u}(t_i, x_j)\frac{\tau^3}{3!} + O(\tau^4)$$

$$u(t_i, x_{j\pm1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} +$$

$$+ u^{iv}(t_i, x_j)\frac{h^4}{24} \pm u^{v}(t_i, x_j)\frac{h^5}{120} + O(h^6)$$

$$f(t_{i+1/2},x_j)=f(t_i,x_j)+\frac{\partial f}{\partial t}(t_i,x_j)\frac{\tau}{2}+\frac{\partial^2 f}{\partial t^2}(t_i,x_j)\frac{\tau^2}{8}+O(\tau^3)$$

$$u(t_{i+1},x_{j\pm1})=u(t_i,x_j)+\cancel{\dot{u}(t_i,x_j)}\tau\pm\cancel{u'(t_i,x_j)}h+$$

$$+\frac{1}{2}\left[\cancel{\ddot{u}(t_i,x_j)}\tau^2\pm2\cancel{\dot{u}'(t_i,x_j)}\tau h+u''(t_i,x_j)h^2\right]+$$

$$+\frac{1}{3!}\left[\cancel{\dddot{u}(t_i,x_j)}\tau^3\pm3\cancel{\ddot{u}'(t_i,x_j)}\tau^2 h+\dot{u}''(t_i,x_j)\tau h^2\pm\cancel{u'''(t_i,x_j)}h^3\right]+$$

$$+\frac{1}{4!}\left[\cancel{\ddddot{u}(t_i,x_j)}\tau^4\pm4\cancel{\dddot{u}'(t_i,x_j)}\tau^3 h+6\ddot{u}''(t_i,x_j)\tau^2h^2\pm4\cancel{u'''(t_i,x_j)}h^3\tau+u^{iv}(t_i,x_j)h^4\right]+\dots$$

$$\psi_j^i=\left[\dot{u}(t_i,x_j)+\underline{\ddot{u}(t_i,x_j)\frac{\tau}{2}}+\underline{\underline{\dddot{u}(t_i,x_j)\frac{\tau^2}{3!}}}+O(\tau^3)\right]-$$

$$-\eta\sigma\left[u''(t_i,x_j)+\underline{\dot{u}''(t_i,x_j)\tau}+\underline{\underline{\ddot{u}''(t_i,x_j)\frac{\tau^2}{2}}}+u^{iv}(t_i,x_j)\frac{h^2}{12}\right]-$$

$$-\eta(1-\sigma)\left[u''(t_i,x_j)+u^{iv}(t_i,x_j)\frac{h^2}{12}\right]-$$

$$\left[f(t_i,x_j)+\underline{\frac{\partial f}{\partial t}(t_i,x_j)\frac{\tau}{2}}+\underline{\underline{\frac{\partial^2 f}{\partial t^2}(t_i,x_j)\frac{\tau^2}{8}}}\right]+O(\tau^3,h^3)$$

$$\psi_j^i = \left[\dot u(t_i,x_j) - \eta u''(t_i,x_j) - f(t_i,x_j)\right] + \frac{\tau}{2}\left[\ddot u(t_i,x_j) - 2\eta\sigma\dot u''(t_i,x_j) - \dot f(t_i,x_j)\right] +$$

$$+\frac{\tau^2}{24}\left[4\dddot u(t_i,x_j) - 12\eta\sigma\ddot u''(t_i,x_j) - 3\ddot f(t_i,x_j)\right] - \eta\frac{h^2}{12}u^{iv}(t_i,x_j) + O(\tau^3,h^3)$$

$$\psi_j^i = \frac{\tau}{2}\left[\ddot u(t_i,x_j) - 2\eta\sigma\dot u''(t_i,x_j) - \dot f(t_i,x_j)\right] +$$

$$+\frac{\tau^2}{24}\left[4\dddot u(t_i,x_j) - 12\eta\sigma\ddot u''(t_i,x_j) - 3\ddot f(t_i,x_j)\right] - \eta\frac{h^2}{12}u^{iv}(t_i,x_j) + O(\tau^3,h^3) = \boxed{O(\tau,h^2)}$$

Дополнительно $\sigma = 1/2$ и

$$\ddot u = \frac{d}{dt}\left(\dot u + f\right) = \eta\dot u'' + \dot f$$

тогда

$$\psi_j^i = \frac{\tau^2}{24}\left[4\dddot u(t_i,x_j) - 12\eta\sigma\ddot u''(t_i,x_j) - 3\ddot f(t_i,x_j)\right] - \eta\frac{h^2}{12}u^{iv}(t_i,x_j) + O(\tau^3,h^3) = \boxed{O(\tau^2,h^2)}$$

При $\sigma = 1/2$ схема называется схемой Крэнка–Николсона

2. Устойчивость

$$\hat{u}_{j-1}\left(-\frac{\eta\sigma}{h^2}\right)+\hat{u}_j\left(\frac{1}{\tau}+\frac{2\eta\sigma}{h^2}\right)+\hat{u}_{j+1}\left(-\frac{\eta\sigma}{h^2}\right)=$$

$$=u_{j-1}\frac{\eta(1-\sigma)}{h^2}+u_j\left(\frac{1}{\tau}-\frac{2\eta(1-\sigma)}{h^2}\right)+u_{j+1}\frac{\eta(1-\sigma)}{h^2}+\tilde{f}$$

$$\alpha_{\max}=\frac{1}{\tau}+\frac{2\eta\sigma}{h^2}$$

тогда по принципу максимума

$$\frac{1}{\tau}+\frac{2\eta\sigma}{h^2}\geq\frac{2\eta\sigma}{h^2}+\frac{2\eta(1-\sigma)}{h^2}+\left|\frac{1}{\tau}-\frac{2\eta(1-\sigma)}{h^2}\right|$$

$$\frac{1}{\tau}-\frac{2\eta(1-\sigma)}{h^2}\geq\left|\frac{1}{\tau}-\frac{2\eta(1-\sigma)}{h^2}\right|$$

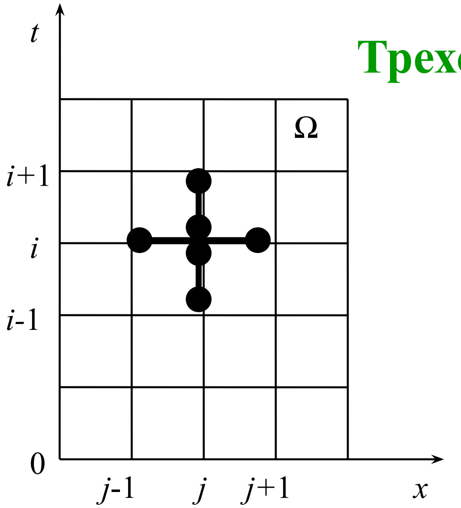
$$\frac{1}{\tau}-\frac{2\eta(1-\sigma)}{h^2}\geq 0$$

$$\boxed{\tau\leq\frac{h^2}{2\eta(1-\sigma)}}$$

Условие устойчивости по правой части

$$\frac{1}{\tau}+\frac{2\eta\sigma}{h^2}-\frac{2\eta\sigma}{h^2}\geq\frac{\omega}{\tau},\quad 0<\omega<1$$

Т.о. схема условно устойчива



Трехслойная схема Ричардсона

$$\frac{\hat{u}_j - \check{u}_j}{2\tau} = \eta \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + f(t_i, x_j)$$

Устойчивость схемы
(методом Неймана)

$$(\delta u_j^{m-1})^{(k)} = (\rho_k)^{m-1} a_k e^{ikx_j}, \quad (\delta u_j^{m})^{(k)} = (\rho_k)^{m} a_k e^{ikx_j}, \quad (\delta u_j^{m+1})^{(k)} = (\rho_k)^{m+1} a_k e^{ikx_j}$$

$$\frac{1}{2\tau}\left((\rho_k)^{m+1} - (\rho_k)^{m-1}\right)e^{ikx_j} = (\rho_k)^m \frac{\eta}{h^2}\left(e^{ik(x_j+h)} - 2e^{ikx_j} + e^{ik(x_j-h)}\right) \Bigg| : \frac{(\rho_k)^{m-1} e^{ikx_j}}{2\tau}$$

$$\rho_k^2 - \rho_k \frac{2\tau\eta}{h^2}(e^{ikh} - 2 + e^{-ikh}) - 1 = 0$$

$$\rho_k^2 + \rho_k \frac{8\tau\eta}{h^2}\sin^2\left(\frac{kh}{2}\right) - 1 = 0$$

Уравнение имеет только действительные корни (т.к. дискриминант положительный)

$$\rho_k = \frac{4\tau\eta}{h^2}\sin^2\left(\frac{kh}{2}\right) \pm \sqrt{\frac{16\tau^2\eta^2}{h^4}\sin^4\left(\frac{kh}{2}\right) + 1}$$

$$\left|(\rho_k)_1 \cdot (\rho_k)_2\right| = 1$$

Схема абсолютно неустойчива !!!

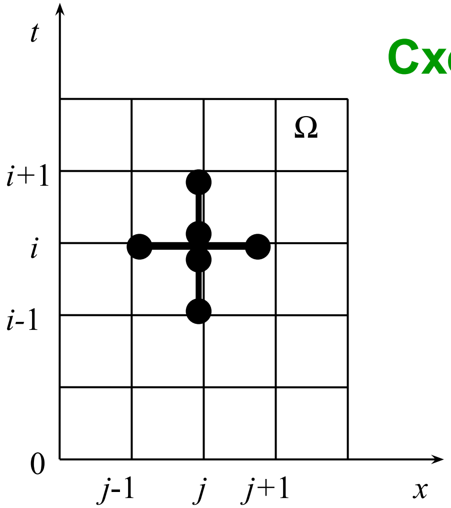


Схема Дюфорта и Франкела

$$\frac{\hat{u}_j - \check{u}_j}{2\tau} = \frac{\eta}{h^2}\left[u_{j+1} - 2\left\{\frac{\hat{u}_j + \check{u}_j}{2}\right\} + u_{j-1}\right] + f(t_i, x_j)$$

1. Погрешность аппроксимации

$$u(t_{i\pm1}, x_j) = u(t_i, x_j) \pm \dot{u}(t_i, x_j)\tau + \ddot{u}(t_i, x_j)\frac{\tau^2}{2} \pm \dddot{u}(t_i, x_j)\frac{\tau^3}{3!} + O(\tau^4)$$

$$u(t_i, x_{j\pm1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} +$$

$$+ u^{iv}(t_i, x_j)\frac{h^4}{24} \pm u^{v}(t_i, x_j)\frac{h^5}{120} + O(h^6)$$

$$\psi_j^i = \frac{u(t_{i+1}, x_j) - u(t_{i-1}, x_j)}{2\tau} - \frac{\eta}{h^2}\left[\underline{u(t_i, x_{j+1})} - 2\left\{\underline{\frac{u(t_{i+1}, x_j) + u(t_{i-1}, x_j)}{2}}\right\} + \underline{u(t_i, x_{j-1})}\right] - f(t_i, x_j) =$$

$$= \left[\dot{u}(t_i, x_j) + \dddot{u}(t_i, x_j)\frac{\tau^2}{6} + O(\tau^3)\right] -$$

$$\eta\left[\cancel{\frac{2u(t_i,x_j)}{h^2}}+u''(t_i,x_j)+u^{iv}(t_i,x_j)\frac{h^2}{12}+O(h^4)-\right.$$

$$\left.\left\{\cancel{\frac{2u(t_i,x_j)}{h^2}}+\ddot{u}(t_i,x_j)\frac{\tau^2}{h^2}+O\left(\frac{\tau^4}{h^2}\right)\right\}\right]-f(t_i,x_j)=$$

$$=\left[\dot{u}(t_i,x_j)-\eta u''(t_i,x_j)-f(t_i,x_j)\right]+\left[\dddot{u}(t_i,x_j)\frac{\tau^2}{6}-\eta u^{iv}(t_i,x_j)\frac{h^2}{12}-\eta\,\ddot{u}(t_i,x_j)\frac{\tau^2}{h^2}\right]+\ldots$$

$$=O\left(\tau^2,h^2,\frac{\tau^2}{h^2}\right)$$

Погрешность аппроксимации зависит от соотношения шагов по времени и координате и мала, если $\lim\limits_{\tau,\,h\to 0}\dfrac{\tau}{h}=0$

2. Устойчивость схемы (методом Неймана)

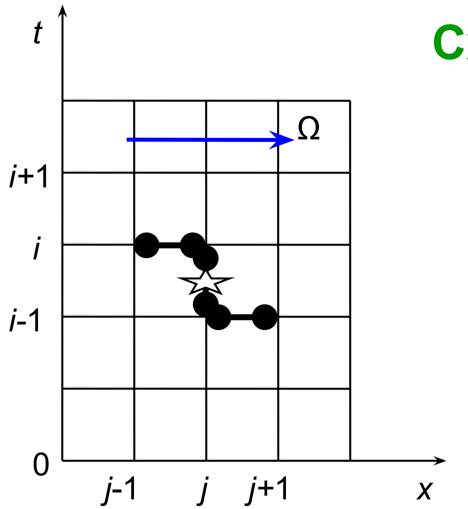
$$\frac{1}{2\tau}\left(\underline{(\rho_k)^{m+1}}-\underline{\underline{(\rho_k)^{m-1}}}\right)e^{ikx_j}=\frac{\eta}{h^2}\left(\underline{\underline{(\rho_k)^m e^{ik(x_j+h)}}}-\left\{\underline{(\rho_k)^{m+1}e^{ikx_j}}+\underline{\underline{(\rho_k)^{m-1}e^{ikx_j}}}\right\}+\underline{\underline{(\rho_k)^m e^{ik(x_j-h)}}}\right)$$

$$\Big| : (\rho_k)^{m-1}e^{ikx_j}$$

$$\left(\frac{1}{2\tau}+\frac{\eta}{h^2}\right)\rho_k^2-\rho_k\frac{\eta}{h^2}\left(e^{ikh}+e^{-ikh}\right)-\left(\frac{1}{2\tau}-\frac{\eta}{h^2}\right)=0$$

$$\rho_k^2-\rho_k\frac{4\tau\eta\cos kh}{2\tau\eta+h^2}-\frac{2\tau\eta-h^2}{2\tau\eta+h^2}=0$$

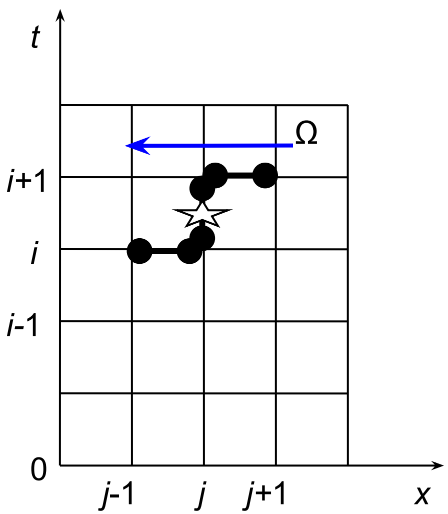
Схема бегущего счета



Для четных слоев

$$\frac{u_j - \hat{u}_j}{2\tau} = \frac{\eta}{h^2}\left[u_{j-1} - u_j - \hat{u}_j + \hat{u}_{j+1}\right] + f(t_{i-1/2}, x_j)$$

$$u_j\left(\frac{1}{\tau} + \frac{\eta}{h^2}\right) = u_{j-1}\frac{\eta}{h^2} + \hat{u}_j\left(\frac{1}{\tau} - \frac{\eta}{h^2}\right) + \hat{u}_{j+1}\frac{\eta}{h^2} + f(t_{i-1/2}, x_j), \quad j = \overline{1, N-1}$$



Для нечетных слоев

$$\frac{\hat{u}_j - u_j}{2\tau} = \frac{\eta}{h^2}\left[u_{j-1} - u_j - \hat{u}_j + \hat{u}_{j+1}\right] + f(t_{i+1/2}, x_j)$$

$$\hat{u}_j\left(\frac{1}{\tau} + \frac{\eta}{h^2}\right) = \hat{u}_{j+1}\frac{\eta}{h^2} + u_j\left(\frac{1}{\tau} - \frac{\eta}{h^2}\right) + u_{j+1}\frac{\eta}{h^2} + f(t_{i+1/2}, x_j), \quad j = \overline{N-1, 1}$$

Для каждого слоя погрешность аппроксимации: $\psi_j^i = O\left(\tau, h, \frac{\tau}{h}, \tau h\right)$

Для двух слоев: $\psi_j^i = O\left(\tau^2, h^2, \frac{\tau^2}{h^2}\right)$

Условие устойчивости схемы по начальным данным $\tau \le \frac{h^2}{\eta}$

УРАВНЕНИЯ ГИПЕРБОЛИЧЕСКОГО ТИПА

$$\frac{\partial^2 u}{\partial t^2} = \lambda^2 \frac{\partial^2 u}{\partial x^2} + f(t,x), \quad \lambda^2 = \frac{F}{\rho}$$

начальные условия

$$u(0,x) = U(x), \quad \frac{\partial u(0,x)}{\partial t} = V(x)$$

и граничные условия

$$u(t,0) = U_0(t), \quad u(t,L) = U_L(t)$$

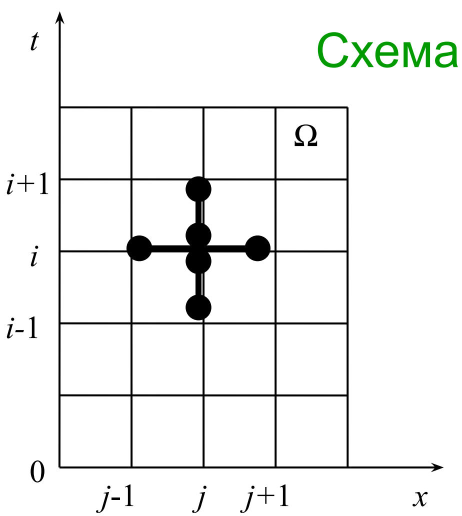


Схема «крест»

$$\frac{\hat{u}_j - 2u_j + \check{u}_j}{\tau^2} = \lambda^2 \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + f(t_i, x_j)$$

1. Погрешность аппроксимации

$$u(t_{i\pm1}, x_j) = u(t_i, x_j) \pm \dot{u}(t_i, x_j)\tau + \ddot{u}(t_i, x_j)\frac{\tau^2}{2} \pm \dddot{u}(t_i, x_j)\frac{\tau^3}{3!} + \ddddot{u}(t_i, x_j)\frac{\tau^4}{6\,!} + O(\tau^5)$$

$$u(t_i, x_{j\pm1}) = u(t_i, x_j) \pm u'(t_i, x_j)h + u''(t_i, x_j)\frac{h^2}{2} \pm u'''(t_i, x_j)\frac{h^3}{6} + u^{iv}(t_i, x_j)\frac{h^4}{24} + O(h^5)$$

$$\psi_j^i = \frac{u(t_{i+1},x_j) - 2u(t_i,x_j) + u(t_{i-1},x_j)}{2\tau} - \frac{\lambda^2}{h^2}\left[u(t_i,x_{j+1}) - 2u(t_i,x_j) + u(t_i,x_{j-1})\right] - f(t_i,x_j) =$$

$$= \left[\ddot{u}(t_i,x_j) + \ddddot{u}(t_i,x_j)\frac{\tau^2}{12} + O(\tau^4)\right] - \lambda^2\left[u''(t_i,x_j) + u^{iv}(t_i,x_j)\frac{h^2}{12} + O(h^4)\right] - f(t_i,x_j) =$$

$$= \left[\ddot{u}(t_i,x_j) - \lambda^2 u''(t_i,x_j) - f(t_i,x_j)\right] + \left[\ddddot{u}(t_i,x_j)\frac{\tau^2}{12} - \lambda^2 u^{iv}(t_i,x_j)\frac{h^2}{12} + O(\tau^4,h^4)\right] =$$

$$O(\tau^2,h^2)$$

Полученная схема *явная, трехслойная*. Значения для нулевого временного слоя – из начальных условий на саму функцию. Значения для первого временного слоя – аппроксимируются из начальных условий на скорость:

$$u(\tau,x_j) = u(0,x_j) + \dot{u}(0,x_j)\tau + \ddot{u}(0,x_j)\frac{\tau^2}{2} + O(\tau^3) =$$

С учетом уравнения

$$= U(x_j) + V(x_j)\tau + \left[\lambda^2 U''(x_j) + f(0,x_j)\right]\frac{\tau^2}{2} + O(\tau^3)$$

тогда для первого временного слоя

$$u_j = U(x_j) + V(x_j)\tau + \left[\lambda^2 U''(x_j) + f(0,x_j)\right]\frac{\tau^2}{2}, \quad j = \overline{1, N-1}$$

2. Устойчивость схемы по начальным данным (методом Неймана)

$$\frac{1}{\tau^2}\left(\delta\hat{u}_j - 2\delta u_j + \delta\check{u}_j\right) - \frac{\lambda^2}{h^2}\left(\delta u_{j+1} - 2\delta u_j + \delta u_{j-1}\right) = 0$$

$$\delta\check{u}_j = a_k e^{ikx_j}, \quad \delta u_j = \rho_k a_k e^{ikx_j}, \quad \delta\hat{u}_j = \rho_k^2 \delta\check{u}_j = \rho_k^2 a_k e^{ikx_j}$$

$$\frac{1}{\tau^2}\left(\rho_k^2 - 2\rho_k + 1\right)e^{ikx_j} - \frac{\lambda^2}{h^2}\left(\rho_k e^{ik(x_j+1)} - 2\rho_k e^{ikx_j} + \rho_k e^{ik(x_j-1)}\right) = 0$$

$$\left[\left(\rho_k^2 - 2\rho_k + 1\right) - \rho_k \frac{\lambda^2\tau^2}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right)\right]e^{ikx_j} = 0$$

$$\rho_k^2 - 2\rho_k\left[1 - 2\frac{\lambda^2\tau^2}{h^2}\sin^2\frac{kh}{2}\right] + 1 = 0$$

$$\rho_{k1} = \left(1 - 2\frac{\lambda^2\tau^2}{h^2}\sin^2\frac{kh}{2}\right) + \sqrt{\left(1 - 2\frac{\lambda^2\tau^2}{h^2}\sin^2\frac{kh}{2}\right)^2 - 1}$$

$$\rho_{k2} = \left(1 - 2\frac{\lambda^2\tau^2}{h^2}\sin^2\frac{kh}{2}\right) - \sqrt{\left(1 - 2\frac{\lambda^2\tau^2}{h^2}\sin^2\frac{kh}{2}\right)^2 - 1}$$

По теореме Виета $\quad |\rho_{k1}| \cdot |\rho_{k2}| = 1$

Схема будет устойчива, если $|\rho_{k1}| = 1 \quad |\rho_{k2}| = 1$

то есть корни будут *комплексные* (дискриминант отрицательный)

$$\left(1 - 2\frac{\lambda^2 \tau^2}{h^2}\sin^2\frac{kh}{2}\right)^2 \le 1$$

$$\left|1 - 2\frac{\lambda^2 \tau^2}{h^2}\sin^2\frac{kh}{2}\right| \le 1$$

$$-2 \le -2\frac{\lambda^2 \tau^2}{h^2}\sin^2\frac{kh}{2} \le 0$$

$$\frac{\lambda^2 \tau^2}{h^2}\sin^2\frac{kh}{2} \le 1$$

$$\frac{\lambda^2 \tau^2}{h^2} \le 1$$

Таким образом, схема условно устойчива при $\tau \le h/\lambda$