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About one Approach for Designing of Algebraic Computations:

Computations in Boolean Algebra.

Designing of algorithms for **algebraic computations** – the main task arising when realising mathematical systems based on symbolic transformations

Mathematical model for this problem – **multisorted algebraic systems (MAS)**

The process of realising of algebraic computations needs carefull preliminary design MAS as hierarchy of its sorts and specifications of algebraic operations interpreters

Theory and detailes in:

- Lvov M.S. Synthesis of Interpreters of Algebraic Operations in Extensions of Multisorted Algebras. (ukr, prepared for print)
- Lvov M.S. Veryfication of Interpreters of Algebraic Operations in Extensions of Multisorted Algebras. (prepared for print)
- Lvov M.S. Method of Inheritance for design of algebraic computations in mathematical educational systems. (ukr, prepared for print)
- Lvov M.S. Method of Morphisms for design of algebraic computations in mathematical educational systems. (ukr, prepared for print)

Approach IEM

Inheritance, Extensions, Morphisms

Base principles and ideas of IEM are quite simple and well known in mathematics and programming

Boolean algebra is very simple, well known and interesting example of subject domain for applying of IEM

Algebraic programming system APS

(V. Peschanenko)

APS uses technologies of algebraic programming, based on rewriting rules systems and strategies of rewriting.

The interpreter of algebraic operation defined by rewriting rules system.

We shall consider problems:

- **of design,**
- **synthesis,**
- **verification**

of interpreters of boolean algebra operations (logical operations).

Problem Definition

Compute the value $Val(F(x_1,...,x_n))$ **of logical expression** $F(x_1,...,x_n)$

in signature of logical operations $<\&,\vee,\neg,0,1>$

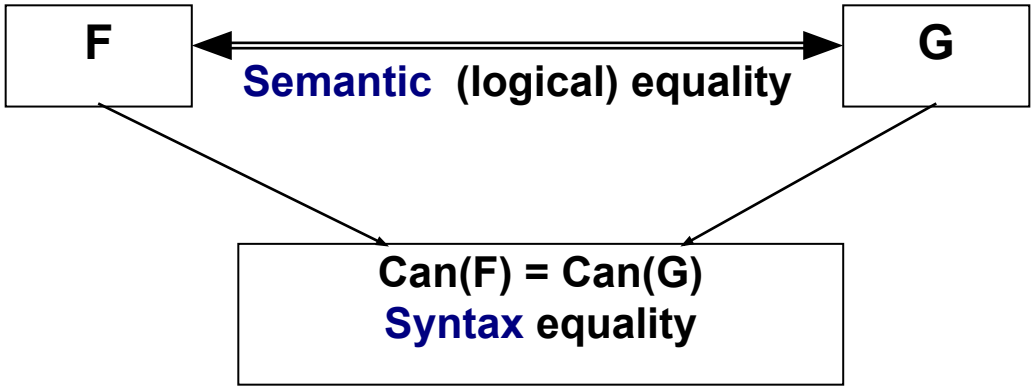
By the value of expression $F(x_1,...,x_n)$ we means its canonical form

i.e. such expression $Can(F)(x_1,...,x_n)$ that

$$F(x_1,...,x_n) \equiv Can(F)(x_1,...,x_n) \quad (1)$$

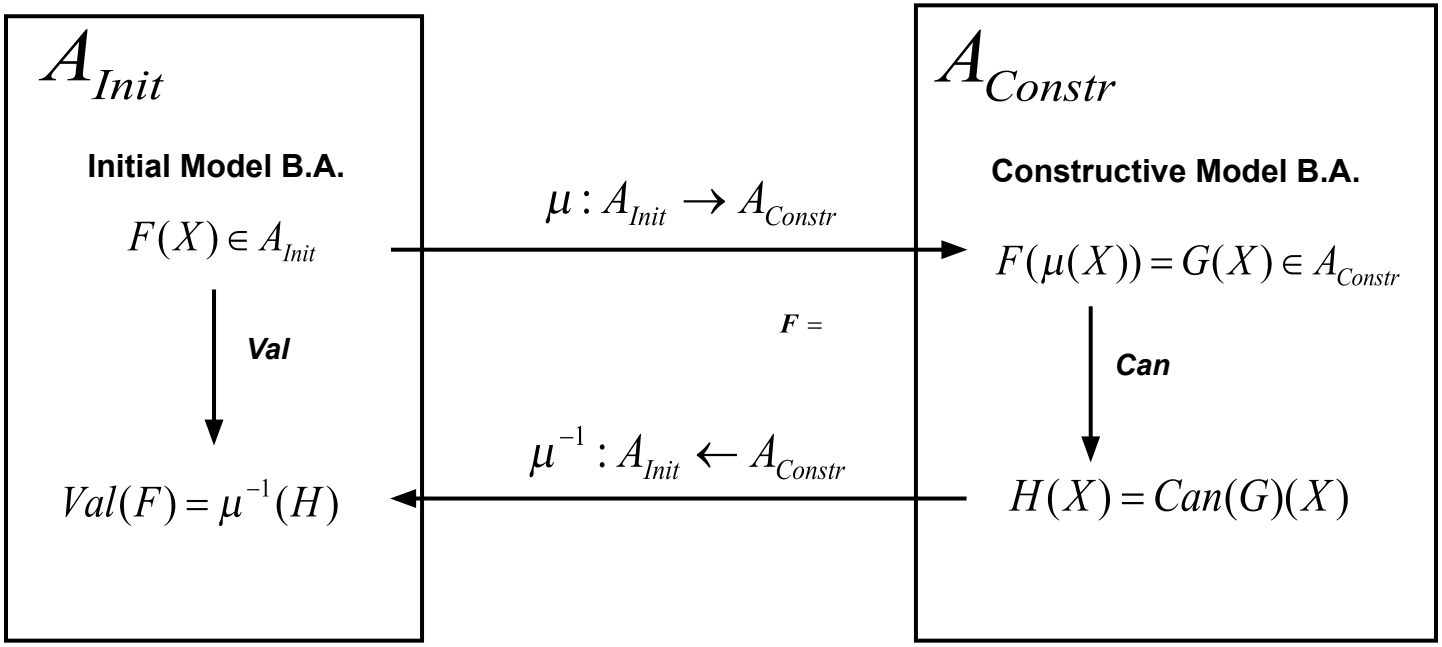
$$F(x_1,...,x_n) \equiv G(x_1,...,x_n) \Rightarrow Can(F)(x_1,...,x_n) = Can(G)(x_1,...,x_n) \quad (2)$$

Sign «=» means syntax equality (equality in free terms algebra)



Traditional logical canonical forms are **PDNF, PCNF.**
We shell use **Recursive Normal Form** (**RNF**).

Paradigm of Algebraic Computations



$$Val(F(X)) = \mu^{-1}(Can(F(\mu(X)))) \qquad (3)$$

Method of Inheritance

In algebraic computations are used:

Classical (axiomatically of constructively defined) algebraic structures,

Algebraic structures, defined by designer

Logical Designing of algebraic computations consists in defining of hierarchy of inheritance of MAS

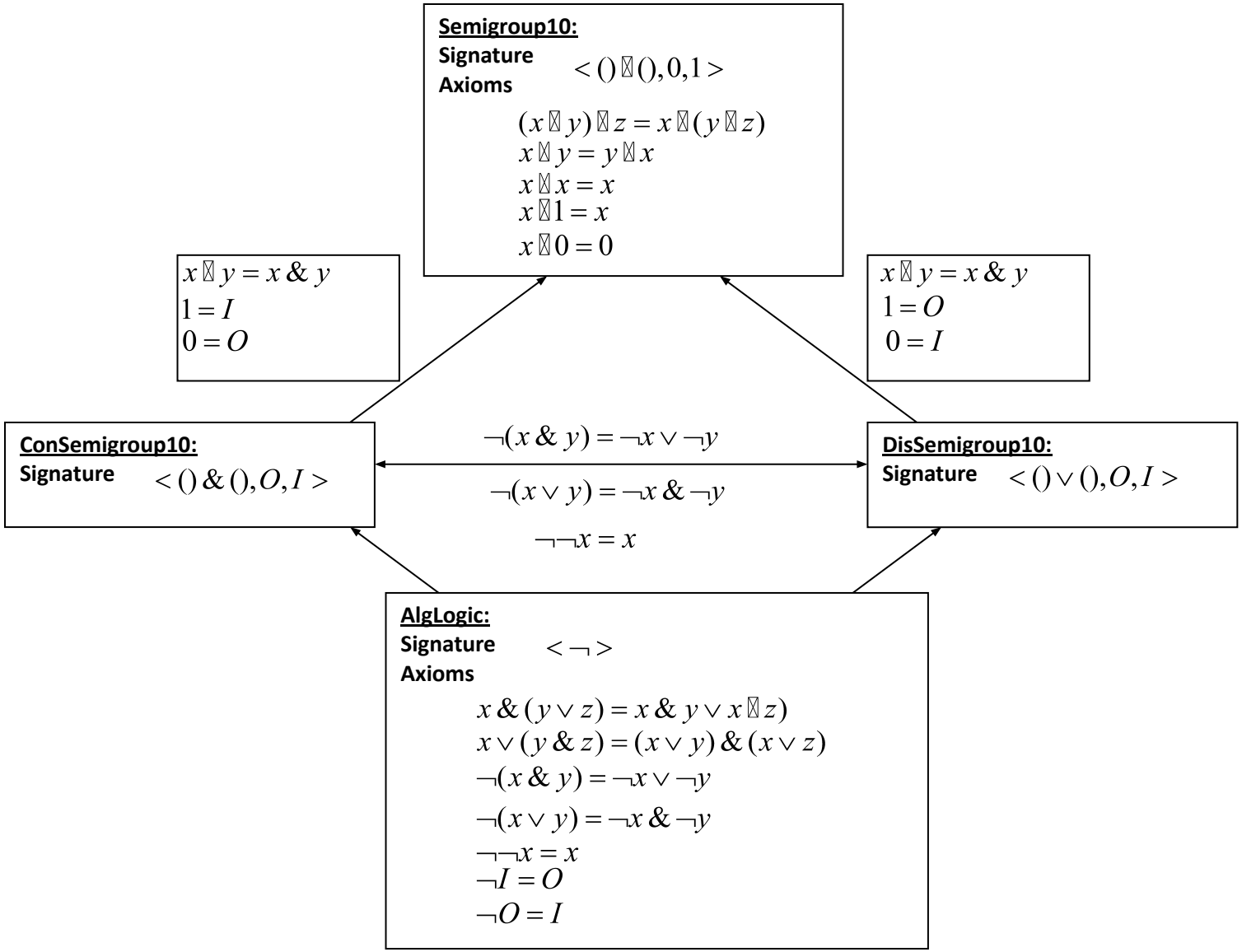


Рис. 3. Diagram of Inheritance for Boolean Algebra

Axiomatic descriptions plays constructive role. Its defines signatures, computations with constants.
Inheritance is used for definitions of algebraic operations interpreters with constants

```
SGOperation := rs(a)
{
      a°1=a, 1°a=a,
      a°0=0, 0°a=0
};

Dis := rs(a)
{
      a∨O = a, O∨a = a,
      a∨I = 0, I∨a = I
};

Con := rs(a)
{
      A&I = a, I&a = a,
      A&O = 0, O&a = O
};
```

Abstract Algorithms

Inheritance is used for descriptions of algorithms on abstract level (independently of algebras support).

Example: **Euclid's Algorithm** (abstract **Euclidian ring**).

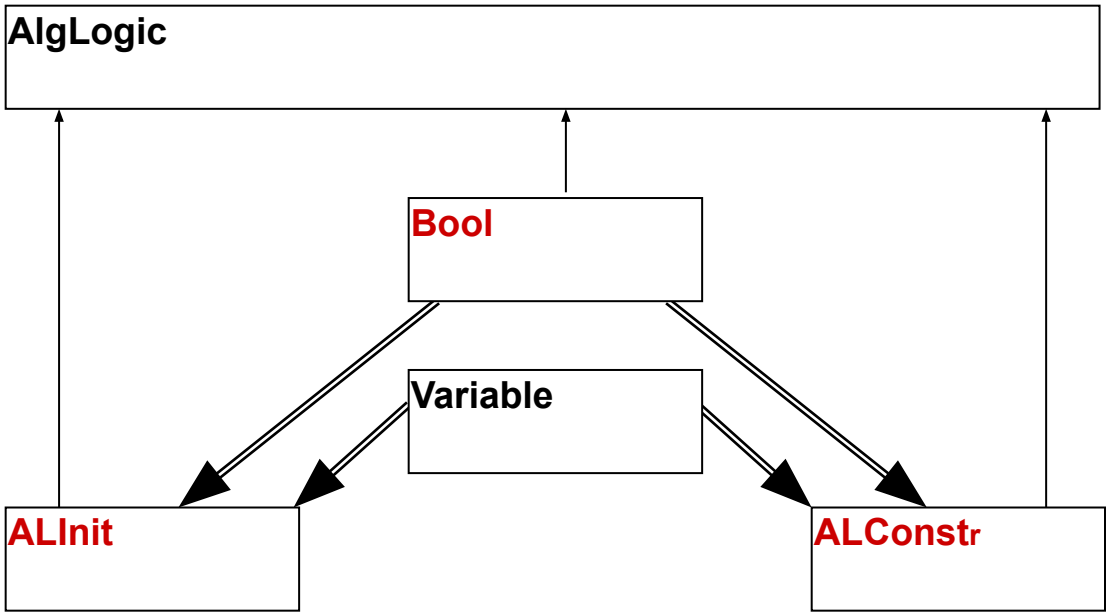
Derived logical operations

Imp := rs(a, b)
{
  $a \rightarrow b = \neg a \vee b$
}

Method of Extensions

Axiomatic definition of sort **AlgLogic is initial for further development**.

Three constructive models of **AlgLogic**:



Initial diagram of Method of extentions for Boolean algebra.

Important:

Constructive definition of algebra A consists in definitions

- **of its support Sup(A),**
- **interpreters of its operations**

Bool и Variable – base sorts.

Definition

Let $Bool = \{O, I\}$, $Variable = \{x_1, x_2, \ldots\}$.

Define sequence $A^{(0)} \subset A^{(1)} \subset \ldots \subset A^{(j)} \subset A^{(j+1)} \subset \ldots$

1. $A^{(0)} = Bool$,

2. $A^{(j+1)} = \{F : F = L(F^{(j)}, G^{(j)}, x_{j+1})\}$,

where $F^{(j)}, G^{(j)} \in A^{(j)}$,

L -special functional symbol, (constructor of ***ALConstr***).

We uses notation $A(x)$ for algebra

$$A(x) = \{F : F = L(F_1, G_1, x), F_1, G_1 \in A, x \notin A\}$$

Elements of $A(x)$ has the view $L(F, G, x)$.

Selectors :

$$Arg_1(L(F,G,x)) = F,\ Arg_2(L(F,G,x)) = G,\ Var(L(F,G,x)) = x$$

3. On ***ALConstr*** relations of inclusions are holds:

$$A \Rightarrow A(x) \qquad L(F,F,x) = F$$

$$Variable \Rightarrow A(x) \qquad L(I,O,x) = x$$

Algebra A_{Constr} is direct limit of increasing sequence $A^{(j)}$:

$$A_{Constr} = \bigcup_{j=0}^{\infty} A^{(j)}$$

Define for A_{Constr} logical operations. Let

$$L(A_1, B_1, x),\ L(A_2, B_2, x) \in \boldsymbol{A(x)},\quad A_1, B_{1,} A_2, B_2 \in \boldsymbol{A}.$$

Base rules:

$$L(A_1, B_1, x) \vee L(A_2, B_2, x) = L(A_1 \vee A_2, B_1 \vee B_2, x), \qquad (10)$$

$$L(A1, B1, x) \,\&\, L(A2, B2, x) = L(A1 \,\&\, A2, B1 \,\&\, B2, x), \qquad (11)$$

$$\neg L(A1, B1, x) = L(\neg A1, \neg B1, x). \qquad (12)$$

Rules (10) - (12) are base for operations on

ALConstr

Additional Rules (partial cases):

In the rule (10) suppose $A_1 = B_1$. then obtain:

$$A_1 \vee L(A_2, B_2, x) = L(A_1, A_1, x) \vee L(A_2, B_2, x) = L(A_1 \vee A_2, A_1 \vee B_2, x).$$

This equality defines the conditional rewriting rule

$$Var(A1) < x \Rightarrow A1 \vee L(A2, B2, x) = L(A1 \vee A2, A1 \vee B2, x). \quad \textbf{(10a)}$$

Analogously

$$L(A1, B1, x) \vee A2 = L(A1, B1, x) \vee L(A2, A2, x) = L(A1 \vee A2, B1 \vee A2, x).$$

$$Var(A2) < x \Rightarrow L(A1, B1, x) \vee A2 = L(A1 \vee A2, B1 \vee A2, x) \quad \textbf{(10b)}$$

Rules (10), (10a), (10b) forms rewriting rules system of interpreter of disjunction for operands of the form L(A, B, x).

Dis := rs(A1, B1, A2, B2, x)
{
L(A1, B1, x) ∨ L(A2, B2 , x) = L(A1 ∨ A2, B1 ∨ B2, x),
Var(A1) < x □ A1 ∨ L(A2, B2 , x) = L(A1 ∨ A2, A1 ∨ B2, x),
Var(A2) < x □ L(A1, B1, x) ∨ A2 = L(A1 ∨ A2, B1 ∨ A2, x)
};

Con := rs(A1, B1, A2, B2, x)
{
L(A1, B1, x) & L(A2, B2 , x) = L(A1 & A2, B1 & B2, x),
Var(A1) < x □ A1 & L(A2, B2 , x) = L(A1 & A2, A1 & B2, x),
Var(A2) < x □ L(A1, B1, x) & A2 = L(A1 & A2, B1 & A2, x)
};

Not := rs(A1, B1, x)
{
¬L(A1, B1, x) = L(¬A1, ¬B1, x)
};

Main Result of Extension's Method

Definition of algebra *ALConstr* as direct limit of increasing sequence of algebras in issue gives the algorithm of synthesis of interpreters for algebraic operations

Full rewriting ruses systems for logical operations: Disjunction

Dis := rs(a, A1, B1, A2, B2, x)
{
  $a \vee O = a$, $O \vee a = a$, // Inherited
  $a \vee I = 0$, $I \vee a = I$,
  $L(A_1, B_1, x) \vee L(A_2, B_2, x) = L(A_1 \vee A_2, B_1 \vee B_2, x)$, // base
  $Var(A_1) < x \Rightarrow A_1 \vee L(A_2, B_2, x) = L(A_1 \vee A_2, A_1 \vee B_2, x)$, // additional
  $Var(A_2) < x \Rightarrow L(A_1, B_1, x) \vee A_2 = L(A_1 \vee A_2, B_1 \vee A_2, x)$
};

Method of morphisms

From diagram 1

Computations in A_{Init} defined by isomorphisms

$$\mu : A_{Init} \to A_{Constr}, \qquad \mu^{-1} : A_{Constr} \to A_{Init}$$

For our considerations μ defined on $Variable \subset A_{Init}$:

$$x = L(I, O, x) \tag{13}$$

Inverse isomorphism μ^{-1} defined by:

$$L(F, G, x) = x \,\&\, F \vee \neg x \,\&\, G \qquad \textbf{// RNF} \tag{14}$$

Base rules (10)-(12) can be obtained from (14) by equivalent transformations

Direct isomorphism

ALInitToConstr := rs(x)

{

IsVar(x)→ x = L(I, O, x)

};

Inverse isomorphism (with some additional rules):

ALConstToInit := rs(x, A, B)

{

L(I, O, x) = x,

L(O, I, x) = ¬x,

L(A, O, x) = x&A,

L(O, B, x) = ¬x&B,

L(A, B, x) = x&A ∨ ¬x&B

};

Veryfication of interpreters of algebraic computations

Axioms from axiomatic definition of algebra $A_{Abstract}$ can be used for veryfication of its constructive model A_{Constr}.

Axioms of $A_{Abstract}$ are identities of A_{Constr}.

The proof may be obtained by mathematical induction for index j of sequence

$$A^{(0)} \subset A^{(1)} \subset \ldots \subset A^{(j)} \subset A^{(j+1)} \subset \ldots.$$

Example

De Morgan's Low:

$$\neg(x \,\&\, y) = \neg x \vee \neg y \qquad \textbf{(1)}$$

Let

$$x = L(A1, B1, u), \quad y = L(A2, B2, u).$$

The result of substituting in (1) :

$$\neg(L(A1, B1, u) \,\&\, L(A2, B2, u) = \neg L(A1, B1, u) \vee \neg L(A2, B2, u) \qquad \textbf{(2)}$$

Atter computing of left and right hand sides using wrt **Dis, Con** and **Not** we obtain:

$$L(\neg(A1 \,\&\, A2), \neg(B1 \,\&\, B2), u) = L(\neg A1 \vee \neg A2, \neg B1 \vee \neg B2, u) \qquad \textbf{(3)}$$

If to equate arguments left and right hand sides we obtain :

$$\neg(A1 \,\&\, A2) = \neg A1 \vee \neg A2, \qquad \neg(B1 \,\&\, B2) = \neg B1 \vee \neg B2 \qquad (4)$$

Satisfiability of (1) reduces for satisfiability of (4).

So to provide induction's base we must check (1) on base algebra *Bool* .

If all additional rules are correct the proof of correctness of (1) is over.

If all axioms of $A_{Abstract}$ are correct, wrt **Dis, Con** and **Not are correct.**

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