E. Kartashova (*RISC, J. Kepler University, Linz, Austria)*
M. Lvov *(RIIT, Kherson State University, Kherson, Ukraine)*

Effective Algorithm for Full Solution of Dispersive Equation for 4-waves Nonlinear Resonances

In present report the algorithm for solving 4-waves dispersive equation are proposed. We consider both asymmetric and symmetric cases. The set of asymmetric solutions is generated by effective algorithm, and the set of symmetric solutions is described as the solutions series defined parametrically.

Problems Definition

The system of equations for the problem:

Let

$$\boldsymbol{k} \overset{df}{=} (\boldsymbol{m}, \boldsymbol{n})$$

$$\boldsymbol{m}, \boldsymbol{n} \in \boldsymbol{Integer}$$

$$\boldsymbol{k}_1 = (\boldsymbol{m}_1, \boldsymbol{n}_1),\ \boldsymbol{k}_2 = (\boldsymbol{m}_2, \boldsymbol{n}_2),\ \boldsymbol{k}_3 = (\boldsymbol{m}_3, \boldsymbol{n}_3),\ \boldsymbol{k}_4 = (\boldsymbol{m}_4, \boldsymbol{n}_4)$$

$$\omega(\boldsymbol{k}) \overset{df}{=} (\boldsymbol{m}^2 + \boldsymbol{n}^2)^{1/4}$$

Given:

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \qquad (1)$$

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4) \qquad (2)$$

$$\boldsymbol{m}, \boldsymbol{n} \in [-\boldsymbol{D}_{\max}, \boldsymbol{D}_{\max}] \qquad (3)$$

Find:

All k_1, k_2, k_3, k_4 **satisfies (1)-(3)**

Previous Publications

1. Elena Kartashova. Fast Computation Algorithm for Discrete Resonances among Gravity Waves. Journal of Low Temperature Physics, Vol. 145, Nos. 1–4, November 2006

2. Elena Kartashova, Alexey Kartashov. LaminatedWave Turbulence: Generic Algorithms I. International Journal of Modern Physics C, Vol. 17, No. 11 (2006), 1579-1596.

3. Elena Kartashova, Alexey Kartashov. LaminatedWave Turbulence: Generic Algorithms II. Communications in computational physics. Vol. 2, no. 4, pp. 783-794

4. Elena Kartashova, Guenther Mayrhofer. Cluster formation in mesoscopic systems.

Physica A 385 (2007) 527–542

5. Elena Kartashova, Sergey Nazarenko, Oleksii Rudenko. Resonant interactions of nonlinear water waves in a finite basin. Physical review e **78**, 016304 _2008_

Problems of present report

The set of all solutions can be classified as

- the set of asymmetric solutions,
- the set of symmetric solutions.

Symmetric solution – parametrically defined series (example)

$$k_1 = (a,b),\ k_2 = (b,a),\ k_3 = (-a,-b),\ k_4 = (-b,-a);\ a,\ b \in N.$$

Asymmetric solution (example)

$$k_1 = (256,512),\ k_2 = (1980,360),\ k_3 = (800,400),\ k_4 = (1436,472)$$

Algorithms described in [1-4] generate both symmetric and asymmetric solutions.

- The number of symmetric solutions is very large, so the time of algorithm's execution is large too.
- The analysis of results in mixed list is essentially complicated

So we must

1. **To build algorithm generates only the base set of asymmetric solutions**
2. **To describe parametrically all series of symmetric solutions**

1. Group of symmetries of model (1)-(3)

Definition 1

Let define group G_{sym} of model (1), (2) on domain (3) as follows:

1. Element of the group G_{sym} is «signed substitution»

$$\begin{pmatrix} (m_1 & n_1) & (m_2 & n_2) & (m_3 & n_3) & (m_4 & n_4) \\ (r_1, & s_1) & (r_2 & s_2) & (r_3 & s_3) & (r_4 & s_4) \end{pmatrix} \tag{4}$$

where $r_j = \begin{cases} \pm m_k \\ \pm n_k \end{cases}, \; s_j = \begin{cases} \pm m_k \\ \pm n_k \end{cases}$

The second line is some substitution of the first line with signs "plus" or "minus".

2. Element of G_{sym}, applied to the model, satisfies this model.

3. Unordered quadruples of vectors are in pairs different

$$(m_1, n_1), (m_2, n_2), (m_3, n_3), (m_4, n_4) \text{ и } (r_1, s_1), (r_2, s_2), (r_3, s_3), (r_4, s_4)$$

Example: substitution

$$\begin{pmatrix} (m_1 & n_1) & (m_2 & n_2) & (m_3 & n_3) & (m_4 & n_4) \\ (n_4 & m_4) & (n_3 & m_3) & (n_2 & m_2) & (n_1 & m_1) \end{pmatrix}$$

satisfies model (1)–(3).

Description of group G_{sym}.

Rotations on angles $k \cdot \pi/2$. Group G_{sym} contains 4 rotations

α: `(m, n) → (-n, m)        // rotation on` $\pi/2$

α^2: `(m, n) → (-m, -n)`

α^3: `(m, n) → (n, -m)`

$\alpha^4 = e$: `(m, n) → (m, n)`

Symmetries of the plane relatively axes OX и OY. Orders = 2.

β: `(m, n) → (m, -n)        // OX symmetry`

γ: `(m, n) → (-m, n)        // OY symmetry`

Symmetries of the plane relatively `x = y, x = -y`. Orders = 2.

δ: `(m, n) → (n, m)  // x = y symmetry`

ε: `(m, n) → (-n, -m)      // x = -y symmetry`

The set of transformations of the plane forms group of 8-th order, well known as group of diedre.

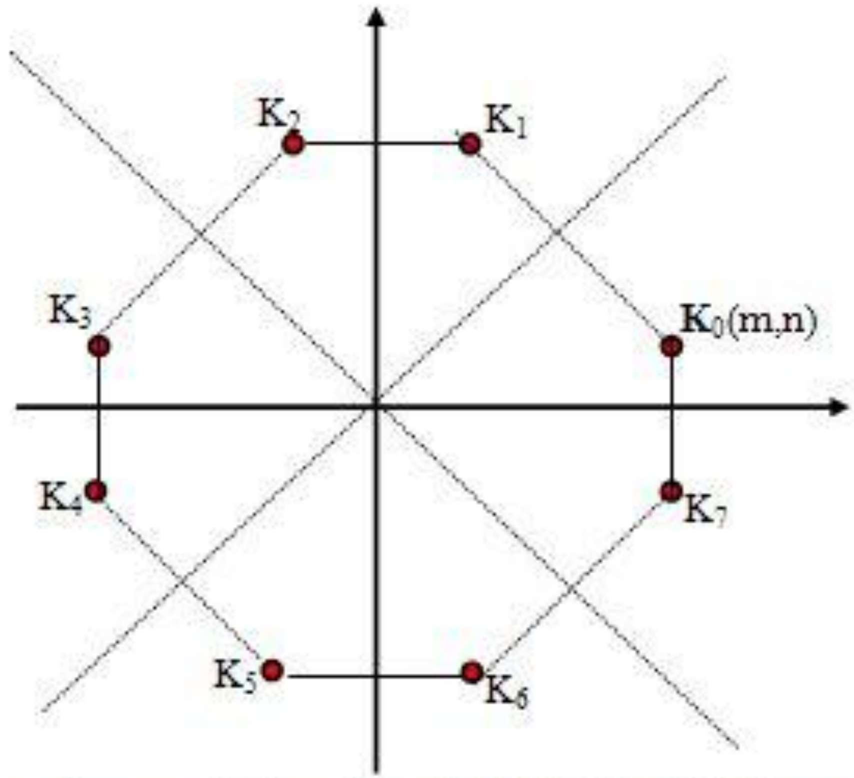

Fig 1. The group of symmetries of model (1)-(3). (Diedre group).

Example: substitution

$$\begin{pmatrix} (m_1 & n_1) & (m_2 & n_2) & (m_3 & n_3) & (m_4 & n_4) \\ (m_1 & -n_1) & (m_2 & -n_2) & (m_3 & -n_3) & (m_4 & -n_4) \end{pmatrix}$$

- result of applying of transformation β к K_0. $K_7 = \beta(K_0)$.

Assertion 1.

Let signed substitution has the view

$$\begin{pmatrix} K_1 & K_2 & K_3 & K_4 \\ \sigma(K_1) & \sigma(K_2) & \sigma(K_3) & \sigma(K_4) \end{pmatrix} \quad (5)$$

Where $\sigma \in G_{sym}$ and K_1, K_2, K_3, K_4 satisfies model (1)-(3). Then quadruple $\sigma(K_1), \sigma(K_2), \sigma(K_3), \sigma(K_4)$ also satisfies model (1)-(3).

Assertion 2.

Let signed substitution has view (4) and quadruple K_1, K_2, K_3, K_4 satisfies (1)-(3). Then quadruple $(r_1, s_1), (r_2, s_2), (r_3, s_3), (r_4, s_4)$ has view $\sigma(K_{j1}), \sigma(K_{j2}), \sigma(K_{j3}), \sigma(K_{j4})$, where substitution (j_1, j_2, j_3, j_4) transforms parallelogram K_1, K_2, K_3, K_4 to it self.

This group described by generators

$$\mathbf{K_1 \rightarrow K_2, K_3 \rightarrow K_4, (K_1, K_2) \rightarrow (K_3, K_4).}$$

So we consider group G_{sym} as group of symmetries of our model (1)-(3)/

Definitions of asymmetric and symmetric solutions

Definition 2. Solution K_1, K_2, K_3, K_4 (of model (1)-(3)) is said to be **asymmetric**, if all quadruples $\sigma(K_1), \sigma(K_2), \sigma(K_3), \sigma(K_4), \sigma \in G_{sym}$ are in pairs different.

Definition 3. Solution K_1, K_2, K_3, K_4 is said to be **symmetric,** if there exists such $\sigma \in G_{sym}, \sigma \neq 1$, then

$$(K_1, K_2, K_3, K_4) = (\sigma(K_1), \sigma(K_2), \sigma(K_3), \sigma(K_4))$$

These definitions are differs from definitions uses in previous publications. Differences are discussed below.

2. Algorithm of generation of base asymmetric solutions

The algorithm consists from two subalgorithms, executing consequently. First algorithm generates the Table TQ of (q)-classes (in terminology of [1]).

Definition 4.

Let $K(m,n)$ - point of the plane. Point K belongs to (q)-class, if

$$m^2 + n^2 = \gamma \cdot \sqrt[4]{q}\ , \qquad (6)$$

where q - natural number, free from 4-th degrees.

As it was shown in previous publications, equation (2) can be rewritten to the view

$$\gamma_1 \cdot \sqrt[4]{q} + \gamma_2 \cdot \sqrt[4]{q} = \gamma_3 \cdot \sqrt[4]{q} + \gamma_4 \cdot \sqrt[4]{q} \qquad \textbf{(asymmetric case ???)} \qquad (7)$$

or to the view

$$\gamma_1 \cdot \sqrt[4]{q_1} + \gamma_2 \cdot \sqrt[4]{q_2} = \gamma_1 \cdot \sqrt[4]{q_1} + \gamma_2 \cdot \sqrt[4]{q_2} \qquad \textbf{(symmetric case)} \qquad (8)$$

Number p is called as the class of point $\boldsymbol{K(m,n)}$.

So (7), (8) means that the arbitrary quadruple - solution of (1)-(3),

a) contains four points from one (q)-class,
b) contains two pairs of points from two different (q)-classes.

Algorithm of generation of base asymmetric solutions. Main idea:

1. The algorithm generates in pairs different **base** quadruples (K_1, K_2, K_3, K_4) belongs to one (q)-class from first octant of the plane: $x \geq 0, y \geq 0, x \geq y$ such that

$$\gamma_1 - \gamma_3 = \gamma_4 - \gamma_2,\ \ \gamma_1 - \gamma_3 \geq 0 \tag{9}$$

In this way we satisfy condition (7) together with equation (2). It is easy to see that these conditions are invariants relatively transformations from G_{sym}.

2. For each such quadruple (K_1, K_2, K_3, K_4) algorithm finds all such quadruples

$$(\boldsymbol{K}_1, \sigma_2(\boldsymbol{K}_2), \sigma_3(\boldsymbol{K}_3), \sigma_4(\boldsymbol{K}_4)),\ \sigma_i \in \boldsymbol{G}_{sym}.$$

that

$$\boldsymbol{k}_1 - \sigma_3(\boldsymbol{k}_3) = \sigma_4(\boldsymbol{k}_4) - \sigma_2(\boldsymbol{k}_2). \tag{10}$$

In this way we satisfy vector equation (1).

Then quadruple

$$(\boldsymbol{k}_1, \sigma_2(\boldsymbol{k}_2), \sigma_3(\boldsymbol{k}_3), \sigma_4(\boldsymbol{k}_4)),\ \sigma_i \in \boldsymbol{G}_{sym} \tag{11}$$

forms the **base** asymmetric solution.

Note then the first point $\mathbf{K}_1$ belongs to first octant. That's why the algorithm will generate only one (base) solution S from each eight different solutions of the view

$$S, \sigma_1(S), \sigma_2(S), \sigma_3(S), \sigma_4(S), \sigma_5(S), \sigma_6(S), \sigma_7(S) \tag{12}$$

That is why we found only the base solution and generating of quadruples made in the view

$$(K_1, \sigma_2(K_2), \sigma_3(K_3), \sigma_4(K_4)), \sigma_i \in G_{sym}.$$

Preliminary computations

On the preliminary stage subalgorithm 1 executes following computations:

1. Form:

   1.1. the table (array) P of all primes $P = \{p : p < \sqrt[4]{2D_{max}^2}\}$.

   1.2 the table (array) P4 of all 4-th degrees of primes from P.

Stage 1. Main Loop

2. For all (m,n), $m \le n \le D_{max}$ compute

   2.1 $A_{m,n} = m^2 + n^2$

   2.2 $\gamma, q : A_{m,n} = \gamma \cdot \sqrt[4]{q}$. // by dividing $A_{m,n}$ on all elements from P4.

   2.3. Insert (m, n, γ) in the Table TQ of (q)- classes

TQ – Table of (q)-classes

Table **TP** is array of pointers on lists of (p) classes.

Elements of the list (p) are ordered by order of enumerating of points (m, n).

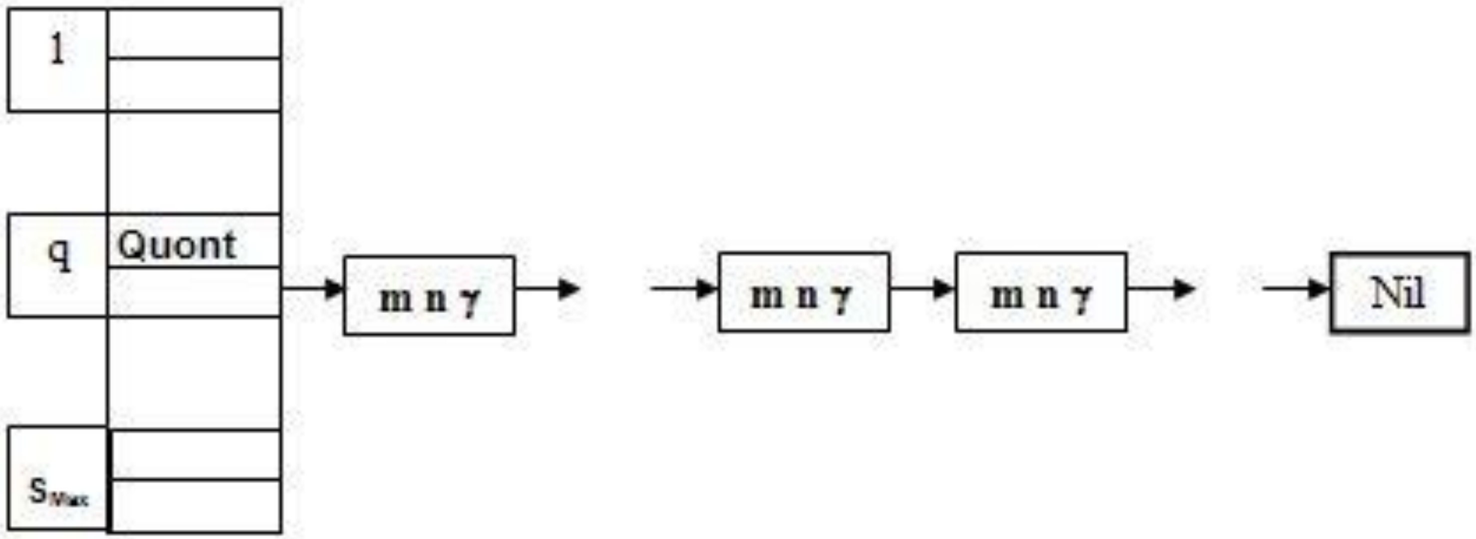

Fig 2. Data structure for (q)-classes – table TP.

Smax = $(D_{max})^2$. Quont – number of elements in (q)-class.

Invariant: $m^2 + n^2 = \gamma^4 \cdot q$.

Stage II (Subalgorithm 2)

1. Forming of the table TG of differences

For all (q) **–classes – lines of Table** TQ

For all pairs $K_j(m_j, n_j)$, $K_1(m_1, n_1)$ **from fixed line of** TQ

If $\gamma_j - \gamma_1 = \gamma$, $\gamma > 0$

Then Insert (TG, γ, K_j, K_1)

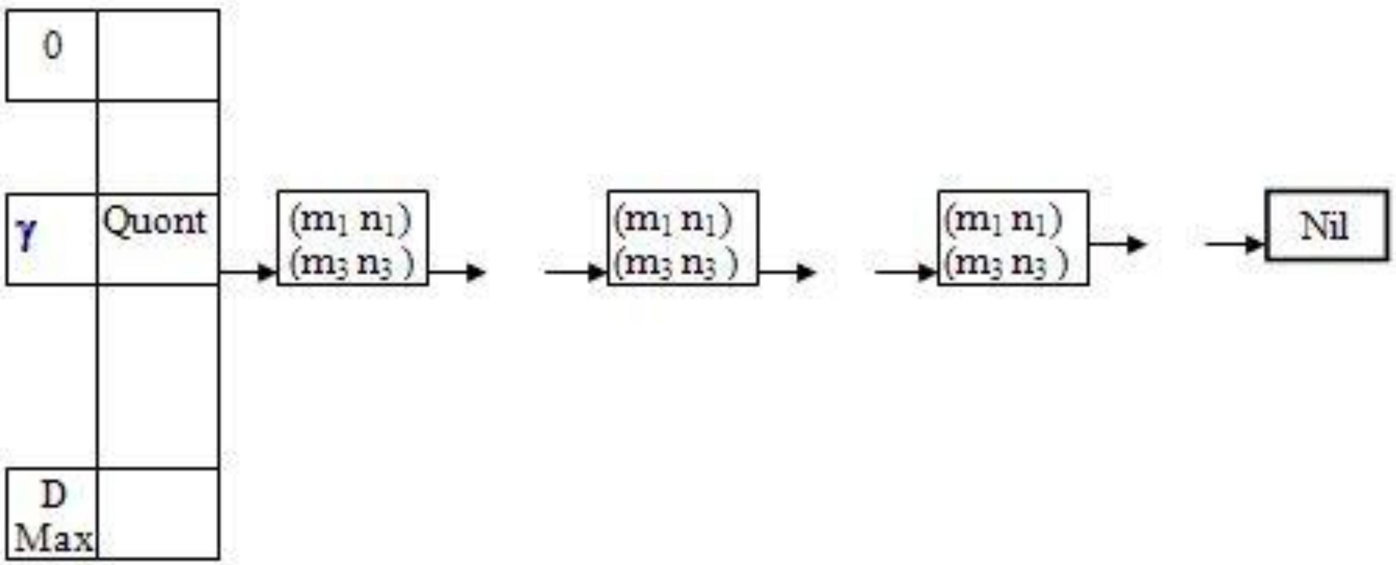

Fig 3. Data structure for (q, γ)-classes – table TG.

Invariant for table TQ:

$$m_1^2 + n_1^2 = \gamma_1^4 \cdot q,\ m_3^2 + n_3^2 = \gamma_3^4 \cdot q,\quad \gamma_1 - \gamma_3 = \gamma,\ \gamma > 0 \tag{15}$$

Each pair of elements from (q)-line of table TG satisfies equation (2).

Stage II (Subalgorithm 2)

2. Searching of asymmetric solutions for fixed pair (K_1, K_3), (K_4, K_2)

Let us consider quadruple (K_1, K_3), (K_4, K_2) belongs to one line of TG−table. Algorithm enumerates all triples of elements of G_{sym}:

For all $(\sigma_2, \sigma_3, \sigma_4)$, $\sigma_i \in \boldsymbol{G}_{sym}$

If $\boldsymbol{k}_1 - \sigma_3(\boldsymbol{k}_3) = \sigma_4(\boldsymbol{k}_4) - \sigma_2(\boldsymbol{k}_2)$ (16)

Then Write(**OutFile**, $(K_1,\ \sigma_2(K_2),\ \sigma_3(K_3)\ \ \sigma_4(K_4))$

Mark, that full account of all triples $(\sigma_2, \sigma_3, \sigma_4)$, $\sigma_i \in \boldsymbol{G}_{sym}$ requires $8*8*8 = 512$ steps. But in the paper we propose an effective algorithm to check (16).

4.Analysis of asymmetric and symmetric solutions

Let K_1, K_2, K_3, K_4 be a quadruple of points from the first octant generated by algorithm as (K_1, K_3), (K_4, K_2) and so satisfies (2) . Then following cases are logically possible:

1. Points K_1, K_2, K_3, K_4 in pairs different
2. Only two points are equal: say, $K_j = K_1$.

   2a. If $K_1 = K_4$, then $K_3 = K_2$, i.e. $(K_1, K_3) = (K_4, K_2)$. This case do not generates by the algorithm. We shall consider it in **3.**

   2b. If $K_1 = K_2$, pairs (K_1, K_3), (K_4, K_1) are different. This case generates by the algorithm.

3. Quadruple is (K_1, K_3), (K_1, K_3). Relation (16) has view

$$A_1 - \sigma_3(A_3) = \sigma_4(A_1) - \sigma_2(A_3) \tag{17}$$

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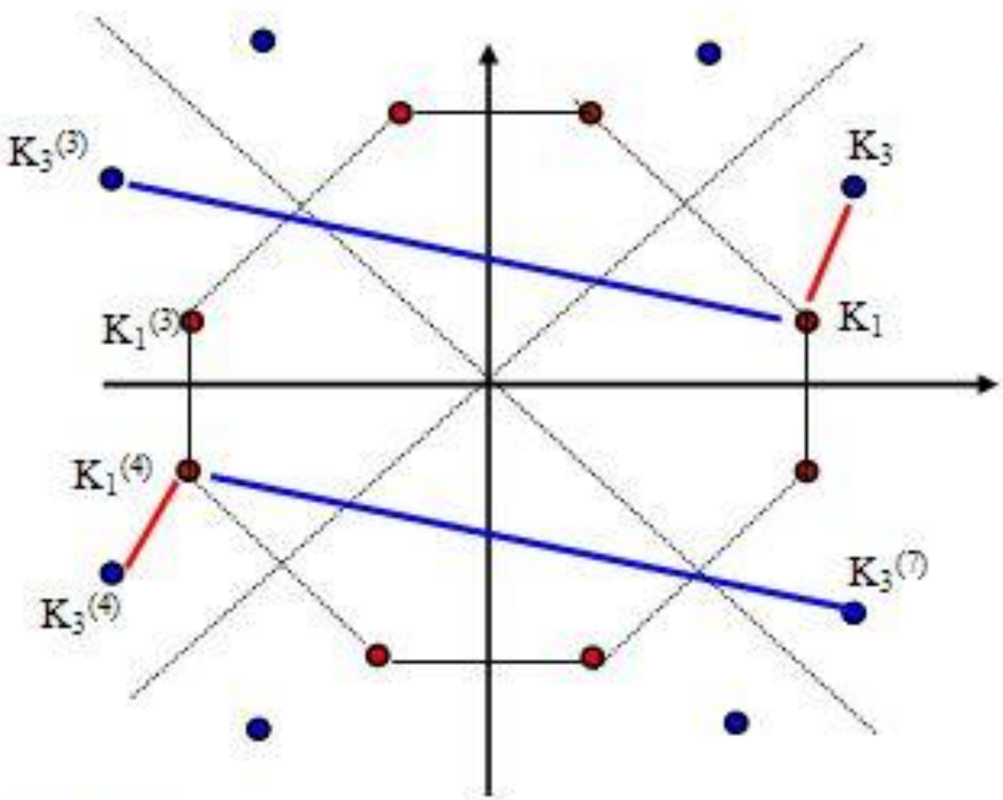

Pairs (K_1, K_3), $(K_1^{(4)}, K_3^{(4)})$ forms solution.

Pairs $(K_1, K_3^{(3)})$, $(K_1^{(4)}, K_3^{(7)})$ forms solution.

Fig 4. Case 3.

3a. If segment (K_1, K_3) is in a general position then all solutions described by

$$K_1 - \sigma(K_3) = \alpha^2 (K_1 - \sigma(K_3)) \tag{19}$$

where α^2 - rotation on 180°, and σ - an arbitrary element of G_{sym}.

Cases 3b – 3e

3b. If $(K_1, K_3) \parallel OX$, then solutions are

$((K_1, K_3), (\beta(K_1), \beta(K_3))), ((K_1, K_3), (\gamma(K_1), \gamma(K_3)))$.

3c. If $(K_1, K_3) \parallel OY$, then solutions are

$((K_1, K_3), (\gamma(K_1)), (\gamma(K_3)), (K_1, K_3), (\beta(K_1), \beta(K_3)))$.

3d. If $(K_1, K_3) \parallel x = y$, then solutions are

$(K_1, K_3), (\delta(K_1), \delta(K_3)), (K_1, K_3), (\varepsilon(K_1), \varepsilon(K_3))$.

3e. $(K_1, K_3) \parallel x = -y$, then solutions are

$(K_1, K_3), (\delta(K_1), \delta(K_3)), (K_1, K_3), (\varepsilon(K_1), \varepsilon(K_3))$.

Case 4.

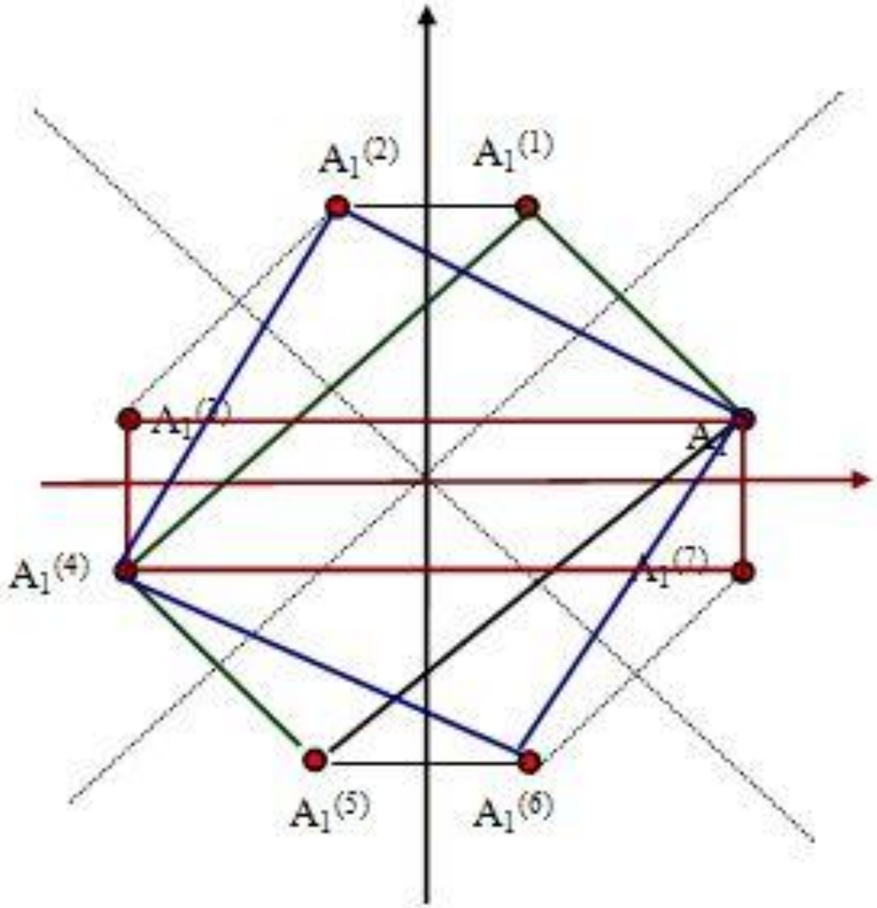

Fig. 5. All 4 points generates from one point κ_1 .

Points κ_1 , κ_2 , κ_3 , κ_4 belongs to the orbit of κ_1

- κ, $\kappa^{(1)}$, $\kappa^{(4)}$, $\kappa^{(5)}$
- κ, $\kappa^{(2)}$, $\kappa^{(4)}$, $\kappa^{(6)}$
- κ, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(7)}$

Series of symmrtric solutions

1. Case 3a.

$$(a,b),\ (c,d),\ (-a,-b),\ (-c,-d). \tag{20}$$

The number of such solutions has the order $O(D^4)$.

2. Case 3b.

$$(a,b),\ (c,b),\ (c,-b),\ (a,-b). \tag{21}$$

$$(a,b),\ (c,b),\ (-a,b),\ (-c,b). \tag{22}$$

The number of such solutions has the order $O(D^3)$.

Case 3c

$$(a,b),\ (a,c),\ (-a,c),\ (-a,b). \tag{23}$$

$$(a,b),\ (a,c),\ (a,-b),\ (a,-c). \tag{24}$$

The number of such solutions has the order $O(D^3)$.

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3. Case 3d:

(a,b), (a+c,b+c), (b+c,a+c), (b,a). (25)

(a,b), (a+c,b+c), (-b-c,-a-c), (-b,-a). (26)

The number of such solutions has the order $O(D^3)$.

Case 3e

(a,b), (a+c,b+c), (b+c,a+c), (b,a). (27)

(a,b), (a+c,b-c), (-b+c,-a-c), (-b,-a). (28)

The number of such solutions has the order $O(D^3)$.

4. Case 4.

(a,b), (b,a), (-a,-b), (-b,-a) (29)

(a,b), (b,a), (-a,-b), (-b,-a) (30)

(a,b), (-b,a), (-a,-b), (b, -a) (31)

The number of such solutions has the order $O(D^2)$. All these solutions included to (20).

Each series defines with precision to transformations from G_{sym}.

6. Series of symmetric solutions for different (p)-classes has the view (8)

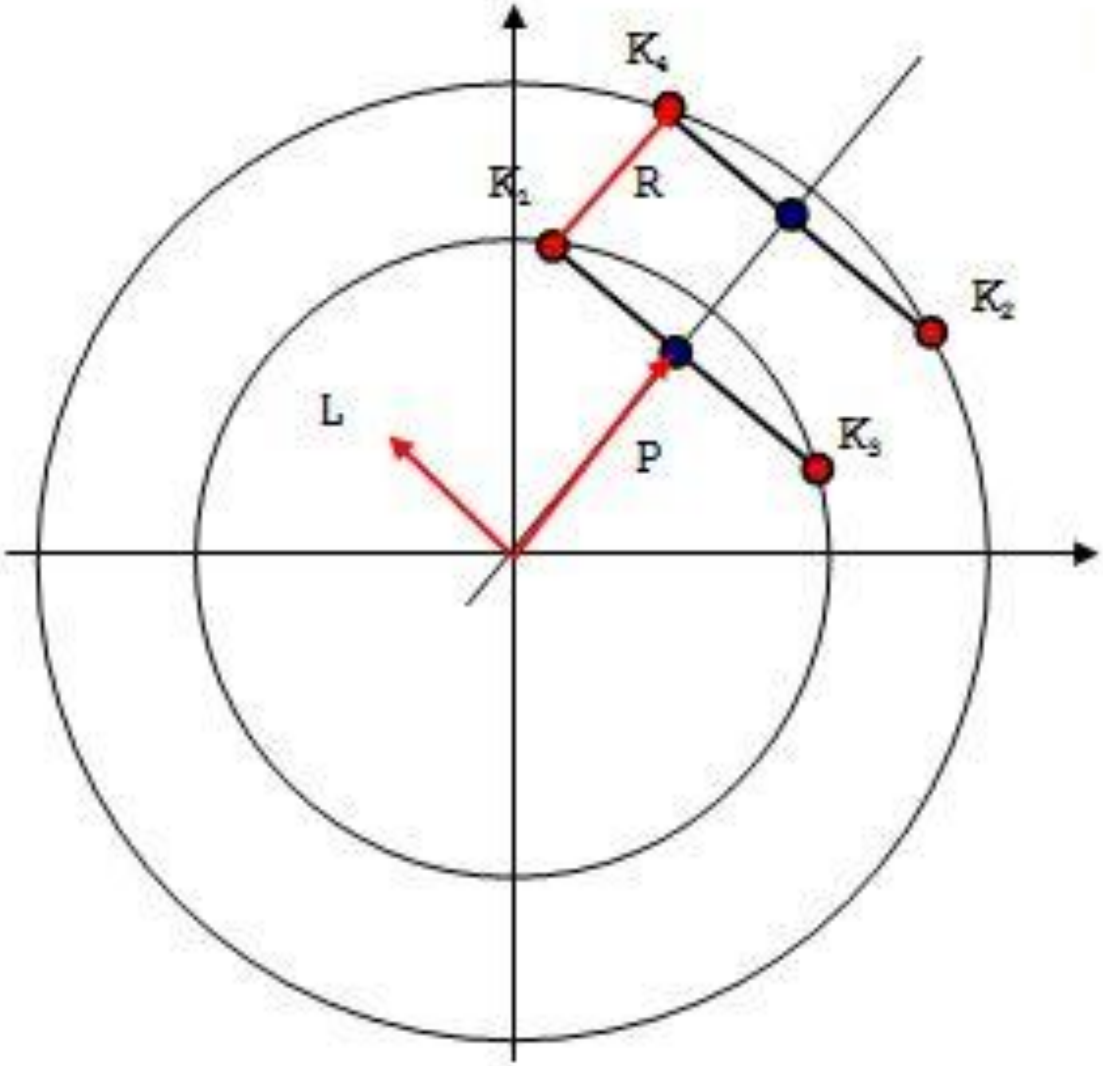
Suppose that the equation (2) after computations has the view (8)

$$\gamma_1 \cdot \sqrt[4]{q_1} + \gamma_2 \cdot \sqrt[4]{q_2} = \gamma_1 \cdot \sqrt[4]{q_1} + \gamma_2 \cdot \sqrt[4]{q_2} \qquad (8)$$

Then there exists at least two pairs $(\bar{k}_1, \bar{k}_3), (k_2, \bar{k}_4)$ such that

$$m_1^2 + n_1^2 = \gamma_1^4 \cdot q_1, \; m_3^2 + n_3^2 = \gamma_1^4 \cdot q_1, \qquad (32)$$

$$m_2^2 + n_2^2 = \gamma_2^4 \cdot q_2, \; m_4^2 + n_4^2 = \gamma_2^4 \cdot q_2$$



$$K_1 - K_3 = K_4 - K_2$$

$$|K_1| = |K_3| = r$$

$$|K_2| = |K_4| = R$$

$$K_1 = P + L,$$

$$K_2 = P - L + R,$$

$$K_3 = P - L,$$

$$K_4 = P + L + R.$$

Fig. 6. Rectangle of solutions.

Let $L = (a,b)$, $d = \gcd(a, b)$.

Then $P = C_1(-b,a)$, $R = C_2(-b,a)$.

$K_1 = (a,b) + C_1/d*(-b,a)$,

$K_3 = (-a,-b) + C_1/d*(-b,a)$,

$K_4 = (a,b) + C_1/d*(-b,a) + C_2/d*(-b,a)$,

$K_2 = (-a,-b) + C_1/d*(-b,a) + C_2/d*(-b,a)$.

All coordinates K_i – integers, and $L = \frac{K_1 - K_3}{2}$.

That's why numbers (a,b) may be semiinteders.

Coefficients C_1, C_2 may be rational with denominator $d = \gcd(a,b)$.

Let

$$C_1 := C_1/d,\ C_2 := C_2/d$$

Then

$K_1 = (a,b) + C_1(-b,a),$

$K_3 = (-a,-b) + C_1(-b,a),$

$K_4 = K_1 + C_2(-b,a),$

$K_2 = K_3 + C_2(-b,a).$

1. $d = \gcd(a,b) = 1.$

1.a a,b are semiintegers $\Rightarrow$ $C1 = 2l+1,\ C2 = 2k.$

1.b $a,b \in Z \Rightarrow C_1, C_2 \in Z$

2. $d = \gcd(a,b) > 1.$

???????????????????????????

Conclusion

1. Все симметричные решения – серии (21)-(31) – частные случаи серии (39). Все эти серии генерируют прямоугольники.

2. Серия (20) – самая большая – генерирует параллелограммы. Она не выражается через (39). Таким образом, существуют симметричные решения, относящиеся к случаю (7) но не относящиеся к случаю (8)

3. Свойство симметричности решения может быть выражено в терминах равенства норм волновых векторов. Существуют асимметричные решения, в которых нормы пары векторов равны (случай 2b).