CSCI 1900
Discrete Structures

Logical Operations
Section 2.1

Statement of Proposition

- Statement of proposition – a declarative sentence that is either true or false, but not both
- Examples:
  - The earth is round: statement that is true
    2+3=5: statement that is true
  - Do you speak English? This is a question, not a statement

More Examples of Statements of Proposition

- 3-x=5: is a declarative sentence, but not a statement since it is true or false depending on the value of x
- Take two aspirins: is a command, not a statement
- The temperature on the surface of the planet Venus is 800^{o}F: is a declarative statement of whose truth is unknown to us
- The sun will come out tomorrow: a statement that is either true or false, but not both, although we will have to wait until tomorrow to determine the answer

Logical Connectives and Compound Statements

- *x, y, z,* … denote variables that can represent real numbers
- *p, q, r,*… denote prepositional variables that can be replaced by statements.
  - *p*: The sun is shining today
  - *q*: It is cold

Negation

- If *p* is a statement, the negation of *p* is the statement *not p*
- Denoted ~*p*
- If *p* is true, ~*p* is false
- If *p* is false, ~*p* is true
- ~*p* is not actually connective, i.e., it doesn't join two of anything
- ***not*** is a unary operation for the collection of statements and ~*p* is a statement if *p* is

Examples of Negation

- If *p*: $2+3 > 1$ then If $\sim p$: $2+3 \leq 1$
- If *q*: It is cold then $\sim q$: It is not the case that it is cold, i.e., It is not cold.

Conjunction

- If p and q are statements, then the ***conjunction*** of p and q is the compound statement “p and q”
- Denoted $p \wedge q$
- $p \wedge q$ is true only if both p and q are true
- Example:
  - p: ETSU parking permits are expensive
  - q: ETSU has plenty of parking
  - $p \wedge q$ = ?

Disjunction

- If p and q are statements, then the ***disjunction*** of p and q is the compound statement “p or q”
- Denoted $p \vee q$
- $p \vee q$ is true if either p or q are true
- Example:
  - p: I am a male
  - q: I am under 40 years old
  - $p \vee q = ?$

Exclusive Disjunction

- If p and q are statements, then the ***exclusive disjunction*** is the compound statement, “either p or q may be true, but both are not true at the same time.”
- Example:
  - p: It is daytime
  - q: It is night time
  - $p \vee q$ (in the exclusive sense) = ?

Inclusive Disjunction

- If p and q are statements, then the ***inclusive disjunction*** is the compound statement, "either p or q may be true or they may both be true at the same time."
- Example:
  - p: It is cold
  - q: It is night time
  - $p \vee q$ (in the inclusive sense) = ?

Exclusive versus Inclusive

- Depending on the circumstances, some disjunctions are inclusive and some of exclusive.
- Examples of Inclusive
  - "I have a dog" or "I have a cat"
  - "It is warm outside" or "It is raining"
- Examples of Exclusive
  - Today is either Tuesday or it is Thursday
  - Pat is either male or female

Compound Statements

- A ***compound statement*** is a statement made from other statements
- For n individual propositions, there are 2^n possible combinations of truth values
- A truth table contains 2^n rows identifying the truth values for the statement represented by the table.
- Use parenthesis to denote order of precedence
- $\wedge$ has precedence over $\vee$

Truth Tables are Important Tools for this Material!

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statement Example
$(p \land q) \lor (\sim p)$

p	q	$p \land q$	$\sim p$	$(p \land q) \lor (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Quantifiers

- Back in Section 1.1, a set was defined {x | P(x)}
- For an element t to be a member of the set, P(t) must evaluate to "true"
- P(x) is called a predicate or a propositional function

Computer Science Functions

- if P(x), then execute certain steps
- while Q(x), do specified actions

Universal quantification of a predicate P(x)

- Universal quantification of predicate P(x) = For all values of x, P(x) is true
- Denoted $\forall x\ P(x)$
- The symbol $\forall$ is called the universal quantifier
- The order in which multiple quantifications are considered does not affect the truth value (e.g., $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$)

Examples:

- P(x): $-(-x) = x$
  - This predicate makes sense for all real numbers x.
  - The universal quantification of P(x), $\forall x\, P(x)$, is a true statement, because for all real numbers, $-(-x) = x$
- Q(x): $x+1<4$
  - $\forall x\, Q(x)$ is a false statement, because, for example, Q(5) is not true

Existential quantification of a predicate P(x)

- Existential quantification of a predicate P(x) is the statement “There exists a value of x for which P(x) is true.”
- Denoted $\exists x\, P(x)$
- Existential quantification may be applied to several variables in a predicate
- The order in which multiple quantifications are considered does not affect the truth value

Applying both universal and existential quantification

- Order of application does matter
- Example: Let ***A*** and ***B*** be n x n matrices
- The statement $\forall \mathbf{A}\ \exists \mathbf{B}\ \ \mathbf{A} + \mathbf{B} = \mathbf{I}_n$
- Reads "for every ***A*** there is a ***B*** such that $\mathbf{A} + \mathbf{B} = \mathbf{I}_n$"
- Prove by coming up for equations for b_{ii} and b_{ij} ($j \neq i$)
- Now reverse the order: $\exists \mathbf{B}\ \forall \mathbf{A}\ \mathbf{A} + \mathbf{B} = \mathbf{I}_n$
- Reads "there exists a ***B*** such that for all ***A*** $\mathbf{A} + \mathbf{B} = \mathbf{I}_n$"
- THIS IS FALSE!

Assigning Quantification to Proposition

- Let p: $\forall x\, P(x)$
- The negation of p is false when p is true and true when p is false
- For p to be false, there must be at least one value of x for which P(x) is false.
- Thus, p is false if $\exists x \sim P(x)$ is true.
- If $\exists x \sim P(x)$ is false, then for every x, ~P(x) is false; that is $\forall x\, P(x)$ is true.

Okay, what exactly did the previous slide say?

- Assume a statement is made that "for all x, P(x) is true."
  - If we can find one case that is not true, then the statement is false.
  - If we cannot find one case that is not true, then the statement is true.
- Example: $\forall$ positive integers, n, P(n) = $n^2 + 41n + 41$ is a prime number.
  - This is false because $\exists$ an integer resulting in a non-prime value, i.e., $\exists$ n such that P(n) is false.

Discrete Structures

Conditional Statements

Section 2.2

Conditional Statement/Implication

- "if *p* then *q*"
- Denoted $p \Rightarrow q$
  - p is called the ***antecedent*** or ***hypothesis***
  - q is called the ***consequent*** or ***conclusion***
- Example:
  - *p*: I am hungry
    q: I will eat
  - *p*: It is snowing
    q: 3+5 = 8

Conditional Statement/Implication (continued)

- In English, we would assume a cause-and-effect relationship, i.e., the fact that p is true would force q to be true.
- If "it is snowing," then "3+5=8" is meaningless in this regard since *p* has no effect at all on *q*
- At this point it may be easiest to view the operator "⇒" as a logic operationsimilar to AND or OR (conjunction or disjunction).

Truth Table Representing Implication

- If viewed as a logic operation, p ⇒ q can only be evaluated as false if ***p is true and q is false***
- This does not say that *p* causes *q*
- Truth table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples where $p \Rightarrow q$ is viewed as a logic operation

- If p is false, then any q supports $p \Rightarrow q$ is true.
  - False ⇒ True = True
  - False ⇒ False = True
- If “2+2=5” then “I am the king of England” is true

Converse and contrapositive

- The converse of $p \Rightarrow q$ is the implication that $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is the implication that $\sim q \Rightarrow \sim p$

Converse and Contrapositive Example

Example: What is the converse and contrapositive of *p*: "it is raining" and *q*: I get wet?

- Implication: If it is raining, then I get wet.
- Converse: If I get wet, then it is raining.
- Contrapositive: If I do not get wet, then it is not raining.

Equivalence or biconditional

- If *p* and *q* are statements, the compound statement *p* ***if and only if*** *q* is called an ***equivalence*** or ***biconditional***
- Denoted $p \Leftrightarrow q$

Equivalence Truth table

- The only time that the expression can evaluate as true is if both statements, p and q, are true or both are false

p	Q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Proof of the Contrapositive

Compute the truth table of the statement
$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Tautology and Contradiction

- A statement that is true for all of its propositional variables is called a ***tautology***. (The previous truth table was a tautology.)
- A statement that is false for all of its propositional variables is called a ***contradiction*** or an ***absurdity***

Contingency

- A statement that can be either true or false depending on its propositional variables is called a ***contingency***
- Examples
  - $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology
  - $p \wedge \sim p$ is an absurdity
  - $(p \Rightarrow q) \wedge \sim p$ is a contingency since some cases evaluate to true and some to false.

Contingency Example

The statement $(p \Rightarrow q) \land (p \lor q)$ is a contingency

p	q	$p \Rightarrow q$	$p \lor q$	$(p \Rightarrow q) \land (p \lor q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Logically equivalent

- Two propositions are ***logically equivalent*** or ***simply equivalent*** if p ⇔ q is a tautology.
- Denoted p ≡ q

Example of Logical Equivalence

Columns 5 and 8 are equivalent, and therefore, p "if and only if" q

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Additional Properties
$(p \Rightarrow q) \equiv ((\sim p) \lor q)$

p	q	$(p \Rightarrow q)$	$\sim p$	$((\sim p) \lor q)$	$(p \Rightarrow q) \Leftrightarrow ((\sim p) \lor q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Additional Properties
$(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$

p	q	$(p \Rightarrow q)$	$\sim q$	$\sim p$	$(\sim q \Rightarrow \sim p)$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

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Methods of Proof

Reading: Kolman, Section 2.3

Past Experience

Up to now we've used the following methods to write proofs:

- Used direct proofs with generic elements, definitions, and given facts
- Used proof by cases such as when we used truth tables

General Description of Process

- $p \Rightarrow q$ denotes "*q* logically follows from *p*"
- Implication may take the form $(p_1 \land p_2 \land p_3 \land \ldots \land p_n) \Rightarrow q$
- q logically follows from $p_1, p_2, p_3, \ldots, p_n$

General Description (continued)

The process is generally written as:

$$p_1$$
$$p_2$$
$$p_3$$
$$\vdots$$
$$\vdots$$
$$p_n$$
$$\underline{\hspace{4cm}}$$
$$\therefore q$$

Components of a Proof

- The p_i's are called ***hypotheses*** or ***premises***
- q is called the ***conclusion***
- Proof shows that if all of the p_i's are true, then q has to be true
- If result is a tautology, then the implication $p \Rightarrow q$ represents a universally correct method of reasoning and is called a ***rule of inference***

Example of a Proof based on a Tautology

- If p implies q and q implies r, then p implies r

$$\begin{array}{l} p \Rightarrow q \\ \underline{q \Rightarrow r} \\ \therefore p \Rightarrow r \end{array}$$

- By replacing the bar under $q \Rightarrow r$ with the "$\Rightarrow$", the proof above becomes $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
- The next slide shows that this is a tautology and therefore is universally valid.

Tautology Example (continued)

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Equivalences

- Some mathematical theorems are equivalences, i.e., $p \Leftrightarrow q$.
- The proof of such a theorem is equivalent with proving both $p \Rightarrow q$ and $q \Rightarrow p$

modus ponens
form (the method of asserting):

$$\begin{array}{l} p \\ \underline{p \Rightarrow q} \\ \therefore q \end{array}$$

- Example:
  - p: a man used the toilet
  - q: the toilet seat is up
  - $p \Rightarrow q$: If a man used the toilet, the seat was left up
- Supported by the tautology $(p \land (p \Rightarrow q)) \Rightarrow q$

modus ponens (continued)

p	q	(p ⇒ q)	p ∧ (p ⇒ q)	(p ∧ (p ⇒ q)) ⇒ q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Invalid Conclusions from Invalid Premises

- Just because the format of the argument is valid does not mean that the conclusion is true. A premise may be false. For example:

  Acorns are money
  <u>If acorns were money, no one would have to work</u>
  ∴ No one has to work

- Argument is valid since it is in modus ponens form
- Conclusion is false because premise p is false

Invalid Conclusion from Invalid Argument

- Sometimes, an argument that looks like modus ponens is actually not in the correct form. For example:
- If tuition was free, enrollment would increase
  Enrollment increased
  ∴ Tuition is free
- Argument is invalid since its form is:

$$\begin{array}{l} p \Rightarrow q \\ \underline{q} \\ \therefore p \end{array}$$

Invalid Argument (continued)

- Truth table shows that this is not a tautology:

p	q	$(p \Rightarrow q)$	$(p \Rightarrow q) \wedge q$	$((p \Rightarrow q) \wedge q) \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Indirect Method

- Another method of proof is to use the tautology:

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

- The form of the proof is:

$$\begin{array}{l} \sim q \\ \underline{\sim q \Rightarrow \sim p} \\ \therefore p \end{array}$$

Indirect Method Example

- p: My e-mail address is available on a web site
- q: I am getting spam
- $p \Rightarrow q$: If my e-mail address is available on a web site, then I am getting spam
- $\sim q \Rightarrow \sim p$: If I am not getting spam, then my e-mail address must not be available on a web site
- This proof says that if I am not getting spam, then my e-mail address is not on a web site.

Another Indirect Method Example

- Prove that if the square of an integer is odd, then the integer is odd too.
- p: n^2 is odd
- q: n is odd
- $\sim q \Rightarrow \sim p$: If n is even, then n^2 is even.

- If n is even, then there exists an integer m for which $n = 2 \times m$. n^2 therefore would equal $(2 \times m)^2 = 4 \times m^2$ which must be even.

Proof by Contradiction

- Another method of proof is to use the tautology $(p \Rightarrow q) \land (\sim q) \Rightarrow (\sim p)$
- The form of the proof is:

$$\begin{array}{l} p \Rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

Proof by Contradiction (continued)

p	q	$(p \Rightarrow q)$	$\sim q$	$(p \Rightarrow q) \wedge \sim q$	$\sim p$	$(p \Rightarrow q) \wedge (\sim q) \Rightarrow (\sim p)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T