CSCI 1900 Discrete Structures

Logical Operations Section 2.1

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Statement of Proposition

- Statement of proposition a declarative sentence that is either true or false, but not both
- Examples:
 - The earth is round: statement that is true
 2+3=5: statement that is true
 - Do you speak English? This is a question, not a statement

More Examples of Statements of Proposition

- 3-x=5: is a declarative sentence, but not a statement since it is true or false depending on the value of x
- Take two aspirins: is a command, not a statement
- The temperature on the surface of the planet Venus is 800°F: is a declarative statement of whose truth is unknown to us
- The sun will come out tomorrow: a statement that is either true or false, but not both, although we will have to wait until tomorrow to determine the answer

Logical Connectives and Compound Statements

- *x, y, z,* ... denote variables that can represent real numbers
- *p*, *q*, *r*,... denote prepositional variables that can be replaced by statements.
 - p: The sun is shining today
 - -q: It is cold

Negation

- If *p* is a statement, the negation of *p* is the statement *not p*
- Denoted ~p
- If p is true, ~p is false
- If p is false, ~p is true
- ~p is not actually connective, i.e., it doesn't join two of anything
- not is a unary operation for the collection of statements and ~p is a statement if p is

Examples of Negation

- If *p*: 2+3 >1 then If ~*p*: 2+3 ≤1
- If q: It is cold then ~q: It is not the case that it is cold, i.e., It is not cold.

Conjunction

- If p and q are statements, then the conjunction of p and q is the compound statement "p and q"
- Denoted $p \land q$
- $p \land q$ is true only if both p and q are true
- Example:
 - p: ETSU parking permits are expensive
 - q: ETSU has plenty of parking
 - $-p \wedge q = ?$

Disjunction

- If p and q are statements, then the *disjunction* of p and q is the compound statement "p or q"
- Denoted p V q
- pVq is true if either p or q are true
- Example:
 - p: I am a male
 - q: I am under 40 years old
 - -pVq = ?

Exclusive Disjunction

- If p and q are statements, then the exclusive disjunction is the compound statement, "either p or q may be true, but both are not true at the same time."
- Example:
 - p: It is daytime
 - q: It is night time
 - -pVq (in the exclusive sense) = ?

Inclusive Disjunction

- If p and q are statements, then the inclusive disjunction is the compound statement, "either p or q may be true or they may both be true at the same time."
- Example:
 - p: It is cold
 - q: It is night time
 - -pVq (in the inclusive sense) = ?

Exclusive versus Inclusive

- Depending on the circumstances, some disjunctions are inclusive and some of exclusive.
- Examples of Inclusive
 - "I have a dog" or "I have a cat"
 - "It is warm outside" or "It is raining"
- Examples of Exclusive
 - Today is either Tuesday or it is Thursday
 - Pat is either male or female

Compound Statements

- A *compound statement* is a statement made from other statements
- For n individual propositions, there are 2ⁿ possible combinations of truth values
- A truth table contains 2ⁿ rows identifying the truth values for the statement represented by the table.
- Use parenthesis to denote order of precedence
- \wedge has precedence over V

Truth Tables are Important Tools for this Material!



Compound Statement Example ($p \land q$) V (~p)



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Quantifiers

- Back in Section 1.1, a set was defined {x | P(x)}
- For an element t to be a member of the set, P(t) must evaluate to "true"
- P(x) is called a predicate or a propositional function

Computer Science Functions

if P(x), then execute certain steps
while Q(x), do specified actions

Universal quantification of a predicate P(x)

- Universal quantification of predicate P(x) = For all values of x, P(x) is true
- Denoted $\forall x P(x)$
- The symbol ∀ is called the universal quantifier
- The order in which multiple quantifications are considered does not affect the truth value (e.g., ∀x ∀y P(x,y) ≡ ∀y ∀x P(x,y)

Examples:

- P(x): -(-x) = x
 - This predicate makes sense for all real numbers x.
 - The universal quantification of P(x), ∀x P(x), is a true statement, because for all real numbers, -(-x) = x
- Q(x): x+1<4
 - ∀x Q(x) is a false statement, because, for example, Q(5) is not true

Existential quantification of a predicate P(x)

- Existential quantification of a predicate P(x) is the statement "There exists a value of x for which P(x) is true."
- Denoted $\exists x P(x)$
- Existential quantification may be applied to several variables in a predicate
- The order in which multiple quantifications are considered does not affect the truth value

Applying both universal and existential quantification

- Order of application does matter
- Example: Let **A** and **B** be n x n matrices
- The statement $\forall A \exists B A + B = I_n$
- Reads "for every A there is a B such that A + B =
 In"
- Prove by coming up for equations for b_{ii} and b_{ii} (j \neq i)
- Now reverse the order: $\exists B \forall A A + B = I_n$
- Reads "there exists a *B* such that for all *A A* + *B* = *I*".
 THIS IS EALSEL
- THIS IS FALSE!

Assigning Quantification to Proposition

- Let p: $\forall x P(x)$
- The negation of p is false when p is true and true when p is false
- For p to be false, there must be at least one value of x for which P(x) is false.
- Thus, p is false if $\exists x \sim P(x)$ is true.
- If ∃x ~P(x) is false, then for every x, ~P(x) is false; that is ∀x P(x) is true.

Okay, what exactly did the previous slide say?

- Assume a statement is made that "for all x, P(x) is true."
 - If we can find one case that is not true, then the statement is false.
 - If we cannot find one case that is not true, then the statement is true.
- Example: ∀ positive integers, n,
 P(n) = n² + 41n + 41 is a prime number.
 - This is false because \exists an integer resulting in a non-prime value, i.e., \exists n such that P(n) is false.

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Conditional Statements Section 2.2

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Conditional Statement/Implication

- "if *p* then *q*"
- Denoted $p \Rightarrow q$
 - p is called the *antecedent* or *hypothesis*
 - q is called the consequent or conclusion
- Example:
 - p: I am hungry
 - q: I will eat
 - *p*: It is snowing
 q: 3+5 = 8

Conditional Statement/Implication (continued)

- In English, we would assume a cause-and-effect relationship, i.e., the fact that p is true would force q to be true.
- If "it is snowing," then "3+5=8" is meaningless in this regard since p has no effect at all on q
- At this point it may be easiest to view the operator "⇒" as a logic operationsimilar to AND or OR (conjunction or disjunction).

Truth Table Representing Implication

- If viewed as a logic operation, p ⇒ q can only be evaluated as false if *p* is true and q is false
- This does not say that p causes q
- Truth table



Examples where $p \Rightarrow q$ is viewed as a logic operation

- If *p* is false, then any *q* supports *p* ⇒ *q* is true.
 - False ⇒ True = True
 - False \Rightarrow False = True
- If "2+2=5" then "I am the king of England" is true

Converse and contrapositive

- The converse of $p \Rightarrow q$ is the implication that $q \Rightarrow p$
- The contrapositive of p ⇒ q is the implication that ~q ⇒ ~p

Converse and Contrapositive Example

Example: What is the converse and contrapositive of *p*: "it is raining" and *q*: I get wet?

- Implication: If it is raining, then I get wet.
- Converse: If I get wet, then it is raining.
- Contrapositive: If I do not get wet, then it is not raining.

Equivalence or biconditional

- If p and q are statements, the compound statement p if and only if q is called an equivalence or biconditional
- Denoted $p \Leftrightarrow q$

Equivalence Truth table

 The only time that the expression can evaluate as true is if both statements, p and q, are true or both are false



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Proof of the Contrapositive

Compute the truth table of the statement $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

р	q	$p \Rightarrow q$	$\sim q$	~p	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
Т	Т	Т	F	F	Т	Т
Т	F	F	Т	F	F	Т
F	Т	Т	F	Т	Т	Т
F	F	Т	Т	Т	Т	Т

Tautology and Contradiction

- A statement that is true for all of its propositional variables is called a *tautology*. (The previous truth table was a tautology.)
- A statement that is false for all of its propositional variables is called a *contradiction* or an *absurdity*

Contingency

- A statement that can be either true or false depending on its propositional variables is called a *contingency*
- Examples
 - $-(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology
 - $-p \wedge \sim p$ is an absurdity
 - $(p \Rightarrow q) \land \sim p$ is a contingency since some cases evaluate to true and some to false.

Contingency Example The statement ($p \Rightarrow q$) \land ($p \lor q$) is a contingency

р	q	$p \Rightarrow q$	$p \lor q$	$(\mathbf{p} \Rightarrow \mathbf{q}) \land (\mathbf{p} \lor \mathbf{q})$
Т	Т	Т	Т	Т
Т	F	F	Т	F
F	Т	Т	Т	Т
F	F	Т	F	F

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Logically equivalent

- Two propositions are *logically equivalent* or *simply equivalent* if p ⇔ q is a tautology.
- Denoted $p \equiv q$

Example of Logical Equivalence

Columns 5 and 8 are equivalent, and therefore, p "if and only if" q

р	q	r	$q \wedge$	$p \lor$	$p \lor$	$p \vee$	$(p \lor q) \land$	$p \lor (q \land r) \Leftrightarrow$
			r	$(q \wedge r)$	q	r	$(p \vee r)$	(p \lor q) \land (p \lor r)
Т	Т	Τ	Т	Т	Т	Т	Т	Т
Т	Т	F	F	Т	Т	Т	Т	Т
Т	F	Τ	F	Т	Т	Т	Т	Т
Т	F	F	F	Т	Т	Т	Т	Т
F	Т	Т	Т	Т	Т	Т	Т	Т
F	Т	F	F	F	Т	F	F	Т
F	F	Т	F	F	F	Т	F	Т
F	F	F	F	F	F	F	F	Т

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Additional Properties $(p \Rightarrow q) \equiv ((\sim p) \lor q)$

р	q	$(p \Rightarrow q)$	~p	((~p) ∨ q)	$(p \Rightarrow q) \Leftrightarrow ((\sim p) \lor q)$
Т	Т	Т	F	Т	Т
Т	F	F	F	F	Т
F	Т	Т	Т	Т	Т
F	F	Т	Т	Т	Т

Additional Properties $(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$

р	q	$(p \Rightarrow q)$	~q	~p	$(\sim q \Rightarrow \sim p)$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
Т	Τ	Т	F	F	Т	Т
Т	F	F	Т	F	F	Т
F	Т	Т	F	Т	Т	Т
F	F	Τ	Т	Т	Τ	Τ

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Methods of Proof Reading: Kolman, Section 2.3

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Past Experience

Up to now we've used the following methods to write proofs:

- Used direct proofs with generic elements, definitions, and given facts
- Used proof by cases such as when we used truth tables

General Description of Process

- $p \Rightarrow q$ denotes "q logically follows from p"
- Implication may take the form $(p_1 \land p_2 \land p_3 \land \dots \land p_n) \Rightarrow q$
- q logically follows from $p_1, p_2, p_3, ..., p_n$

General Description (continued)

The process is generally written as:



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Components of a Proof

- The p's are called hypotheses or premises
- q is called the *conclusion*
- Proof shows that if all of the p_i's are true, then q has to be true
- If result is a tautology, then the implication
 p ⇒ q represents a universally correct
 method of reasoning and is called a *rule of inference*

Example of a Proof based on a Tautology

- If p implies q and q implies r, then p implies r
 - $p \Rightarrow q$
 - $\underline{q} \Rightarrow \underline{r}$
 - ∴p ⇒ r
- By replacing the bar under q ⇒ r with the "⇒", the proof above becomes ((p ⇒ q) ∧ (q ⇒ r)) ⇒ (p ⇒ r)
- The next slide shows that this is a tautology and therefore is universally valid.

	Tautology Example (continued)									
р	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \land (q \Rightarrow r)$	$p \Rightarrow r$	$\begin{array}{c} ((p \Rightarrow q) \land (q \Rightarrow r)) \\ \Rightarrow (p \Rightarrow r) \end{array}$			
Т	Т	Τ	Т	Т	Т	Т	Т			
Τ	Т	F	Т	F	F	F	Т			
Τ	F	Τ	F	Т	F	Т	Т			
Т	F	F	F	Т	F	F	Т			
F	Т	Т	Т	Т	Т	Т	Т			
F	Т	F	Т	F	F	Т	Т			
F	F	Т	Т	Т	Т	Т	Т			
F	F	F	Т	Т	T	Т	Т			

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Equivalences

- Some mathematical theorems are equivalences, i.e., p ⇔ q.
- The proof of such a theorem is equivalent with proving both $p \Rightarrow q$ and $q \Rightarrow p$

modus ponens form (the method of asserting): $p = p \Rightarrow q$ $\cdot \cdot q$

• Example:

- p: a man used the toilet
- q: the toilet seat is up
- $-p \Rightarrow q$: If a man used the toilet, the seat was left up
- Supported by the tautology $(p \land (p \Rightarrow q)) \Rightarrow q$

	modus ponens (continued)								
р	q	$(p \Rightarrow q)$	$p \land (p \Rightarrow q)$	$\left (\mathbf{p} \land (\mathbf{p} \Rightarrow \mathbf{q})) \Rightarrow \mathbf{q} \right $					
T	Т	Т	Т	T					
Т	F	F	F	T					
F	Т	Т	F	T					
F	F	T	F	T					

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Invalid Conclusions from Invalid Premises

 Just because the format of the argument is valid does not mean that the conclusion is true.
 A premise may be false. For example:

Acorns are money <u>If acorns were money, no one would have to work</u> ...No one has to work

- Argument is valid since it is in modus ponens form
- Conclusion is false because premise p is false

Invalid Conclusion from Invalid Argument

- Sometimes, an argument that looks like modus ponens is actually not in the correct form. For example:
- If tuition was free, enrollment would increase
 <u>Enrollment increased</u>
 - Tuition is free
- Argument is invalid since its form is:

p ⇒ q q ..p

Invalid Argument (continued)

Truth table shows that this is not a tautology:

p	q	$(p \Rightarrow q)$	$(p \Rightarrow q) \land$	$((\mathbf{p} \Rightarrow \mathbf{q}) \land \mathbf{q}) \Rightarrow$
			q	p
Т	Т	Т	Т	Т
Т	F	F	F	Т
F	Т	Т	Т	F
F	F	Т	F	T

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Indirect Method

Another method of proof is to use the tautology:

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

• The form of the proof is:

$$\sim q$$

$$\sim q \Rightarrow \sim p$$

∴ p

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Indirect Method Example

- p: My e-mail address is available on a web site
- q: I am getting spam
- p ⇒ q: If my e-mail address is available on a web site, then I am getting spam
- ~q ⇒ ~p: If I am not getting spam, then my e-mail address must not be available on a web site
- This proof says that if I am not getting spam, then my e-mail address is not on a web site.

Another Indirect Method Example

- Prove that if the square of an integer is odd, then the integer is odd too.
- p: n² is odd
- q: n is odd
- $\sim q \Rightarrow \sim p$: If n is even, then n^2 is even.
- If n is even, then there exists an integer m for which n = 2×m. n^2 therefore would equal $(2 \times m)^2 = 4 \times m^2$ which must be even.

Proof by Contradiction

- Another method of proof is to use the tautology (p ⇒ q) ∧ (~q) ⇒ (~p)
- The form of the proof is:

p ⇒ q <u>~q</u> ∴~p

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Proof by Contradiction (continued)

p	q	$(p \Rightarrow$	~q	$(p \Rightarrow q) \land$	~p	$(p \Rightarrow q) \land (\sim q) \Rightarrow$
		q)		~q		(~p)
Τ	Т	Т	F	F	F	Т
Τ	F	F	Т	F	F	T
F	Т	Т	F	F	Т	Т
F	F	Τ	Т	Τ	Т	Τ

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