

CSCI 1900

Discrete Structures

Logical Operations

Section 2.1

Statement of Proposition

- Statement of proposition – a declarative sentence that is either true or false, but not both
- Examples:
 - The earth is round: statement that is true
 - $2+3=5$: statement that is true
 - Do you speak English? This is a question, not a statement

More Examples of Statements of Proposition

- $3-x=5$: is a declarative sentence, but not a statement since it is true or false depending on the value of x
- Take two aspirins: is a command, not a statement
- The temperature on the surface of the planet Venus is 800°F : is a declarative statement of whose truth is unknown to us
- The sun will come out tomorrow: a statement that is either true or false, but not both, although we will have to wait until tomorrow to determine the answer

Logical Connectives and Compound Statements

- x, y, z, \dots denote variables that can represent real numbers
- p, q, r, \dots denote propositional variables that can be replaced by statements.
 - p : The sun is shining today
 - q : It is cold

Negation

- If p is a statement, the negation of p is the statement *not* p
- Denoted $\sim p$
- If p is true, $\sim p$ is false
- If p is false, $\sim p$ is true
- $\sim p$ is not actually connective, i.e., it doesn't join two of anything
- ***not*** is a unary operation for the collection of statements and $\sim p$ is a statement if p is

Examples of Negation

- If p : $2+3 > 1$ then If $\sim p$: $2+3 \leq 1$
- If q : It is cold then $\sim q$: It is not the case that it is cold, i.e., It is not cold.

Conjunction

- If p and q are statements, then the ***conjunction*** of p and q is the compound statement “ p and q ”
- Denoted $p \wedge q$
- $p \wedge q$ is true only if both p and q are true
- Example:
 - p : ETSU parking permits are expensive
 - q : ETSU has plenty of parking
 - $p \wedge q = ?$

Disjunction

- If p and q are statements, then the ***disjunction*** of p and q is the compound statement “ p or q ”
- Denoted $p \vee q$
- $p \vee q$ is true if either p or q are true
- Example:
 - p : I am a male
 - q : I am under 40 years old
 - $p \vee q = ?$

Exclusive Disjunction

- If p and q are statements, then the ***exclusive disjunction*** is the compound statement, “either p or q may be true, but both are not true at the same time.”
- Example:
 - p : It is daytime
 - q : It is night time
 - $p \vee q$ (in the exclusive sense) = ?

Inclusive Disjunction

- If p and q are statements, then the ***inclusive disjunction*** is the compound statement, “either p or q may be true or they may both be true at the same time.”
- Example:
 - p : It is cold
 - q : It is night time
 - $p \vee q$ (in the inclusive sense) = ?

Exclusive versus Inclusive

- Depending on the circumstances, some disjunctions are inclusive and some of exclusive.
- Examples of Inclusive
 - “I have a dog” or “I have a cat”
 - “It is warm outside” or “It is raining”
- Examples of Exclusive
 - Today is either Tuesday or it is Thursday
 - Pat is either male or female

Compound Statements

- A ***compound statement*** is a statement made from other statements
- For n individual propositions, there are 2^n possible combinations of truth values
- A truth table contains 2^n rows identifying the truth values for the statement represented by the table.
- Use parenthesis to denote order of precedence
- \wedge has precedence over \vee

Truth Tables are Important Tools for this Material!

p	q	$p \wedge q$	p	q	$p \vee q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

Compound Statement Example

$$(p \wedge q) \vee (\sim p)$$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Quantifiers

- Back in Section 1.1, a set was defined $\{x \mid P(x)\}$
- For an element t to be a member of the set, $P(t)$ must evaluate to “true”
- $P(x)$ is called a predicate or a propositional function

Computer Science Functions

- if $P(x)$, then execute certain steps
- while $Q(x)$, do specified actions

Universal quantification of a predicate $P(x)$

- Universal quantification of predicate $P(x) =$
For all values of x , $P(x)$ is true
- Denoted $\forall x P(x)$
- The symbol \forall is called the universal quantifier
- The order in which multiple quantifications are considered does not affect the truth value (e.g., $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$)

Examples:

- $P(x): -(-x) = x$
 - This predicate makes sense for all real numbers x .
 - The universal quantification of $P(x)$, $\forall x P(x)$, is a true statement, because for all real numbers, $-(-x) = x$
- $Q(x): x+1 < 4$
 - $\forall x Q(x)$ is a false statement, because, for example, $Q(5)$ is not true

Existential quantification of a predicate $P(x)$

- Existential quantification of a predicate $P(x)$ is the statement “There exists a value of x for which $P(x)$ is true.”
- Denoted $\exists x P(x)$
- Existential quantification may be applied to several variables in a predicate
- The order in which multiple quantifications are considered does not affect the truth value

Applying both universal and existential quantification

- Order of application does matter
- Example: Let A and B be $n \times n$ matrices
- The statement $\forall A \exists B A + B = I_n$
- Reads “for every A there is a B such that $A + B = I_n$ ”
- Prove by coming up for equations for b_{ii} and b_{ij} ($j \neq i$)
- Now reverse the order: $\exists B \forall A A + B = I_n$
- Reads “there exists a B such that for all A $A + B = I_n$ ”
- THIS IS FALSE!

Assigning Quantification to Proposition

- Let $p: \forall x P(x)$
- The negation of p is false when p is true and true when p is false
- For p to be false, there must be at least one value of x for which $P(x)$ is false.
- Thus, p is false if $\exists x \sim P(x)$ is true.
- If $\exists x \sim P(x)$ is false, then for every x , $\sim P(x)$ is false; that is $\forall x P(x)$ is true.

Okay, what exactly did the previous slide say?

- Assume a statement is made that “for all x , $P(x)$ is true.”
 - If we can find one case that is not true, then the statement is false.
 - If we cannot find one case that is not true, then the statement is true.
- Example: \forall positive integers, n ,
 $P(n) = n^2 + 41n + 41$ is a prime number.
 - This is false because \exists an integer resulting in a non-prime value, i.e., $\exists n$ such that $P(n)$ is false.

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Conditional Statements

Section 2.2

Conditional Statement/Implication

- "if p then q "
- Denoted $p \Rightarrow q$
 - p is called the ***antecedent*** or ***hypothesis***
 - q is called the ***consequent*** or ***conclusion***
- Example:
 - p : I am hungry
 q : I will eat
 - p : It is snowing
 q : $3+5 = 8$

Conditional Statement/Implication (continued)

- In English, we would assume a cause-and-effect relationship, i.e., the fact that p is true would force q to be true.
- If “it is snowing,” then “ $3+5=8$ ” is meaningless in this regard since p has no effect at all on q
- At this point it may be easiest to view the operator “ \Rightarrow ” as a logic operation similar to AND or OR (conjunction or disjunction).

Truth Table Representing Implication

- If viewed as a logic operation, $p \Rightarrow q$ can only be evaluated as false if ***p is true and q is false***
- This does not say that p causes q
- Truth table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples where $p \Rightarrow q$ is viewed as a logic operation

- If p is false, then any q supports $p \Rightarrow q$ is true.
 - $\text{False} \Rightarrow \text{True} = \text{True}$
 - $\text{False} \Rightarrow \text{False} = \text{True}$
- If “ $2+2=5$ ” then “I am the king of England” is true

Converse and contrapositive

- The converse of $p \Rightarrow q$ is the implication that $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is the implication that $\sim q \Rightarrow \sim p$

Converse and Contrapositive Example

Example: What is the converse and contrapositive of p : "it is raining" and q : I get wet?

- Implication: If it is raining, then I get wet.
- Converse: If I get wet, then it is raining.
- Contrapositive: If I do not get wet, then it is not raining.

Equivalence or biconditional

- If p and q are statements, the compound statement p ***if and only if*** q is called an ***equivalence*** or ***biconditional***
- Denoted $p \Leftrightarrow q$

Equivalence Truth table

- The only time that the expression can evaluate as true is if both statements, p and q , are true or both are false

p	Q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Proof of the Contrapositive

Compute the truth table of the statement
 $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Tautology and Contradiction

- A statement that is true for all of its propositional variables is called a ***tautology***. (The previous truth table was a tautology.)
- A statement that is false for all of its propositional variables is called a ***contradiction*** or an ***absurdity***

Contingency

- A statement that can be either true or false depending on its propositional variables is called a ***contingency***
- Examples
 - $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology
 - $p \wedge \sim p$ is an absurdity
 - $(p \Rightarrow q) \wedge \sim p$ is a contingency since some cases evaluate to true and some to false.

Contingency Example

The statement $(p \Rightarrow q) \wedge (p \vee q)$ is a contingency

p	q	$p \Rightarrow q$	$p \vee q$	$(p \Rightarrow q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Logically equivalent

- Two propositions are *logically equivalent* or *simply equivalent* if $p \Leftrightarrow q$ is a tautology.
- Denoted $p \equiv q$

Example of Logical Equivalence

Columns 5 and 8 are equivalent, and therefore, p “if and only if” q

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Additional Properties

$$(p \Rightarrow q) \equiv ((\sim p) \vee q)$$

p	q	$(p \Rightarrow q)$	$\sim p$	$((\sim p) \vee q)$	$(p \Rightarrow q) \Leftrightarrow ((\sim p) \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Additional Properties

$$(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$$

p	q	$(p \Rightarrow q)$	$\sim q$	$\sim p$	$(\sim q \Rightarrow \sim p)$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

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Methods of Proof

Reading: Kolman, Section 2.3

Past Experience

Up to now we've used the following methods to write proofs:

- Used direct proofs with generic elements, definitions, and given facts
- Used proof by cases such as when we used truth tables

General Description of Process

- $p \Rightarrow q$ denotes "q logically follows from p"
- Implication may take the form $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \Rightarrow q$
- q logically follows from $p_1, p_2, p_3, \dots, p_n$

General Description (continued)

The process is generally written as:

$$p_1$$
$$p_2$$
$$p_3$$
$$\vdots$$
$$\vdots$$
$$p_n$$
$$\dots q$$

Components of a Proof

- The p_i 's are called ***hypotheses*** or ***premises***
- q is called the ***conclusion***
- Proof shows that if all of the p_i 's are true, then q has to be true
- If result is a tautology, then the implication $p \Rightarrow q$ represents a universally correct method of reasoning and is called a ***rule of inference***

Example of a Proof based on a Tautology

- If p implies q and q implies r , then p implies r

$$p \Rightarrow q$$

$$\underline{q \Rightarrow r}$$

$$\therefore p \Rightarrow r$$

- By replacing the bar under $q \Rightarrow r$ with the “ \Rightarrow ”, the proof above becomes $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
- The next slide shows that this is a tautology and therefore is universally valid.

Tautology Example (continued)

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Equivalences

- Some mathematical theorems are equivalences, i.e., $p \Leftrightarrow q$.
- The proof of such a theorem is equivalent with proving both $p \Rightarrow q$ and $q \Rightarrow p$

modus ponens

form (the method of asserting):

$$\begin{array}{l} p \\ p \Rightarrow q \\ \hline \therefore q \end{array}$$

- Example:
 - p : a man used the toilet
 - q : the toilet seat is up
 - $p \Rightarrow q$: If a man used the toilet, the seat was left up
- Supported by the tautology $(p \wedge (p \Rightarrow q)) \Rightarrow q$

modus ponens (continued)

p	q	$(p \Rightarrow q)$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Invalid Conclusions from Invalid Premises

- Just because the format of the argument is valid does not mean that the conclusion is true. A premise may be false. For example:

Acorns are money

If acorns were money, no one would have to work

∴ No one has to work

- Argument is valid since it is in modus ponens form
- Conclusion is false because premise p is false

Invalid Conclusion from Invalid Argument

- Sometimes, an argument that looks like modus ponens is actually not in the correct form. For example:
- If tuition was free, enrollment would increase
Enrollment increased
∴ Tuition is free
- Argument is invalid since its form is:

$$p \Rightarrow q$$

$$\underline{q}$$

$$\therefore p$$

Invalid Argument (continued)

- Truth table shows that this is not a tautology:

p	q	$(p \Rightarrow q)$	$(p \Rightarrow q) \wedge q$	$((p \Rightarrow q) \wedge q) \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Indirect Method

- Another method of proof is to use the tautology:

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

- The form of the proof is:

$$\begin{array}{l} \sim q \\ \hline \sim q \Rightarrow \sim p \\ \hline \therefore p \end{array}$$

Indirect Method Example

- p : My e-mail address is available on a web site
- q : I am getting spam
- $p \Rightarrow q$: If my e-mail address is available on a web site, then I am getting spam
- $\sim q \Rightarrow \sim p$: If I am not getting spam, then my e-mail address must not be available on a web site
- This proof says that if I am not getting spam, then my e-mail address is not on a web site.

Another Indirect Method Example

- Prove that if the square of an integer is odd, then the integer is odd too.
- p : n^2 is odd
- q : n is odd
- $\sim q \Rightarrow \sim p$: If n is even, then n^2 is even.

- If n is even, then there exists an integer m for which $n = 2 \times m$. n^2 therefore would equal $(2 \times m)^2 = 4 \times m^2$ which must be even.

Proof by Contradiction

- Another method of proof is to use the tautology $(p \Rightarrow q) \wedge (\sim q) \Rightarrow (\sim p)$
- The form of the proof is:

$$p \Rightarrow q$$

$$\underline{\sim q}$$

$$\therefore \sim p$$

Proof by Contradiction (continued)

p	q	$(p \Rightarrow q)$	$\sim q$	$(p \Rightarrow q) \wedge \sim q$	$\sim p$	$(p \Rightarrow q) \wedge (\sim q) \Rightarrow (\sim p)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T