

Далее, из $\text{rot } \vec{u} = \vec{0} \Leftrightarrow \vec{u} = \text{grad} f$. $\text{grad}(f + C) = \text{grad} f$.

$$\vec{u} = \{P; Q, R\}, \text{rot } \vec{u} = \vec{0}, \text{ т.е. } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \in G \subset E_3$$

$$(x_0, y_0, z_0) \in G. \rightarrow f(x, y, z) = \int_{x_0}^x P(\xi, y, z) d\xi + \varphi(y, z), \text{ где, из } \frac{\partial f}{\partial x} = P(x, y, z)$$

$$Q(x, y, z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_{x_0}^x P(\xi, y, z) d\xi + \frac{\partial \varphi}{\partial y} = \left(\text{об-ва упр-лов,} \right) = \int_{x_0}^x \frac{\partial P(\xi, y, z)}{\partial y} d\xi + \frac{\partial \varphi}{\partial y} =$$

$$= \int_{x_0}^x \frac{\partial Q(\xi, y, z)}{\partial \xi} d\xi + \frac{\partial \varphi}{\partial y} = Q(x, y, z) - Q(x_0, y, z) + \frac{\partial \varphi}{\partial y}, \text{ т.е. } \frac{\partial \varphi}{\partial y} = Q(x_0, y, z) \Rightarrow$$

$$\Rightarrow f(x, y, z) = \int_{x_0}^x P(\xi, y, z) d\xi + \int_{y_0}^y Q(x_0, \eta, z) d\eta + \psi(z). \quad R(x, y, z) = \frac{\partial f}{\partial z} =$$

$$= \int_{x_0}^x \frac{\partial P(\xi, y, z)}{\partial z} d\xi + \int_{y_0}^y \frac{\partial Q(x_0, \eta, z)}{\partial z} d\eta + \psi'(z) = \int_{x_0}^x \frac{\partial R(\xi, y, z)}{\partial \xi} d\xi + \int_{y_0}^y \frac{\partial R(x_0, \eta, z)}{\partial \eta} d\eta + \psi'(z) =$$

$$= R(x, y, z) - R(x_0, y, z) + R(x_0, y, z) - R(x_0, y_0, z) + \psi'(z), \text{ т.е. } \psi'(z) = R(x_0, y_0, z) \Rightarrow$$

$$f(x, y, z) = \int_{x_0}^x P(\xi, y, z) d\xi + \int_{y_0}^y Q(x_0, \eta, z) d\eta + \int_{z_0}^z R(x_0, y_0, \xi) d\xi + C$$

$$f(x, y, z) = \int_{x_0}^x P(\xi, y, z) d\xi + \int_{y_0}^y Q(x_0, \eta, z) d\eta + \int_{z_0}^z R(x_0, y_0, \zeta) d\zeta + C$$

Пример. $\vec{u} = -\frac{\vec{r}}{r^3} = \left\{ -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\}$ $\left(\begin{array}{l} \vec{r} = \{x, y, z\} \\ r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \end{array} \right)$

$\text{rot } \vec{u} = \vec{0}$ (conserv.), $\vec{u} = \text{grad } f$, $f(x, y, z) = -\int_{x_0}^x \frac{\xi d\xi}{(\xi^2+y^2+z^2)^{3/2}} - \int_{y_0}^y \frac{\eta d\eta}{(x_0^2+\eta^2+z^2)^{3/2}} -$

$$- \int_{z_0}^z \frac{\zeta d\zeta}{(x_0^2+y_0^2+\zeta^2)^{3/2}} + C = \frac{1}{\sqrt{\xi^2+y^2+z^2}} \Big|_{\xi=x_0}^{\xi=x} + \frac{1}{\sqrt{x_0^2+\eta^2+z^2}} \Big|_{\eta=y_0}^{\eta=y} + \frac{1}{\sqrt{x_0^2+y_0^2+\zeta^2}} \Big|_{\zeta=z_0}^{\zeta=z} + C =$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x_0^2+y^2+z^2}} + \frac{1}{\sqrt{x_0^2+y^2+z^2}} - \frac{1}{\sqrt{x_0^2+y_0^2+z^2}} + \frac{1}{\sqrt{x_0^2+y_0^2+z^2}} - \frac{1}{\sqrt{x_0^2+y_0^2+z_0^2}} + C$$

$$= \frac{1}{r} - \frac{1}{r_0} + C = \frac{1}{r} + \text{const}$$

О векторном потенциале

Потенциальное поле: $\oint_{\Gamma} (\vec{u}, d\vec{r}) = 0$ $\operatorname{rot} \vec{u} = \vec{0}$ $\vec{u} = \operatorname{grad} f$

Соленоидальное поле: $\oint_S (\vec{u}, d\vec{S}) = 0$ $\operatorname{div} \vec{u} = 0$?

О векторном потенциале

Потенциальное поле: $\oint_{\Gamma} (\vec{u}, d\vec{r}) = 0$ $\text{rot } \vec{u} = \vec{0}$ $\vec{u} = \text{grad } f$

Соленоидальное поле: $\oint_S (\vec{u}, d\vec{S}) = 0$ $\text{div } \vec{u} = 0$ $\vec{u} = \text{rot } \vec{v}$

\vec{v} - векторный потенциал

$$\vec{u} = \{P, Q, R\}, \quad \vec{v} = \{P_1, Q_1, R_1\}, \quad \text{rot } \vec{v} = \left\{ \underbrace{\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}}_P, \underbrace{\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}}_Q, \underbrace{\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}}_R \right\} = \vec{u}$$

$$0 = \text{div } \vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial^2 R_1}{\partial x \partial y} - \frac{\partial^2 Q_1}{\partial x \partial z} + \frac{\partial^2 P_1}{\partial y \partial z} + \frac{\partial^2 R_1}{\partial x \partial y} + \frac{\partial^2 Q_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y \partial z}$$

$$P = \frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}, \quad Q = \frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}, \quad R = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \quad \text{Положим } R_1(x, y, z) \equiv 0$$

$$\Rightarrow P = -\frac{\partial Q_1}{\partial z}, \quad Q = \frac{\partial P_1}{\partial z}, \quad R = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}. \quad (x_0, y_0, z_0) \in G \Rightarrow P_1(x, y, z) \stackrel{z}{=} \int_{z_0}^z Q(x, y, \xi) d\xi$$

$$Q_1(x, y, z) = -\int_{z_0}^z P(x, y, \xi) d\xi + \varphi(x, y) \Rightarrow R(x, y, z) = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} = -\int_{z_0}^z \frac{\partial P(x, y, \xi)}{\partial x} d\xi +$$

$$+ \frac{\partial \varphi}{\partial x} - \int_{z_0}^z \frac{\partial Q(x, y, \xi)}{\partial y} d\xi = \frac{\partial \varphi}{\partial x} - \int_{z_0}^z \left(\frac{\partial P(x, y, \xi)}{\partial x} + \frac{\partial Q(x, y, \xi)}{\partial y} \right) d\xi = \frac{\partial \varphi}{\partial x} + \int_{z_0}^z \frac{\partial R(x, y, \xi)}{\partial \xi} d\xi =$$

$$= \frac{\partial \varphi}{\partial x} + R(x, y, z) - R(x, y, z_0), \quad \text{т.е. } \frac{\partial \varphi}{\partial x} = R(x, y, z_0) \Rightarrow \varphi(x, y) = \int_{x_0}^x R(\xi, y, z_0) d\xi + \psi(y)$$

$\text{дерив} \equiv 0$

$$\text{Итак, } P_1(x, y, z) = \int_{z_0}^z Q(x, y, \xi) d\xi; \quad Q_1(x, y, z) = -\int_{z_0}^z P(x, y, \xi) d\xi + \int_{x_0}^x R(\xi, y, z_0) d\xi; \quad R_1(x, y, z) \equiv 0$$

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Пример $\vec{u} = \{y, z, x\}$ $\text{div } \vec{u} = 0$ Найдем \vec{v} : $\vec{u} = \text{rot } \vec{v}$ $x_0 = y_0 = z_0 = 0$

$P_1(x, y, z) = \int_0^z \xi d\xi = \frac{z^2}{2}$; $Q_1(x, y, z) = -\int_0^z y d\xi + \int_0^x \xi d\xi = \frac{x^2}{2} - yz$; $R_1(x, y, z) \equiv 0$

$\vec{v} = \left\{ \frac{z^2}{2}, \frac{x^2}{2} - yz, 0 \right\}$ $\text{rot } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z^2}{2} & \frac{x^2}{2} - yz & 0 \end{vmatrix} = \{y, z, x\} = \vec{u}$, т.е. \vec{v} - векторный потенциал

Итак, $P_1(x, y, z) = \int_{z_0}^z Q(x, y, \xi) d\xi$; $Q_1(x, y, z) = -\int_{z_0}^z P(x, y, \xi) d\xi + \int_{x_0}^x R(\xi, y, z_0) d\xi$; $R_1(x, y, z) \equiv 0$

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$P_1(x, y, z) = \int_0^z \xi d\xi = \frac{z^2}{2}$; $Q_1(x, y, z) = -\int_0^z y d\xi + \int_0^x \xi d\xi = \frac{x^2}{2} - yz$; $R_1(x, y, z) \equiv 0$

$\vec{v} = \left\{ \frac{z^2}{2}; \frac{x^2}{2} - yz; 0 \right\}$ $\text{rot } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z^2}{2} & \frac{x^2}{2} - yz & 0 \end{vmatrix} = \{y, z, x\} = \vec{u}$, т.е. \vec{v} - векторный потенциал

Задача. Проверить, что для поля $\vec{u} = -\frac{\vec{r}}{r^3}$ $\text{div } \vec{u} = 0$ и найти векторный потенциал

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2$ $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона

$\nabla^{(2)}$ / $\nabla^{(1)}$	grad	div	rot
grad			
div			
rot			

Дифференциальные операции второго порядка

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$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad	X		
div		X	X
rot	X		

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$\nabla_{(2)}$ / $\nabla_{(1)}$	grad	div	rot
grad	X	Δ	
div		X	X
rot	X		

$$\operatorname{div} \operatorname{grad} f = \operatorname{div} \left\{ \frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z} \right\} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \quad (\text{однаковое})$$

$$\Delta - \text{оператор Лапласа, } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

скал. пр-ие

$$\text{Формально: } \operatorname{div} \operatorname{grad} f = (\nabla, \nabla f) = (\nabla, \nabla) f = \Delta f, \text{ т.к. } (\nabla, \nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta \text{ можно применить и к векторному полю: } \Delta \vec{u} = \Delta \{P; Q; R\} \equiv \{ \Delta P; \Delta Q; \Delta R \}$$

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2$ $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \diagdown $\nabla_{(1)}$	grad	div	rot
grad	X	Δ	$\vec{0}$
div	X	X	X
rot	X	X	X

$$\text{rot grad } f = \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\} & \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} & \left\{ \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\} \\ \text{rot} & = & = \{0; 0; 0\} = \vec{0} \end{matrix}$$

Формально: $\text{rot grad } f = [\nabla, \nabla f] = [\nabla, \nabla] f = \vec{0}$, так $[\nabla, \nabla] = \vec{0}$

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2$ $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \nabla_{(1)}	grad	div	rot
grad		Δ	$\vec{\sigma}$
div			
rot		0	

$$\operatorname{div} \operatorname{rot} \vec{u} = \operatorname{div} \operatorname{rot} \{P, Q, R\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{div} \left\{ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} =$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial x \partial y} + \frac{\partial^2 Q}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial z} = 0$$

Формально: $\operatorname{div} \operatorname{rot} \vec{u} = (\nabla, [\nabla, \vec{u}]) = 0$ (смешанное произведение)

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2$ $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона





$\nabla_{(2)}$	$\nabla_{(1)}$	grad	div	rot
grad	X		Δ	$\vec{\sigma}$
div	grad div	X	X	X
rot	X		0	

$$\begin{aligned} \text{grad div } \vec{u} &= \text{grad div} \{P; Q; R\} = \text{grad} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) = \\ &= \left\{ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x \partial y} + \frac{\partial^2 R}{\partial x \partial z}, \frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial y \partial z}, \frac{\partial^2 P}{\partial x \partial z} + \frac{\partial^2 Q}{\partial y \partial z} + \frac{\partial^2 R}{\partial z^2} \right\} \end{aligned}$$

Формально: $\text{grad div } \vec{u} = \nabla(\nabla, \vec{u})$ - далее не преобразуется

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2$ $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ - оператор Гамильтона

$\nabla_{(2)} \backslash \nabla_{(1)}$	grad	div	rot
grad		Δ	$\vec{0}$
div	grad div		
rot		0	grad div - Δ

$$\text{rot rot } \vec{u} = [\nabla, [\nabla, \vec{u}]] = \nabla(\nabla, \vec{u}) - (\nabla, \nabla)\vec{u} = \text{grad div } \vec{u} - \Delta \vec{u}$$

Задача. Получить эту формулу неформально (т.е. выкладкой)