Д-зать, что $\operatorname{rot}\vec{u} = \vec{O} \Leftrightarrow \vec{u} = \operatorname{grad} f$. $\operatorname{grad}(f + C) = \operatorname{grad} f$.

$\vec{u} = \{P; Q, R\}$, $\operatorname{rot}\vec{u} = \vec{O}$, т.е. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ в $G \subset \mathbb{E}_3$

$(x_0, y_0, z_0) \in G$. $f(x,y,z) = \int\limits_{x_0}^{x} P(\xi, y, z)\,d\xi + \varphi(y,z)$. Ясно, что $\frac{\partial f}{\partial x} = P(x,y,z)$

$$Q(x,y,z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int\limits_{x_0}^{x} P(\xi, y, z)\,d\xi + \frac{\partial \varphi}{\partial y} = \left(\begin{matrix}\text{св-ва инт-лов,}\\ \text{завис. от парам.}\end{matrix}\right) = \int\limits_{x_0}^{x} \frac{\partial P(\xi, y, z)}{\partial y}\,d\xi + \frac{\partial \varphi}{\partial y} =$$

$$= \int\limits_{x_0}^{x} \frac{\partial Q(\xi, y, z)}{\partial \xi}\,d\xi + \frac{\partial \varphi}{\partial y} = Q(x,y,z) - Q(x_0, y, z) + \frac{\partial \varphi}{\partial y}, \text{ т.е. } \frac{\partial \varphi}{\partial y} = Q(x_0, y, z) \Rightarrow$$

$$\Rightarrow f(x,y,z) = \int\limits_{x_0}^{x} P(\xi, y, z)\,d\xi + \int\limits_{y_0}^{y} Q(x_0, \eta, z)\,d\eta + \psi(z). \quad R(x,y,z) = \frac{\partial f}{\partial z} =$$

$$= \int\limits_{x_0}^{x} \frac{\partial P(\xi, y, z)}{\partial z}\,d\xi + \int\limits_{y_0}^{y} \frac{\partial Q(x_0, \eta, z)}{\partial z}\,d\eta + \psi'(z) = \int\limits_{x_0}^{x} \frac{\partial R(\xi, y, z)}{\partial \xi}\,d\xi + \int\limits_{y_0}^{y} \frac{\partial R(x_0, \eta, z)}{\partial \eta}\,d\eta + \psi'(z) =$$

$$= R(x,y,z) - R(x_0, y, z) + R(x_0, y, z) - R(x_0, y_0, z) + \psi'(z), \text{ т.е. } \psi'(z) = R(x_0, y_0, z) \Rightarrow$$

$$f(x,y,z) = \int\limits_{x_0}^{x} P(\xi, y, z)\,d\xi + \int\limits_{y_0}^{y} Q(x_0, \eta, z)\,d\eta + \int\limits_{z_0}^{z} R(x_0, y_0, \zeta)\,d\zeta + C$$

$$f(x,y,z)=\int_{x_0}^{x}P(\xi,y,z)\,d\xi+\int_{y_0}^{y}Q(x_0,\eta,z)\,d\eta+\int_{z_0}^{z}R(x_0,y_0,\zeta)\,d\zeta+C$$

Пример. $\vec{u}=-\dfrac{\vec{r}}{r^3}=\left\{-\dfrac{x}{(x^2+y^2+z^2)^{3/2}};-\dfrac{y}{(x^2+y^2+z^2)^{3/2}};-\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right\}$ $\left(\begin{array}{l}\vec{r}=\{x,y,z\}\\ r=|\vec{r}|=\sqrt{x^2+y^2+z^2}\end{array}\right)$

$\operatorname{rot}\vec{u}=\vec{0}$ (соленоид.), $\vec{u}=\operatorname{grad}f$, $f(x,y,z)=-\displaystyle\int_{x_0}^{x}\frac{\xi\,d\xi}{(\xi^2+y^2+z^2)^{3/2}}-\int_{y_0}^{y}\frac{\eta\,d\eta}{(x_0^2+\eta^2+z^2)^{3/2}}-$

$$-\int_{z_0}^{z}\frac{\zeta\,d\zeta}{(x_0^2+y_0^2+\zeta^2)^{3/2}}+C=\frac{1}{\sqrt{\xi^2+y^2+z^2}}\Bigg|_{\xi=x_0}^{\xi=x}+\frac{1}{\sqrt{x_0^2+\eta^2+z^2}}\Bigg|_{\eta=y_0}^{\eta=y}+\frac{1}{\sqrt{x_0^2+y_0^2+\zeta^2}}\Bigg|_{\zeta=z_0}^{\zeta=z}+C=$$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{1}{\sqrt{x_0^2+y^2+z^2}}+\frac{1}{\sqrt{x_0^2+y^2+z^2}}-\frac{1}{\sqrt{x_0^2+y_0^2+z^2}}+\frac{1}{\sqrt{x_0^2+y_0^2+z^2}}-\frac{1}{\sqrt{x_0^2+y_0^2+z_0^2}}+C=$$

$$=\frac{1}{r}-\frac{1}{r_0}+C=\frac{1}{r}+\text{const}$$

О векторном потенциале

Потенциальное поле: $\oint\limits_{\Gamma}(\vec{u}, d\vec{r}) = 0$ $\quad$ $\operatorname{rot}\vec{u} = \vec{0}$ $\quad$ $\vec{u} = \operatorname{grad} f$

Соленоидальное поле: $\oiint\limits_{S}(\vec{u}, d\vec{S}) = 0$ $\quad$ $\operatorname{div}\vec{u} = 0$ $\quad$?

О векторном потенциале

Потенциальное поле: $\oint\limits_{\Gamma}(\vec{u}, d\vec{r}) = 0 \qquad \operatorname{rot}\vec{u} = \vec{0} \qquad \vec{u} = \operatorname{grad} f$

Соленоидальное поле: $\oiint\limits_{S}(\vec{u}, d\vec{S}) = 0 \qquad \operatorname{div}\vec{u} = 0 \qquad \vec{u} = \operatorname{rot}\vec{v}$

$\vec{v}$ – векторный потенциал

$$\vec{u} = \{P, Q, R\},\ \vec{v} = \{P_1, Q_1, R_1\},\ \operatorname{rot}\vec{v} = \Big\{\underbrace{\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}}_{P};\ \underbrace{\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}}_{Q};\ \underbrace{\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}}_{R}\Big\} = \vec{u}$$

$$0 = \operatorname{div}\vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial^2 R_1}{\partial x \partial y} - \frac{\partial^2 Q_1}{\partial x \partial z} + \frac{\partial^2 P_1}{\partial y \partial z} + \frac{\partial^2 R_1}{\partial x \partial y} + \frac{\partial^2 Q_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y \partial z}$$

$$P = \frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z},\ Q = \frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x},\ R = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}$$ Положим $R_1(x, y, z) \equiv 0$

$$\Rightarrow P = -\frac{\partial Q_1}{\partial z},\ Q = \frac{\partial P_1}{\partial z},\ R = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}. \quad (x_0, y_0, z_0) \in G \Rightarrow P_1(x, y, z) = \int\limits_{z_0}^{z} Q(x, y, \xi)\,d\xi$$

$$Q_1(x, y, z) = -\int\limits_{z_0}^{z} P(x, y, \xi)\,d\xi + \varphi(x, y) \Rightarrow R(x, y, z) = \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} = -\int\limits_{z_0}^{z} \frac{\partial P(x, y, \xi)}{\partial x}\,d\xi +$$

$$+ \frac{\partial \varphi}{\partial x} - \int\limits_{z_0}^{z} \frac{\partial Q(x, y, \xi)}{\partial y}\,d\xi = \frac{\partial \varphi}{\partial x} - \int\limits_{z_0}^{z}\left(\frac{\partial P(x, y, \xi)}{\partial x} + \frac{\partial Q(x, y, \xi)}{\partial y}\right)d\xi = \frac{\partial \varphi}{\partial x} + \int\limits_{z_0}^{z} \frac{\partial R(x, y, \xi)}{\partial \xi}\,d\xi =$$

$$= \frac{\partial \varphi}{\partial x} + R(x, y, z) - R(x, y, z_0), \text{ т.е. } \frac{\partial \varphi}{\partial x} = R(x, y, z_0) \Rightarrow \varphi(x, y) = \int\limits_{x_0}^{x} R(\xi, y, z_0)\,d\xi + \underbrace{\psi(y)}_{\text{берём} \equiv 0}$$

Итак, $P_1(x, y, z) = \int\limits_{z_0}^{z} Q(x, y, \xi)\,d\xi;\ Q_1(x, y, z) = -\int\limits_{z_0}^{z} P(x, y, \xi)\,d\xi + \int\limits_{x_0}^{x} R(\xi, y, z_0)\,d\xi;\ R_1(x, y, z) \equiv 0$

Итак, $P_1(x,y,z)=\int\limits_{z_0}^{z}Q(x,y,\xi)d\xi$; $Q_1(x,y,z)=-\int\limits_{z_0}^{z}P(x,y,\xi)d\xi+\int\limits_{x_0}^{x}R(\xi,y,z_0)d\xi$; $R_1(x,y,z)\equiv 0$

Пример $\vec{u}=\{y,z,x\}$ $\operatorname{div}\vec{u}=0$ Найдём $\vec{v}$: $\vec{u}=\operatorname{rot}\vec{v}$ $x_0=y_0=z_0=0$

$P_1(x,y,z)=\int\limits_0^z\xi d\xi=\frac{z^2}{2}$; $Q_1(x,y,z)=-\int\limits_0^z y d\xi+\int\limits_0^x\xi d\xi=\frac{x^2}{2}-yz$; $R_1(x,y,z)\equiv 0$

$\vec{v}=\left\{\frac{z^2}{2};\frac{x^2}{2}-yz;0\right\}$ $\operatorname{rot}\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{z^2}{2} & \frac{x^2}{2}-yz & 0\end{vmatrix}=\{y;z;x\}=\vec{u}$, т.е. $\vec{v}$ - векторный потенциал

Итак, $P_1(x,y,z)=\int\limits_{z_0}^{z}Q(x,y,\xi)d\xi$; $Q_1(x,y,z)=-\int\limits_{z_0}^{z}P(x,y,\xi)d\xi+\int\limits_{x_0}^{x}R(\xi,y,z_0)d\xi$; $R_1(x,y,z)\equiv 0$

Пример $\vec{u}=\{y,z,x\}$ $\operatorname{div}\vec{u}=0$ Найдём $\vec{v}:\vec{u}=\operatorname{rot}\vec{v}$ $x_0=y_0=z_0=0$

$P_1(x,y,z)=\int\limits_0^z\xi d\xi=\frac{z^2}{2}$; $Q_1(x,y,z)=-\int\limits_0^z y d\xi+\int\limits_0^x\xi d\xi=\frac{x^2}{2}-yz$; $R_1(x,y,z)\equiv 0$

$\vec{v}=\left\{\frac{z^2}{2};\frac{x^2}{2}-yz;0\right\}$ $\operatorname{rot}\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{z^2}{2} & \frac{x^2}{2}-yz & 0\end{vmatrix}=\{y;z;x\}=\vec{u}$, т.е. $\vec{v}$ - векторный потенциал

Задача. Проверить, что для поля $u=-\frac{\vec{r}}{r^3}$ $\operatorname{div}\vec{u}=0$ и найти векторный потенциал

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad			
div			
rot			

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad			
div			
rot			

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad		Δ	
div			
rot			

$$\operatorname{div} \operatorname{grad} f = \operatorname{div}\left\{\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right\} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \quad \text{(обозначение)}$$

Δ – оператор Лапласа, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

скал. пр-ие

Формально: $\operatorname{div}\operatorname{grad} f = (\nabla, \nabla f) = (\nabla, \nabla) f = \Delta f$, т.к. $(\nabla, \nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Δ можно применять и к векторному полю: $\Delta \vec{u} = \Delta\{P; Q; R\} \equiv \{\Delta P; \Delta Q; \Delta R\}$

Дифференциальные операции второго порядка

$f, \vec{a} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ — оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad		Δ	$\vec{0}$
div			
rot			

$$\operatorname{rot}\operatorname{grad} f = \operatorname{rot}\left\{\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left\{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}; \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}; \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right\} =$$

$$= \{0; 0; 0\} = \vec{0}$$

Формально: $\operatorname{rot}\operatorname{grad} f = [\nabla, \nabla f] = [\nabla, \nabla] f = \vec{0}$, т.к. $[\nabla, \nabla] = \vec{0}$

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \qquad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad		Δ	$\vec{0}$
div			
rot		0	

$$\operatorname{div}\operatorname{rot}\vec{u} = \operatorname{div}\operatorname{rot}\{P; Q, R\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{div}\left\{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}; \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} =$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial x \partial y} + \frac{\partial^2 Q}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial z} = 0$$

Формально: $\operatorname{div}\operatorname{rot}\vec{u} = (\nabla, [\nabla, \vec{u}]) = 0$ (смешанное произведение)

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ – оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad		Δ	$\vec{0}$
div	grad div		
rot		0	

$$\operatorname{grad}\operatorname{div}\vec{u} = \operatorname{grad}\operatorname{div}\{P;Q;R\} = \operatorname{grad}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right) =$$

$$= \left\{\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x \partial y}+\frac{\partial^2 R}{\partial x \partial z}; \frac{\partial^2 P}{\partial x \partial y}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 R}{\partial y \partial z}; \frac{\partial^2 P}{\partial x \partial z}+\frac{\partial^2 Q}{\partial y \partial z}+\frac{\partial^2 R}{\partial z^2}\right\}$$

Формально: $\operatorname{grad}\operatorname{div}\vec{u} = \nabla(\nabla, \vec{u})$ – далее не преобразуется

Дифференциальные операции второго порядка

$f, \vec{u} \in C_2 \quad \nabla = \left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\}$ - оператор Гамильтона

$\nabla_{(2)}$ \ $\nabla_{(1)}$	grad	div	rot
grad		Δ	$\vec{0}$
div	grad div		
rot		0	graddiv $-\ \Delta$

$$\operatorname{rot}\operatorname{rot}\vec{u} = [\nabla, [\nabla, \vec{u}]] = \nabla(\nabla, \vec{u}) - (\nabla, \nabla)\vec{u} = \operatorname{grad}\operatorname{div}\vec{u} - \Delta\vec{u}$$

<u>Задание</u>. Получить эту формулу неформально (т.е. выкладкой)