Linear Algebra
Lecture 2
Solution Sets of Linear Systems.
Applications of Linear Systems.
Linear Independence.

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Learning Objectives:

1. Solving Homogeneous Systems.
2. Solving Nonhomogeneous Systems.
3. Applications.
4. Represent Linear Independence of sets of vectors.

Previously…

We have seen that a linear system of m equations in n unknowns can be rephrased as a matrix-vector equation

$$A\mathbf{x} = \mathbf{b}\ ,$$

where A is the $m \times n$ real matrix of coefficients,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

is the vector whose components are the n variables of the system, **b** is the column vector of constants, and A**x** is the matrix-vector product, defined as the linear combination of the columns of A using $x_1, \ldots, x_n$ as the scalar weights.

1.5. Solution Sets of Linear Systems.

Now we seek to understand the solution sets of such equations:

the hope is to be able to use the tools developed thus far to describe the set of all $\mathbf{x} \in \mathbf{R}^n$ satisfying a given equation $A\mathbf{x} = \mathbf{b}$.

To do this, we turn first to the easiest case to study: the case when $\mathbf{b} = \mathbf{0}$. Thus, we are asking about linear combinations of the column vectors of A which equal $\mathbf{0}$, or equivalently, intersections of linear subsets of R^n that all pass through the origin.

We will then discover that describing the solutions to $A\mathbf{x}=\mathbf{0}$ help unlock a general solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$.

Homogeneous Linear Systems

Definition

A system of m real linear equations in n variables is called *homogenous* if there exists an $m \times n$ matrix A such that the system can be described by the matrix-vector equation

$$\mathrm{A}\mathbf{x} = \mathbf{0}\,,$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector whose components are the n variables of the system, and $\mathbf{0} \in \mathbb{R}^m$ is the zero vector with m components.

Observation

A homogeneous system is always consistent. In particular, it always has at least one (obvious) solution: the *trivial solution* $\mathbf{x} = \mathbf{0} \in \mathbb{R}^n$.

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Homogeneous Linear Systems

Example

Find the solutions to the homogeneous system $\mathrm{A}\mathbf{x} = \mathbf{0}$ where $\mathbf{x}, \mathbf{0} \in \mathbb{R}^3$ and

$$\mathrm{A} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ 4 & 5 & -5 \end{bmatrix}.$$

Solution: To solve $\mathrm{A}\mathbf{x} = \mathbf{0}$, we can row reduce the augmented matrix $\left[\mathrm{A} \mid \mathbf{0}\right]$.

Homogeneous Linear Systems

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 4 & 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which leaves the variable z free, and gives equations $x - 5z = 0$ and $y + 3z = 0$.
Thus,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

Write $z = t$ and let

$$\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

We can thus describe the solution to this system as the set

$$\{t\mathbf{v} \mid t \in \mathbb{R}\} = \mathrm{Span}\{\mathbf{v}\}.$$

Homogeneous Linear Systems

When we write our solution explicitly in the form

$$\mathbf{x} = t\mathbf{v} = t\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

we say that the solution is in *parametric vector form*.

Nonhomogeneous Systems

We will now begin to tackle the general case of A**x** = **b** for nonzero **b**, which is called the nonhomogeneous case.

Before we prove the general result, let's look at a familiar example that contains all of the pieces.

Example:

Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Nonhomogeneous Systems

SOLUTION Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \quad\; - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad\quad &= 2 \\ 0 &= 0 \end{aligned}$$

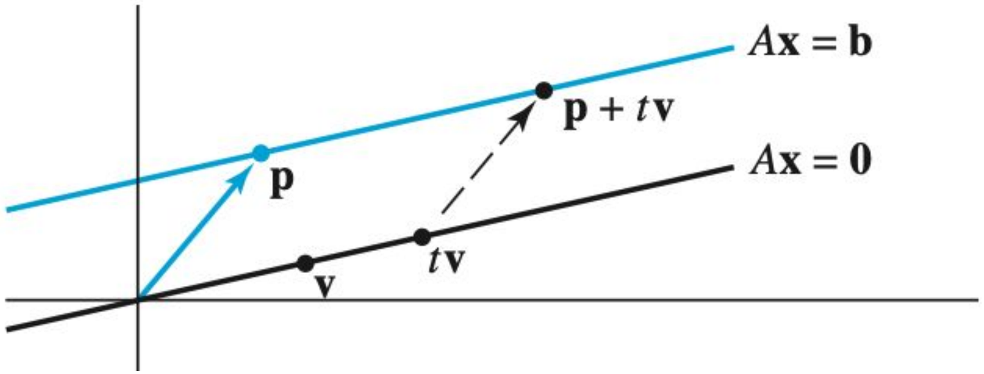
Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

Nonhomogeneous Systems

To describe the solution set of A**x=b** geometrically, we can think of vector addition as a *translation*. Given **v** and **p** in R^2or R^3, the effect of adding **p** to **v** is to *move* **v** in a direction parallel to the line through **p** and **0**. We say that **v** is **translated by p** to **v+p**.

Suppose L is the line through **0** and **v**, described by equation **x**=t**v**. Adding **p** to each point on L produces the translated line described by equation **x**=p+t**v**. We call this **the equation of the line through p parallel to v**. Thus *the solution set of* A**x=b** *is a line through* **p** *parallel to the solution set of* A**x=0**. Figure below illustrates this case.



Nonhomogeneous Systems

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 6 says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ for the translation. Figure 6 illustrates the case in which there are two free variables. Even when $n > 3$, our mental image of the solution set of a consistent system $A\mathbf{x} = \mathbf{b}$ (with $\mathbf{b} \neq \mathbf{0}$) is either a single nonzero point or a line or plane not passing through the origin.

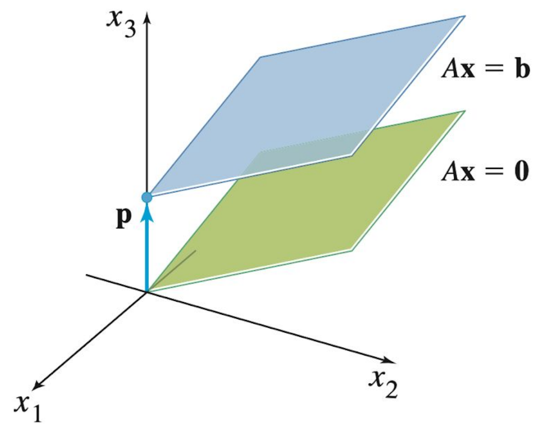


FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

1.6. Applications of Linear Algebra in SE

Any applications in software engineering where a large amount of equations need to be calculated quickly, linear algebra is most likely being used. These applications would include things like graphics software, visual gaming, physics, and signal processing.

Another application is in **computer graphics.** Using very simple linear algebra, as well as parts of other branches of mathematics, you can easily make objects move around in a virtual world, make them larger or smaller.

Web development hardly requires any knowledge of linear algebra. Building strong backends to web frontends requires no knowledge of linear algebra (in most cases, randomization can achieve good load balancing if you are building backend farms).

Example in a Network Flow

Urban planners and traffic engineers monitor the pattern of traffic flow in a grid of city streets. Electrical engineers calculate current flow through electrical circuits. And economists analyze the distribution of products from manufacturers to consumers through a network of wholesalers and retailers. For many networks, the systems of equations involve hundreds or even thousands of variables and equations.

The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction.

Example in a Network Flow

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

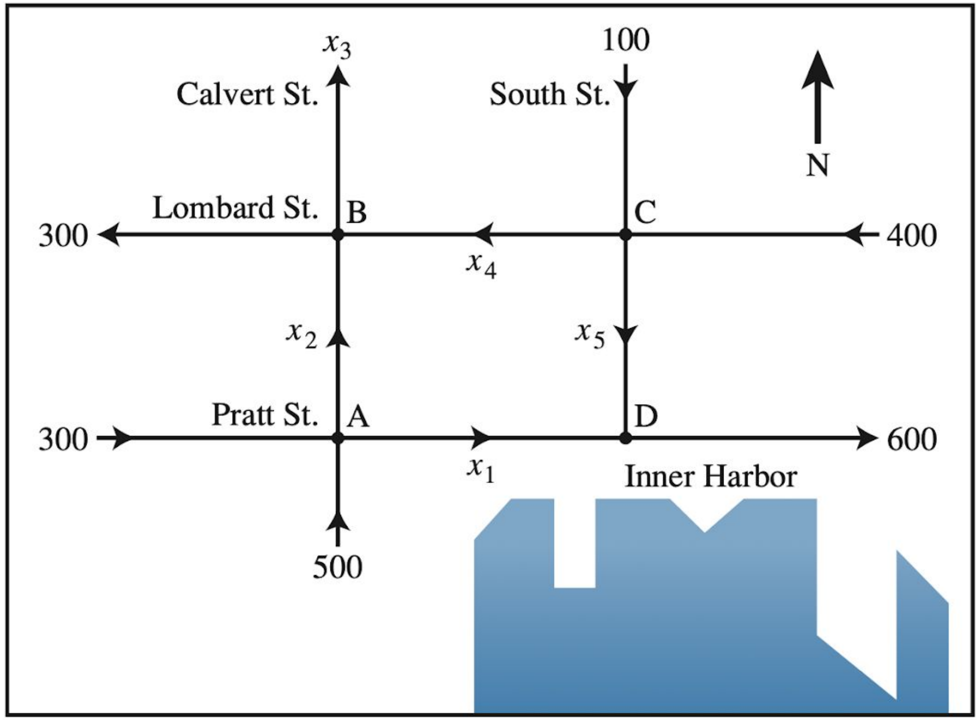


FIGURE 2 Baltimore streets.

Example in a Network Flow

SOLUTION Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in Figure 2. At each intersection, set the flow in equal to the flow out.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

$$\begin{array}{rcl} x_1 + x_2 \qquad\qquad\qquad & = & 800 \\ x_2 - x_3 + x_4 \qquad & = & 300 \\ x_4 + x_5 & = & 500 \\ x_1 \qquad\qquad\qquad + x_5 & = & 600 \\ x_3 \qquad\qquad & = & 400 \end{array} \rightarrow \begin{array}{rcl} x_1 \qquad\qquad\qquad + x_5 & = & 600 \\ x_2 \qquad\qquad - x_5 & = & 200 \\ x_3 \qquad\qquad & = & 400 \\ x_4 + x_5 & = & 500 \end{array} \rightarrow \begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$

1.7. Linear Independence

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Solution

a. We must determine if there is a nontrivial solution of equation (1) above. Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (1). Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent (and not linearly independent).

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - 2x_3 &= 0 \\ x_2 + \ \ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3—say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

Linear Independence of Matrix Columns

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

EXAMPLE 2 Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

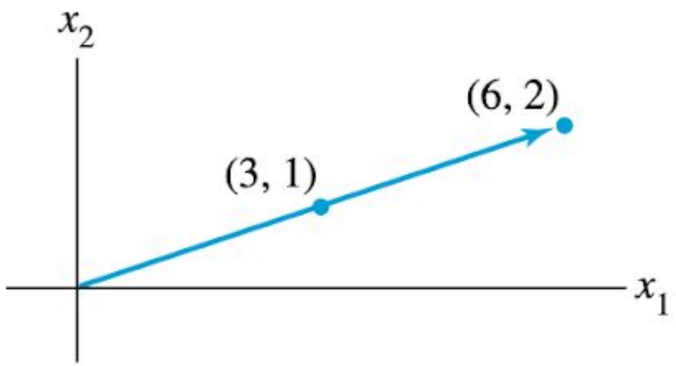
SOLUTION To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

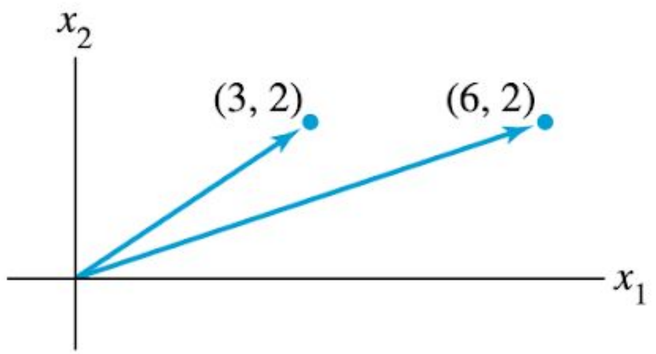
At this point, it is clear that there are three basic variables and no free variables. So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of A are linearly independent. ■

Sets of One or Two Vectors

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.



Linearly dependent



Linearly independent

Sets of Two or More Vectors

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

THEOREM 8

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

THEOREM 9

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Lecture Summary

1. Homogeneous and Nonhomogeneous Linear Systems (trivial/nontrivial solutions)
2. Applications
3. Linear Independence