Azərbaycan Dövlət Neft və Sənaye Universiteti

Optimal Control

AZƏRBAYCAN DÖVLƏT NEFT VƏ SƏNAYE UNİVERSİTETİ

1920

Variational Approach to the Fixed-Time, Free-Endpoint Problem

We now want to see how far a variational approach – i.e. an approach based on analyzing the first (and second) variation of the cost functional – can take us in studying the optimal control problem formulated in the previous lecture.

❖ Preliminaries

Consider the optimal control problem:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad (0)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in U \subset \mathbb{R}^m$ is the control, $t \in \mathbb{R}$ is the time, t_0 is the initial time, $\mathbf{x}_0$ is the initial state; the cost functional is

$$\mathrm{I}(\mathbf{u}) := \int_{t_0}^{t_1} L\big(t, \mathbf{x}(t), \mathbf{u}(t)\big)\mathrm{d}t + K\big(t_1, \mathbf{x}_f\big)$$

where $t_1, \mathbf{x}_f$ are the final time and state;

with the following additional specifications:

- **the target set is $S = \{t_1\} \times \mathbb{R}^n$, where t_1 is a fixed time (so this is a fixed-time, free-endpoint problem);**
- **the set $U = \mathbb{R}^m$ (the control is unconstrained);**
- **and the terminal cost is $K = K(\mathbf{x}_f)$, with no direct dependence on the final time (just for simplicity).**

We can rewrite the cost in terms of the fixed final time t_1 as

$$\mathbf{I}(\mathbf{u}) := \int_{t_0}^{t_1} L\big(t, \mathbf{x}(t), \mathbf{u}(t)\big)\mathrm{d}t + K\big(\mathbf{x}(t_1)\big) \qquad (1)$$

Our goal is to derive necessary conditions for optimality.

Let $\mathbf{u}^*$ be an optimal control, by which we presently mean that it provides a global minimum.

In other words, $\mathbf{I}(\mathbf{u}^*) \leq \mathbf{I}(\mathbf{u})$ for all piecewise continuous controls $\mathbf{u}$.

Let $\mathbf{x}^*$ be the corresponding optimal trajectory.

We would like to consider nearby trajectories of the familiar form

$$\mathbf{x} = \mathbf{x}^* + \alpha\,\boldsymbol{\eta} \qquad (2)$$

but we must make sure that these perturbed trajectories are still solutions of the system (0), for suitably chosen controls.

Unfortunately, the class of perturbations η that are admissible in this sense is difficult to characterize if we start with (2).

Note also that the cost I, whose first variation we will be computing, is a function of u and not of x.

Thus, in the optimal control context it is more natural to directly perturb the control instead.

And then define perturbed state trajectories in terms of perturbed controls.

To this end, we consider perturbed controls of the form

$$\mathbf{u} = \mathbf{u}^* + \alpha\, \xi \qquad (3)$$

where ξ is a piecewise continuous function from $[t_0; t_1]$ to $\mathbb{R}^m$ and α is a real parameter as usual.

We now want to find (if possible) a function

$$\eta\colon [t_0; t_1] \to \mathbb{R}^n$$

for which the solutions of (0) corresponding to the perturbed controls (3), for a fixed ξ, are given by (2).

Actually, we do not have any reason to believe that the perturbed trajectory depends linearly on α as in (2).

Thus we should replace (2) by the more general (and more realistic) expression

$$\mathbf{x} = \mathbf{x}^* + \alpha\, \boldsymbol{\eta} + o(\alpha) \qquad (4)$$

It is obvious that $\boldsymbol{\eta}(t_0) = \mathbf{0}$ since the initial condition does not change.

Next, we derive a differential equation for $\boldsymbol{\eta}$.

Let us use the more detailed notation $\mathbf{x}(t; \alpha)$ for the solution of diff. eq. (0) at time t corresponding to the perturbed control (3).

The function $\mathbf{x}(.;\alpha)$ coincides with the right-hand side of (4) if and only if

$$\mathbf{x}_{\alpha}(t,0) = \boldsymbol{\eta}(t) \qquad (5)$$

for all t. (we are assuming here that the partial derivative $\mathbf{x}_{\alpha}$ exists.)

Next, we differentiate the quantity (5) with respect to time and interchange the order of partial derivatives.

We have

$$\dot{\boldsymbol{\eta}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_{\alpha}(t,0) = \mathbf{x}_{\alpha,t}(t,0) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \dot{\mathbf{x}}(t,\alpha)$$

$$\ldots = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} f\big(t, \mathbf{x}(t,\alpha), \mathbf{u}^*(t) + \alpha\, \boldsymbol{\xi}(t)\big)$$

$$= f_{\mathbf{x}}\big(t, \mathbf{x}(t,0), \mathbf{u}^*(t)\big) \cdot \mathbf{x}_{\alpha}(t,0) + f_{\mathbf{u}}\big(t, \mathbf{x}(t,0), \mathbf{u}^*(t)\big) \cdot \boldsymbol{\xi}(t)$$

According to (4) and (5) we obtain

$$\dot{\boldsymbol{\eta}}(t) = f_{\mathbf{x}}\big(t, \mathbf{x}^*(t), \mathbf{u}^*(t)\big) \cdot \boldsymbol{\eta}(t) + f_{\mathbf{u}}\big(t, \mathbf{x}^*(t), \mathbf{u}^*(t)\big) \cdot \boldsymbol{\xi}(t)$$

The last expression can be written more compactly (remembering also the initial condition $\boldsymbol{\eta}(t_0) = \mathbf{0}$) as

$$\dot{\boldsymbol{\eta}} = \boldsymbol{f}_{\mathbf{x}}(t, \mathbf{x}^*, \mathbf{u}^*) \cdot \boldsymbol{\eta} + \boldsymbol{f}_{\mathbf{u}}(t, \mathbf{x}^*, \mathbf{u}^*) \cdot \boldsymbol{\xi}$$

$$:= \boldsymbol{f}_{\mathbf{x}}\Big|_* \cdot \boldsymbol{\eta} + \boldsymbol{f}_{\mathbf{u}}\Big|_* \cdot \boldsymbol{\xi}, \qquad \boldsymbol{\eta}(t_0) = \mathbf{0}. \quad (6)$$

Here and below, we use the shorthand notation $|_*$ to indicate that a function is being evaluated along the optimal trajectory.

The linear time-varying system (6) is nothing but the linearization of the original system (0) in the neighborhood of the optimal trajectory.

To emphasize the linearity of the system (6) we can introduce the notation $\boldsymbol{A}_*(\boldsymbol{t}) := \boldsymbol{f}_{\mathbf{x}}|_*(\boldsymbol{t})$ and $\boldsymbol{B}_*(\boldsymbol{t}) := \boldsymbol{f}_{\mathbf{u}}|_*(\boldsymbol{t})$ for the matrices appearing in it, bringing it to the form

$$\dot{\eta} = \boldsymbol{A}_*(\boldsymbol{t}) \cdot \eta + \boldsymbol{B}_*(\boldsymbol{t}) \cdot \xi, \qquad \eta(\boldsymbol{t}_0) = \mathbf{0}. \tag{7}$$

The optimal control $\mathbf{u}^*$ minimizes the cost given by (1), and the control system (0) can be viewed as imposing the pointwise-in-time (non-integral) constraint

$$\dot{\mathbf{x}}(t) - f\big(t, \mathbf{x}(t), \mathbf{u}(t)\big) = \mathbf{0}.$$

Motivated by Lagrange's idea for treating such constraints in calculus of variations, expressed by an augmented cost, let us rewrite our cost as indicated below.

$$\mathbf{I}(\mathbf{u}) := K\big(\mathbf{x}(t_1)\big) +$$

$$+ \int_{t_0}^{t_1} \{L\big(t, \mathbf{x}(t), \mathbf{u}(t)\big) + \mathbf{p}(t) \cdot \big[\dot{\mathbf{x}}(t) - f\big(t, \mathbf{x}(t), \mathbf{u}(t)\big)\big]\} \mathrm{d}t$$

for some C^1 function $\mathbf{p}: [t_0; t_1] \to \mathbb{R}^n$ to be selected later.*

As we will see momentarily, $\mathbf{p}(.)$ is also closely related to the momentum.

Clearly, the extra term inside the integral does not change the value of the cost.

We will be working in the Hamiltonian framework, which is why we continue to use the same symbol $\mathbf{p}$ by which we denoted the momentum earlier (while some other sources prefer $\boldsymbol{\lambda}$ and $\boldsymbol{\psi}$).

We will henceforth use the more explicit notation $\langle .,. \rangle$ for the inner product defined in $\mathbb{R}^n$.

Let us introduce the Hamiltonian

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) := \langle \mathbf{p}, f(t, \mathbf{x}, \mathbf{u}) \rangle - L(t, \mathbf{x}, \mathbf{u}) \tag{8}$$

Note that this definition matches our earlier definition of the Hamiltonian in calculus of variations, where we had

$$H(x, \mathbf{y}, \mathbf{y}', \mathbf{p}) = \langle \mathbf{p}, \mathbf{y}' \rangle - L(x, \mathbf{y}, \mathbf{y}');$$

We just need to remember that after we changed the notation from calculus of variations to optimal control, the independent variable x became t, the dependent variable $\mathbf{y}$ became $\mathbf{x}$, its derivative $\mathbf{y}'$ became $\dot{\mathbf{x}}$ and is given by (0), and the third argument of L is taken to be $\mathbf{u}$ rather than $\dot{\mathbf{x}}$ (which with the current definition of H makes even more sense).

We can rewrite the cost in terms of the Hamiltonian as

$$\mathbf{I}(\mathbf{u}) := \boldsymbol{K}\big(\mathbf{x}(\boldsymbol{t_1})\big) +$$

$$+ \int_{t_0}^{t_1} \big[\langle \mathbf{p}(\boldsymbol{t}), \dot{\mathbf{x}}(\boldsymbol{t})\rangle - \boldsymbol{H}\big(\boldsymbol{t}, \boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{p}(\boldsymbol{t})\big)\big]\mathbf{d}t \quad (9)$$

➢ First Variation

We want to compute and analyze the first variation $\delta\mathbf{I}|_{\mathbf{u}^*}$ of the cost functional $\mathbf{I}(\mathbf{u})$ at the optimal control function $\mathbf{u}^*$.

To do this, in view of the definition

$$\mathbf{I}(\mathbf{y}+\alpha\eta) = \mathbf{I}(\mathbf{y}) + \delta\mathbf{I}\Big|_{\mathbf{y}}(\eta)\cdot\alpha + o(\alpha),$$

we must isolate the first-order terms with respect to α in the cost difference between the perturbed control (3) and the optimal control.

$$\mathbf{I}(\mathbf{u}) - \mathbf{I}(\mathbf{u}^*) = \mathbf{I}(\mathbf{u}^* + \alpha\xi) - \mathbf{I}(\mathbf{u}^*) =$$

$$= \delta\mathbf{I}\Big|_{\mathbf{u}^*}(\xi)\cdot\alpha + o(\alpha) \qquad (10)$$

The formula (9) suggests to regard the difference $\mathbf{I}(\mathbf{u}) - \mathbf{I}(\mathbf{u}^*)$ as being composed of three distinct terms, which we now inspect in more detail.

We will let the approximate equality $\approx$ denote equality up to terms of order $o(\alpha)$.

For the terminal cost, we have

$$K\big(\mathbf{x}(t_1)\big) - K\big(\mathbf{x}^*(t_1)\big) =$$

$$K\big(\mathbf{x}^*(t_1) + \alpha\,\boldsymbol{\eta}(t_1) + o(\alpha)\big) - K\big(\mathbf{x}^*(t_1)\big) \approx$$

$$\big\langle K_{\mathbf{x}}\big(\mathbf{x}^*(t_1)\big), \alpha\,\boldsymbol{\eta}(t_1)\big\rangle \qquad \textbf{(11)}$$

For the Hamiltonian, omitting the t −arguments inside $\mathbf{x}$ and $\mathbf{u}$ for brevity, we have

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) - H(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}) =$$

$$H(t, \mathbf{x}^* + \alpha\,\boldsymbol{\eta}, \mathbf{u}^* + \alpha\,\boldsymbol{\xi}, \mathbf{p}) - H(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}) \approx \cdots$$

$$\ldots \approx \langle H_{\mathbf{x}}(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}), \alpha\, \boldsymbol{\eta} \rangle + \langle H_{\mathbf{u}}(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}), \alpha\, \boldsymbol{\xi} \rangle \tag{12}$$

As for the inner product $\langle \mathbf{p}, \dot{\mathbf{x}} - \dot{\mathbf{x}}^* \rangle$, we use integration by parts as we did several times in calculus of variations:

$$\int_{t_0}^{t_1} \langle \mathbf{p}(t), \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^*(t) \rangle \mathrm{d}t = \langle \mathbf{p}(t), \mathbf{x}(t) - \mathbf{x}^*(t) \rangle \Big|_{t_0}^{t_1} -$$

$$- \int_{t_0}^{t_1} \langle \dot{\mathbf{p}}(t), \mathbf{x}(t) - \mathbf{x}^*(t) \rangle \mathrm{d}t \approx \cdots$$

$$\ldots \approx \langle \mathbf{p}(t_1), \alpha\, \boldsymbol{\eta}(t_1) \rangle - \int_{t_0}^{t_1} \langle \dot{\mathbf{p}}(t), \alpha\, \boldsymbol{\eta}(t) \rangle \mathrm{d}t \quad (13)$$

where we used the fact that $\mathbf{x}(t_0) = \mathbf{x}^*(t_0)$.

Combining the formulas (9) − (13), we readily see that the first variation is given by

$$\delta \mathbf{I}\Big|_{\mathbf{u}^*}(\boldsymbol{\xi}) = -\int_{t_0}^{t_1} \big(\langle \dot{\mathbf{p}} + H_{\mathbf{x}}(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}), \boldsymbol{\eta} \rangle + \\ + \langle H_{\mathbf{u}}(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}), \boldsymbol{\xi} \rangle \big) \mathrm{d}t + \big\langle K_{\mathbf{x}}\big(\mathbf{x}^*(t_1)\big) + \mathbf{p}(t_1), \boldsymbol{\eta}(t_1) \big\rangle$$

$$(14)$$

Note that here η is related to ξ via the system (6).

The familiar first-order necessary condition for optimality says that we must have $\delta I|_{u^*}(\xi) = 0$ for all ξ.

This condition is true for every function p, but becomes particularly revealing if we make a special choice of p.

Namely, let $\mathbf{p}$ be the solution of the differential equation

$$\dot{\mathbf{p}} = -H_{\mathbf{x}}(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}) \qquad (15)$$

satisfying the boundary condition

$$\mathbf{p}(t_1) = -K_{\mathbf{x}}\left(\mathbf{x}^*(t_1)\right) \qquad (16)$$

Note that this boundary condition specifies the value of $\mathbf{p}$ at the end of the interval $[t_0; t_1]$, i.e. it is a final (or terminal) condition rather than an initial condition.

In case of no terminal cost we treat K as being equal to 0, which corresponds to $\mathbf{p}(t_1) = \mathbf{0}$.

We label the function $\mathbf{p}$ defined by (15) and (16) as $\mathbf{p}^*$ from now on, to reflect the fact that it is associated with the optimal trajectory.

We also extend the notation $|_*$ to mean evaluation along the optimal trajectory with $\mathbf{p} = \mathbf{p}^*$, so that, for example, $H|_*(t) = H(t, \mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*)$.

Setting $\mathbf{p} = \mathbf{p}^*$ and using the equations (15) and (16) to simplify the right-hand side of (14), we are left with (for all perturbations $\boldsymbol{\xi}$)

$$\delta \mathbf{I}\Big|_{\mathbf{u}^*}(\boldsymbol{\xi}) = -\int_{t_0}^{t_1} \left\langle H_{\mathbf{u}}\Big|_*, \boldsymbol{\xi} \right\rangle \mathrm{d}t = \mathbf{0} \qquad (17)$$

We already know from Main Lemma that this implies $H_{\mathbf{u}}|_* = \mathbf{0}$ or, in more detail,

$$H_{\mathbf{u}}\big(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)\big) = \mathbf{0} \quad \forall t \in [t_0, t_1]. \qquad (18)$$

The meaning of this condition is that the Hamiltonian has a stationary point as a function of $\mathbf{u}$ along the optimal trajectory.

More precisely, the function $H\left(t,\mathbf{x}^*(t),\mathbf{u},\mathbf{p}^*(t)\right)$ has a stationary point at $\mathbf{u}^*(t)$ for all t.

This is just a reformulation of the property already discussed by us in the context of calculus of variations.

In light of the definition (8) of the Hamiltonian, we can rewrite our control system (0) more compactly as $\dot{\mathbf{x}} = H_{\mathbf{p}}(t, \mathbf{x}, \mathbf{u}, \mathbf{p})$.

Thus the joint evolution of $\mathbf{x}^*$ and $\mathbf{p}^*$ is governed by the system

$$\begin{aligned} \dot{\mathbf{x}}^* &= H_{\mathbf{p}}|_* \\ \dot{\mathbf{p}}^* &= -H_{\mathbf{x}}|_* \end{aligned} \tag{19}$$

which you can recognize as the system of Hamilton's canonical equations.

Let us examine the differential equation for $\mathbf{p}^*$ in (19) in more detail.

We can expand it with the help of (8) as

$$\dot{\mathbf{p}}^* = -(f_{\mathbf{x}})^T\Big|_* \cdot \mathbf{p}^* + L_{\mathbf{x}}\Big|_*$$

where we recall that $f_{\mathbf{x}}$ is the Jacobian matrix of f with respect to $\mathbf{x}$.

This is a linear time-varying system of the form

$$\dot{\mathbf{p}}^* = -A_*^T \cdot \mathbf{p}^* + L_{\mathbf{x}}|_* \tag{20}$$

Here $A_*(.)$ is the same as in the differential equation (7) derived earlier for the first-order state perturbation η.

Definition

Two linear systems $\dot{\mathbf{x}}^* = A\,\mathbf{x}$ and $\dot{\mathbf{z}} = -A^T\,\mathbf{z}$ are said to be **adjoint** to each other, and for this reason $\mathbf{p}$ is called the **adjoint vector**.

Note also that we can think of $\mathbf{p}$ as acting on the state or, more precisely, on the state velocity vector, since it always appears inside inner products such as $\langle \mathbf{p}, \dot{\mathbf{x}} \rangle$.

For this reason, $\mathbf{p}$ is also called the costate of the system.

Let's summarize the results obtained so far and see how to apply them in practice.

The basic optimal control problem can be stated as follows:

Find the control vector $\mathbf{u} = \{u_1, u_2, \dots, u_m\}$ which minimizes the functional, called the performance index,

$$J[\mathbf{u}] = \int_{t_0}^{t_1} L(\mathbf{x}, \mathbf{u}, t)\,\mathrm{d}t + K(t_1, \mathbf{x}(t_1)) \qquad (21)$$

where $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is called the state vector.

Here t is the time parameter; t_1 the terminal time, and L is a function of $\mathbf{x}$, $\mathbf{u}$, and t.

The state variables x_i and the control variables u_j are related as

$$\frac{dx_i}{dt} = f_i(t;\ x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_m)$$

$$(i = 1, 2, \dots, n)$$

or

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{u}) \qquad (22)$$

In many problems, the system is linear and Eq. (22) can be stated as

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u} \tag{23}$$

where $A(.)$ is an $n \times n$ matrix and $B(.)$ is an $n \times m$ matrix.

Further, while finding the control vector $\mathbf{u}$, the state vector $\mathbf{x}$ is to be transferred from a known initial vector $\mathbf{x}_0$ at $t = t_0$ to a terminal vector $\mathbf{x}_f$ at $t = t_1$, where some (or all or none) of the state variables are specified.

Necessary Conditions for Optimal Control

To derive the necessary conditions for the optimal control, we consider the following simple problem: find u which minimizes

$$\mathrm{J}[u] = \int_0^T L(t, x, u)\,\mathrm{d}t \qquad (24)$$

subject to

$$\dot{x} = f(t, x, u) \qquad (25)$$

with the boundary condition $x(0) = x_0$.

To solve this optimal control problem, we introduce a Lagrange multiplier λ and define an augmented functional as follows:

$$\mathbf{J}^{*}[u] = \int_{0}^{T} \{L(t, x, u) + \lambda[f(t, x, u) - \dot{x}]\} \mathrm{d}t \quad (26)$$

Since the integrand

$$F = L + \lambda(f - \dot{x}) \quad (27)$$

is a function of the two variables x and u, we can write the Euler–Lagrange equations for it.

If we introduce the notations

$$y_1 = x, \qquad y_1' = \dot{x}; \qquad y_2 = u, \qquad y_2' = \dot{u}$$

the Euler–Lagrange equations take the form

$$\frac{\partial F}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{y}_1}\right) = 0 \;\rightarrow\; \frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \quad (28)$$

$$\frac{\partial F}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{y}_2}\right) = 0 \;\rightarrow\; \frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{u}}\right) = 0 \quad (29)$$

In view of relation (27), Eqs. (28) and (29) can be expressed as

$$\frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} + \dot{\lambda} = 0 \tag{30}$$

$$\frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} = 0 \tag{31}$$

A new functional H, called the Hamiltonian, is defined as

$$H = L + \lambda f \tag{32}$$

and Eqs. (30) and (31) can be rewritten as

$$-\frac{\partial H}{\partial x} = \dot{\lambda} \tag{33}$$

$$\frac{\partial H}{\partial u} = 0 \tag{34}$$

Equations (33) and (34) represent two first-order differential equations.

The integration of these equations leads to two constants whose values can be found from the known boundary conditions of the problem.

If two boundary conditions are specified as $x(0) = x_0$ and $x(T) = x_T$, the two integration constants can be evaluated without any difficulty.

On the other hand, if only one boundary condition is specified as, say, $x(0) = x_0$, the free-end condition is used as

$$\partial F / \partial \dot{x} = 0 \quad \text{or} \quad \lambda = 0 \quad \text{at} \quad t = T$$

Example 1. Find the optimal control u that makes the functional

$$J = \int_0^1 (x^2 + u^2)\mathrm{d}t \qquad (E_1)$$

stationary with

$$\dot{x} = u \qquad (E_2)$$

and $x(0) = 1$.

Note that the value of x is not specified at $t = 1$.

Solution. The Hamiltonian can be expressed as

$$H = L + \lambda u = x^2 + u^2 + \lambda u \qquad (E_3)$$

and Eqs. (33) and (34) give

$$-2x = \dot{\lambda} \qquad (E_4)$$

$$2u + \lambda = 0 \qquad (E_5)$$

Differentiation of Eq. (E_5) leads to

$$2\dot{u} + \dot{\lambda} = 0 \qquad (E_6)$$

Solution.(continued)

Equations ($\mathbf{E_4}$) and ($\mathbf{E_6}$) yield

$$\dot{u} = x \tag{E_7}$$

Since $\dot{x} = u$ [according to Eq. ($\mathbf{E_2}$)], we obtain

$$\ddot{x} = \dot{u} \quad \rightarrow \quad \ddot{x} = x$$

that is

$$\ddot{x} - x = 0 \tag{E_8}$$

The solution of Eq. ($\mathbf{E_8}$) is given by

$$x(t) = c_1 \sinh t + c_2 \cosh t \tag{E_9}$$

Solution . (continued) Here c_1 and c_2 are constants. By using the initial condition $x(0) = 1$, we obtain $c_2 = 1$.

Since x is not fixed at the terminal point $t = T = 1$, we use the condition $\lambda = 0$ at $t = 1$ in Eq. (E_5) and obtain $u(t = 1) = 0$.

But

$$u = \dot{x} = c_1 \cosh t + \sinh t \qquad (\mathrm{E}_{10})$$

Solution.(continued) Thus

$$u(1) = 0 = c_1 \cosh 1 + \sinh 1$$

or

$$c_1 = -\sinh 1 \,/\cosh 1 \qquad (E_{11})$$

and hence the optimal control is

$$u(t) = -\frac{\sinh 1}{\cosh 1}\cosh t + \sinh t = -\frac{\sinh(1-t)}{\cosh 1} \qquad (E_{12})$$

The corresponding state trajectory is given by

- $$x(t) = \dot{u} = \cosh(1-t)/\cosh 1 \qquad (E_{13})$$

Necessary Conditions for a General Problem

We shall now consider the basic optimal control problem stated earlier: find the optimal control vector u that minimizes

$$\mathbf{J}[\mathbf{u}] = \int_{t_0}^{t_1} L(t, \mathbf{x}, \mathbf{u})\mathbf{d}t \qquad (35)$$

subject to

$$\dot{x}_i = f_i(t, \mathbf{x}, \mathbf{u}) \qquad (36)$$

$$(\mathbf{i} = 1, 2, \ldots, n)$$

Now we introduce a Lagrange multiplier λ_i, also known as the **adjoint variable**, for the i^{th} constraint equation in (36) and form an augmented functional as follows

$$J^*[u] = \int_{t_0}^{t_1} \left[L + \sum_{i=1}^{n} \lambda_i (f_i - \dot{x}_i) \right] dt \qquad (37)$$

The Hamiltonian functional H is defined as

$$H = L + \sum_{i=1}^{n} \lambda_i f_i \qquad (38)$$

So that

$$\mathbf{J}^* = \int_{t_0}^{t_1} \left(H - \sum_{\mathrm{i}=1}^{n} \lambda_{\mathrm{i}}\, \dot{x}_{\mathrm{i}} \right) \mathrm{d}t \tag{39}$$

Since the integrand

$$F = H - \sum_{\mathrm{i}=1}^{n} \lambda_{\mathrm{i}}\, \dot{x}_{\mathrm{i}} \tag{40}$$

depends on $\mathbf{x}$, $\mathbf{u}$, and t, there are $n + m$ dependent variables ($\mathbf{x}$ and $\mathbf{u}$) and hence the Euler–Lagrange equations involve $n + m$ Eqs.

The Euler–Lagrange equations become

$$\frac{\partial F}{\partial x_\mathrm{i}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}_\mathrm{i}}\right) = 0 \quad (\mathrm{i} = 1, 2, \ldots, n) \qquad (41)$$

$$\frac{\partial F}{\partial u_\mathrm{j}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{u}_\mathrm{j}}\right) = 0 \quad (\mathrm{j} = 1, 2, \ldots, m) \qquad (42)$$

In view of relation (40), Eqs. (41) and (42) can be rewritten as

$$-\frac{\partial H}{\partial x_{\mathrm{i}}} = \dot{\lambda}_{\mathrm{i}} \;\; (\mathrm{i} = 1, 2, \dots, n) \qquad (43)$$

$$\frac{\partial H}{\partial u_{\mathrm{j}}} = 0 \;\; (\mathrm{j} = 1, 2, \dots, m) \qquad (44)$$

Equations (43) are known as adjoint equations.

The optimum solutions for $\mathbf{x}$, $\mathbf{u}$, and $\boldsymbol{\lambda}$ can be obtained by solving Eqs. (36), (43), and (44).

There are totally $2n + m$ equations with n x_i's, n λ_i's, and m u_j's as unknowns.

If we know the initial conditions $x_i(t_0)$ for $i = 1, 2, \ldots, n$ and the terminal conditions $x_j(t_1)$ for $j = 1, 2, \ldots, l$ $(l < n)$, we will have the terminal values of the remaining variables, namely, $x_j(t_1)$ for $j = l + 1, l + 2, \ldots, n$ free.

Hence we will have to use the free end conditions

$$\lambda_{\mathrm{j}}(t_1) = 0 \;\; (\mathrm{j} = l + 1, l + 2, \dots, n) \qquad (45)$$

Equations (45) are called the **transversality conditions**.

The purpose of the next exercise is to recover earlier conditions from calculus of variations, namely, the Euler-Lagrange equation and the Lagrange multiplier condition (for multiple degrees of freedom and several non-integral constraints) from the preliminary necessary conditions for optimality derived so far, expressed by the existence of an adjoint vector $\mathbf{p}^*$ satisfying (18) and (19).

Exercise 2.

The standard (unconstrained) calculus of variations problem with n degrees of freedom can be rewritten in the optimal control language by considering the control system

$$\dot{x}_{\mathrm{i}}^{*} = u_{\mathrm{i}} \quad (\mathrm{i} = 1, 2, \dots, n)$$

together with the cost

$$\mathrm{I}(\mathrm{x}) = \int_{t_0}^{t_1} L\big(t, \mathrm{x}(t), \dot{\mathrm{x}}(t)\big)\mathrm{d}t$$

Exercise 2.(continued)

Assuming that a given trajectory satisfies (18) and (19) for this system, prove that the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}_\mathrm{i}} = L_{x_\mathrm{i}}$$

are satisfied along this trajectory.

***

Thank you for attention

text

text

text