



An Accredited Institution of the University of Westminster (UK)

LECTURE 9

MATRIX ALGEBRA AND SIMULTANEOUS LINEAR EQUATIONS

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- The meaning and properties of matrices;
- The arithmetic operations on matrices;
- The applications of matrices to reality

- A **Matrix** is simply a rectangular array of numbers arranged in rows and columns.
- The size of a matrix is indicated by the number of its rows and the number of its columns



- The whole matrix is labeled by a *capital letter*
- The individual numbers (elements) contained in the matrix are labeled by *lower case letters* with a suffix to identify their locations within the matrix.

Examples of matrices

Examples:

$$1) \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \text{ 2x2 matrix}$$

$$2) \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} 3 & 5 & 7 \\ 2 & 4 & 6 \end{pmatrix} \text{ 2x3 matrix}$$

$$3) \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix} ?$$

Addition (and Subtraction) of matrices

- You can add (subtract) two matrices of the same size (equal number of rows and columns).
- The sum (difference) of two equal-sized matrices results in the new matrix of the same size as the two matrices being added.

Example:

$$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} (2+1) & (3+(-2)) & (4+1) \\ (5+2) & (6+3) & (7+5) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 5 \\ 7 & 9 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} (2-1) & (3-(-2)) & (4-1) \\ (5-2) & (6-3) & (7-5) \end{pmatrix} = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

Exercise: Addition and Subtraction

$$1) \begin{pmatrix} 12 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 9 & 6 \end{pmatrix} =$$

$$2) \begin{pmatrix} 12 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 9 & 6 & 2 \end{pmatrix} =$$

$$3) \begin{pmatrix} 12 & 4 & 3 \\ 5 & 6 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 9 & 6 & 2 \end{pmatrix} =$$

$$4) (12 \ 4) - (1 \ 2 \ 3) =$$

Scalar multiplication

- Multiply each element of the matrix by the number.

Example:

$$3 \cdot \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} (3 \cdot 3) & (3 \cdot 4) \\ (3 \cdot 2) & (3 \cdot 5) \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 6 & 15 \end{pmatrix}$$

Exercise: Scalar multiplication

$$1) \frac{1}{2} \cdot \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix} =$$

$$2) \frac{\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}}{2} =$$

Matrix multiplication

- Two matrices can be multiplied only if the number of columns of the 1st matrix equals to the number of the rows of the 2nd matrix.
- Multiply rows of the 1st matrix by columns of the 2nd matrix

Example:

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} (2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3) & (2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1) \\ (1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3) & (1 \cdot 3 + 3 \cdot 2 + 2 \cdot 1) \end{pmatrix} = \begin{pmatrix} 20 & 16 \\ 13 & 11 \end{pmatrix}$$

- The **transpose of matrix** can be obtained by interchanging the rows and columns
- The first row of the matrix A is the first column of matrix A transposed

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}^T = (a_{11} \quad a_{21})$$

Zero & Identity matrix

- **Zero matrix** is a matrix with all elements **0**.
- **Identity matrix** is a **square matrix** with elements of **1s** on the main diagonal from top left to bottom right and **0s** on other positions

Zero matrix

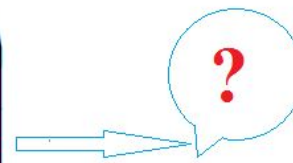
$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Determinant of a matrix

- A numerical value of matrix
- Can be a negative number
- Exists for a square matrix only
- Determinant for 2x2 matrix is calculated as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Inverse of a (2x2) matrix (1)

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ then}$$

$$A^{-1} = \frac{1}{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Inverse of a (2x2) matrix (2)

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then

Calculate:

$$A^{-1} = \frac{1}{(1 \cdot 4) - (2 \cdot 3)} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

Check:

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} = \begin{pmatrix} (1 \cdot (-2) + 2 \cdot 1.5) & (1 \cdot 1 + 2 \cdot (-0.5)) \\ (3 \cdot (-2) + 4 \cdot 1.5) & (3 \cdot 1 + 4 \cdot (-0.5)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Simultaneous linear equations

- A set of linear equations (or functions) considered together, where all the equations have the same unknowns

Example:
$$\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases}$$

$(x; y) = (2; 3)$ is a solution

- The price of a product at which the quantity demanded is equal to the quantity supplied

$$Q_D = Q_S$$

What kind of relationship does price have with quantity demanded and supplied?

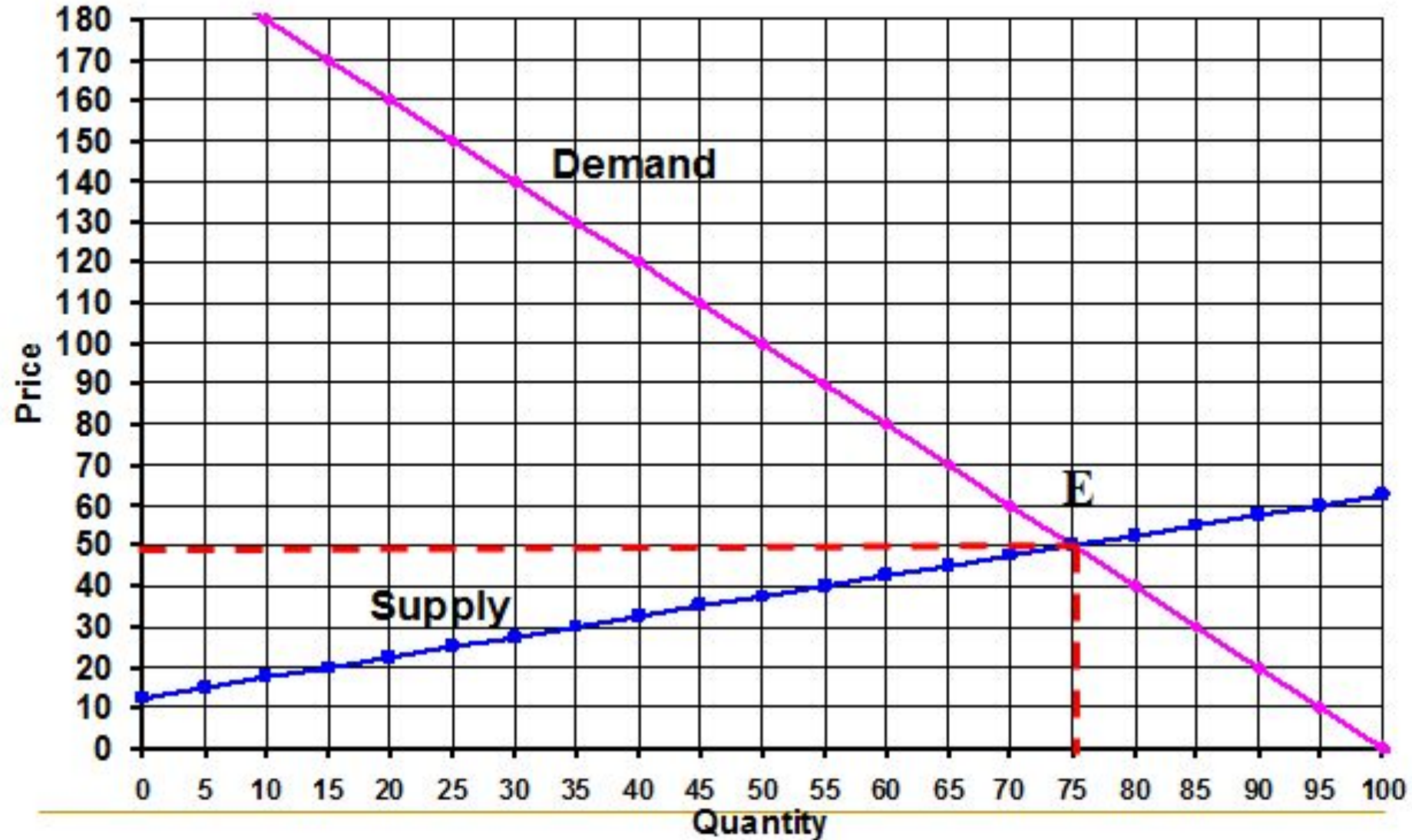
Equilibrium price in practice

If the quantity demanded (Q_D) and quantity supplied (Q_S) have following functions (in terms of price) respectively and at equilibrium price $Q_D = Q_S$, hence

$$\begin{cases} Q_D = -0.5P + 100 \\ Q_S = 2P - 25 \end{cases} \Rightarrow \begin{cases} Q = -0.5P + 100 \\ Q = 2P - 25 \end{cases}$$

$(P; Q) = (50; 75)$ is the equilibrium point

Equilibrium point in graph



Breakeven point in theory

The level of output for which the total revenue is equal to the total cost,

$$TR = TC$$

Saying simply, in such situation, you make
no losses or no gains

Breakeven in practice

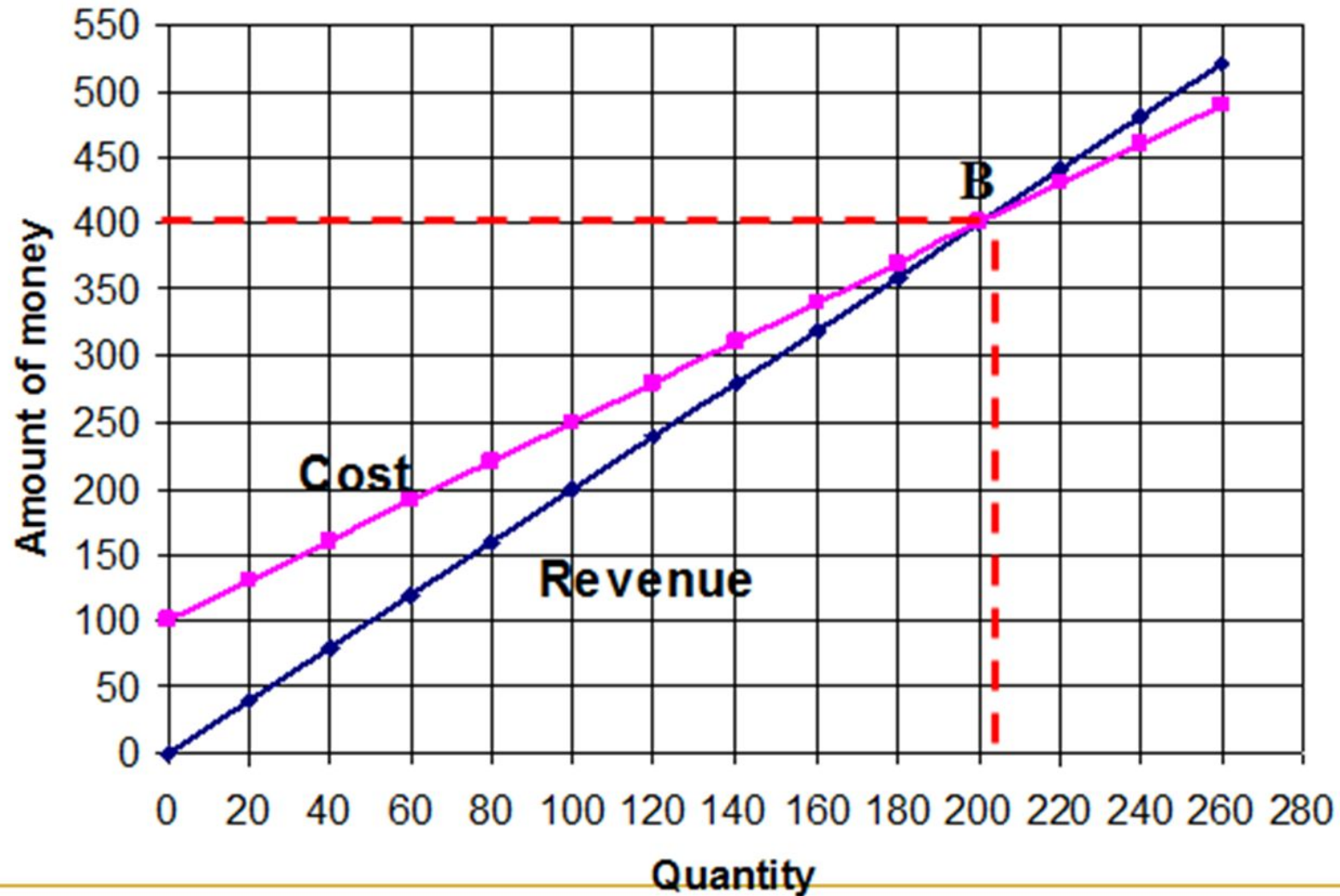
If the total revenue and total cost shown below are confirmed to be true, find the break even point

$$\begin{cases} R = 2Q \\ C = 1.5Q + 100 \end{cases}$$

where, **R** is the revenue,
 Q is the quantity,
 C is the total cost

$(Q; R) = (Q; C) = (200; 400)$ is the breakeven point

Breakeven point in graph



Three methods of solving SLE

- Gauss's method
- Matrix inverse's method
- Cramer's method

Let's solve the following Simultaneous linear equations using each of the three methods above:

$$\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases}$$

Gauss' method (elimination)

- eliminate x or y in the SLEs

$$\begin{cases} 2x + 5y = 19 & | \cdot 1 \\ x + 2y = 8 & | \cdot (-2) \end{cases} \Rightarrow \begin{cases} 2x + 5y = 19 \\ -2x - 4y = -16 \end{cases} \Big| (+) \Rightarrow$$

$$\Rightarrow \begin{cases} 2x + 5y = 19 \\ y = 3 \end{cases} \Rightarrow \begin{cases} 2x + 5 \cdot 3 = 19 \\ y = 3 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 3 \end{cases}$$

Thus, $(x; y) = (2; 3)$ is the solution of the SLEs

Matrix inverse method

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; X = \begin{pmatrix} x \\ y \end{pmatrix}; B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Matrix inverse method

$$\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases} \quad A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \quad B = \begin{pmatrix} 19 \\ 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} =$$

$$= \frac{1}{4 - 5} \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 19 \\ 8 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 19 \\ 8 \end{pmatrix} = \begin{pmatrix} -38 + 40 \\ 19 - 16 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Cramer's method

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; A_x = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}; A_y = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$$

$$x = \frac{|A_x|}{|A|} \qquad y = \frac{|A_y|}{|A|}$$

Cramer's method

Example:

$$\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases} \quad A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 19 & 5 \\ 8 & 2 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 2 & 19 \\ 1 & 8 \end{pmatrix}$$

$$|A| = -1 \quad |A_x| = -2 \quad |A_y| = -3$$

$$x = \frac{|A_x|}{|A|} = \frac{-2}{-1} = 2 \quad y = \frac{|A_y|}{|A|} = \frac{-3}{-1} = 3$$