WESTMINSTER

INTERNATIONAL UNIVERSITY IN TASHKENT

An Accredited Institution of the University of Westminster (UK)

LECTURE 9

MATRIX ALGEBRA AND SIMULTANEOUS LINEAR EQUATIONS

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Lecture outline

- The meaning and properties of matrices;
- The arithmetic operations on matrices;
- The applications of matrices to reality

Matrix

- A **Matrix** is simply a rectangular array of numbers arranged in rows and columns.
- The size of a matrix is indicated by the number of its rows and the number of its columns

- The whole matrix is labeled by a *capital letter*
- The individual numbers (elements) contained in the matrix are labeled by *lower case letters* with a suffix to identify their locations within the matrix.

Examples of matrices

Examples:

1) $$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$ 2x2 matrix

2) $$B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}=\begin{pmatrix} 3 & 5 & 7 \\ 2 & 4 & 6 \end{pmatrix}$$ x3 matrix

3) $$C=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}$$?

Addition (and Subtraction) of matrices

- You can add (subtract) two matrices of the same size (equal number of rows and columns).
- The sum (difference) of two equal-sized matrices results in the new matrix of the same size as the two matrices being added.

Example:

$$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} (2+1) & (3+(-2)) & (4+1) \\ (5+2) & (6+3) & (7+5) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 5 \\ 7 & 9 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} (2-1) & (3-(-2)) & (4-1) \\ (5-2) & (6-3) & (7-5) \end{pmatrix} = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

Exercise: Addition and Subtraction

1) $\begin{pmatrix} 12 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 9 & 6 \end{pmatrix} =$

2) $\begin{pmatrix} 12 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 9 & 6 & 2 \end{pmatrix} =$

3) $\begin{pmatrix} 12 & 4 & 3 \\ 5 & 6 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 9 & 6 & 2 \end{pmatrix} =$

4) $\begin{pmatrix} 12 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} =$

Scalar multiplication

- Multiply each element of the matrix by the number.

Example:

$$3 \cdot \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} (3 \cdot 3) & (3 \cdot 4) \\ (3 \cdot 2) & (3 \cdot 5) \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 6 & 15 \end{pmatrix}$$

Exercise: Scalar multiplication

$$1)\ \frac{1}{2}\cdot\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix} =$$

$$2)\ \frac{\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}}{2} =$$

Matrix multiplication

- Two matrices can be multiplied only if the number of columns of the 1st matrix equals to the number of the rows of the 2nd matrix.
- Multiply rows of the 1st matrix by columns of the 2nd matrix

Example:

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} (2\cdot 1+3\cdot 2+4\cdot 3) & (2\cdot 3+3\cdot 2+4\cdot 1) \\ (1\cdot 1+3\cdot 2+2\cdot 3) & (1\cdot 3+3\cdot 2+2\cdot 1) \end{pmatrix} = \begin{pmatrix} 20 & 16 \\ 13 & 11 \end{pmatrix}$$

Matrix Transpose

- The transpose of matrix can be obtained by interchanging the rows and columns
- The first row of the matrix A is the first column of matrix A transposed

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix}$$

Zero & Identity matrix

- Zero matrix is a matrix with all elements **0.**
- Identity matrix is a **square matrix** with elements of **1s** on the main diagonal from top left to bottom right and **0s** on other positions

Zero matrix

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \text{?}$$

Determinant of a matrix

- A numerical value of matrix
- Can be a negative number
- Exists for a square matrix only
- Determinant for 2x2 matrix is calculated as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Inverse of a (2x2) matrix (1)

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

$$A^{-1} = \frac{1}{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Inverse of a (2x2) matrix (2)

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then

Calculate:

$$A^{-1} = \frac{1}{(1 \cdot 4) - (2 \cdot 3)} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

Check:

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} = \begin{pmatrix} (1 \cdot (-2) + 2 \cdot 1.5) & (1 \cdot 1 + 2 \cdot (-0.5)) \\ (3 \cdot (-2) + 4 \cdot 1.5) & (3 \cdot 1 + 4 \cdot (-0.5)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Simultaneous linear equations

- A set of linear equations (or functions) considered together, where all the equations have the same unknowns

Example: $\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases}$

$(x; y) = (2; 3)$ is a solution

Equilibrium price in theory

- The price of a product at which the quantity demanded is equal to the quantity supplied

$$Q_D = Q_S$$

What kind of relationship does price have with quantity demanded and supplied?

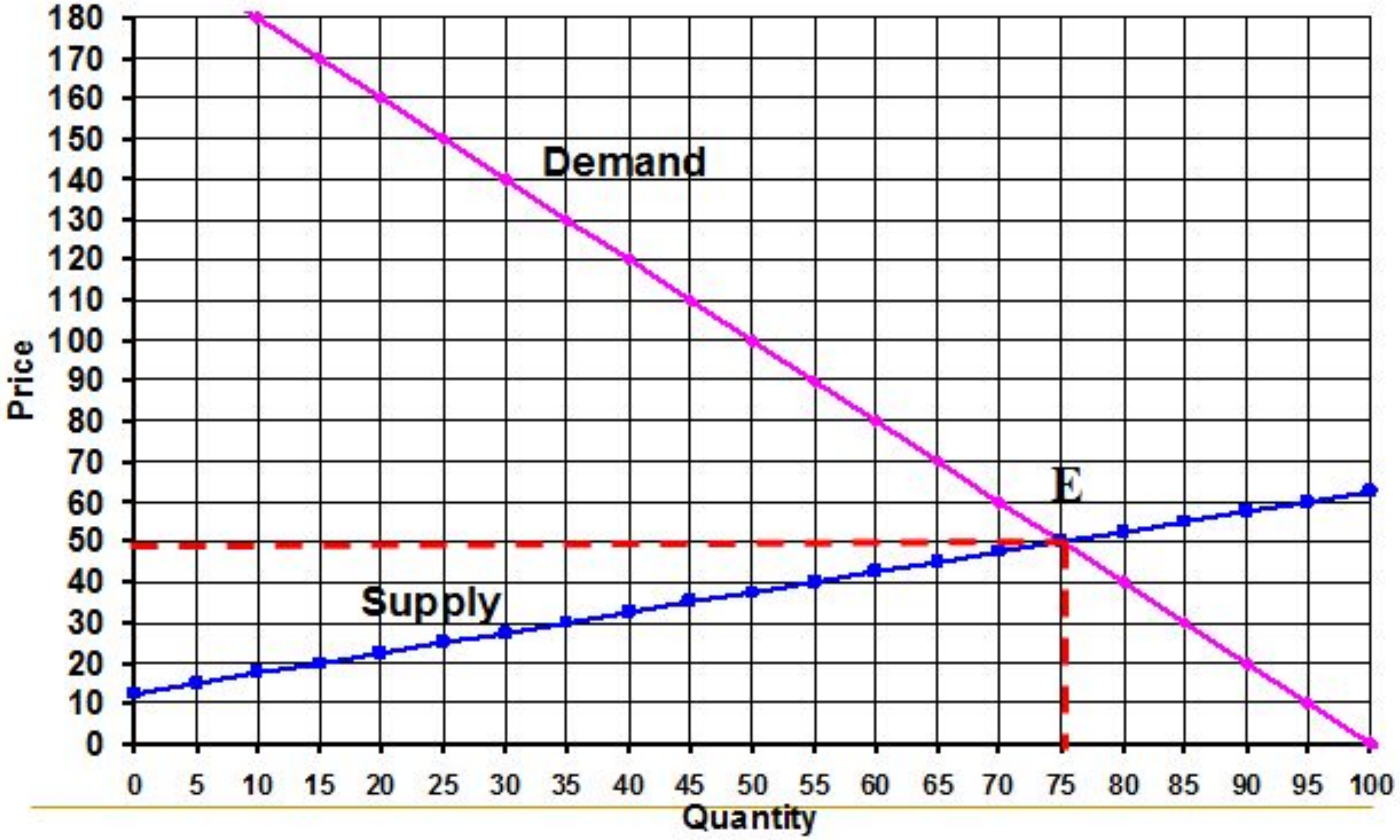
Equilibrium price in practice

If the quantity demanded (Q_D) and quantity supplied (Q_S) have following functions (in terms of price) respectively and at equilibrium price $Q_D = Q_S$, hence

$$\begin{cases} Q_D = -0.5P + 100 \\ Q_S = \quad 2P - 25 \end{cases} \Rightarrow \begin{cases} Q = -0.5P + 100 \\ Q = \quad 2P - 25 \end{cases}$$

$(P;\ Q) = (50;\ 75)$ is the equilibrium point

Equilibrium point in graph



Breakeven point in theory

The level of output for which the total revenue is equal to the total cost,

TR = TC

Saying simply, in such situation, you make ***no losses*** or ***no gains***

Breakeven in practice

If the total revenue and total cost shown below are confirmed to be true, find the break even point

$$\begin{cases} R = 2Q \\ C = 1.5Q + 100 \end{cases}$$

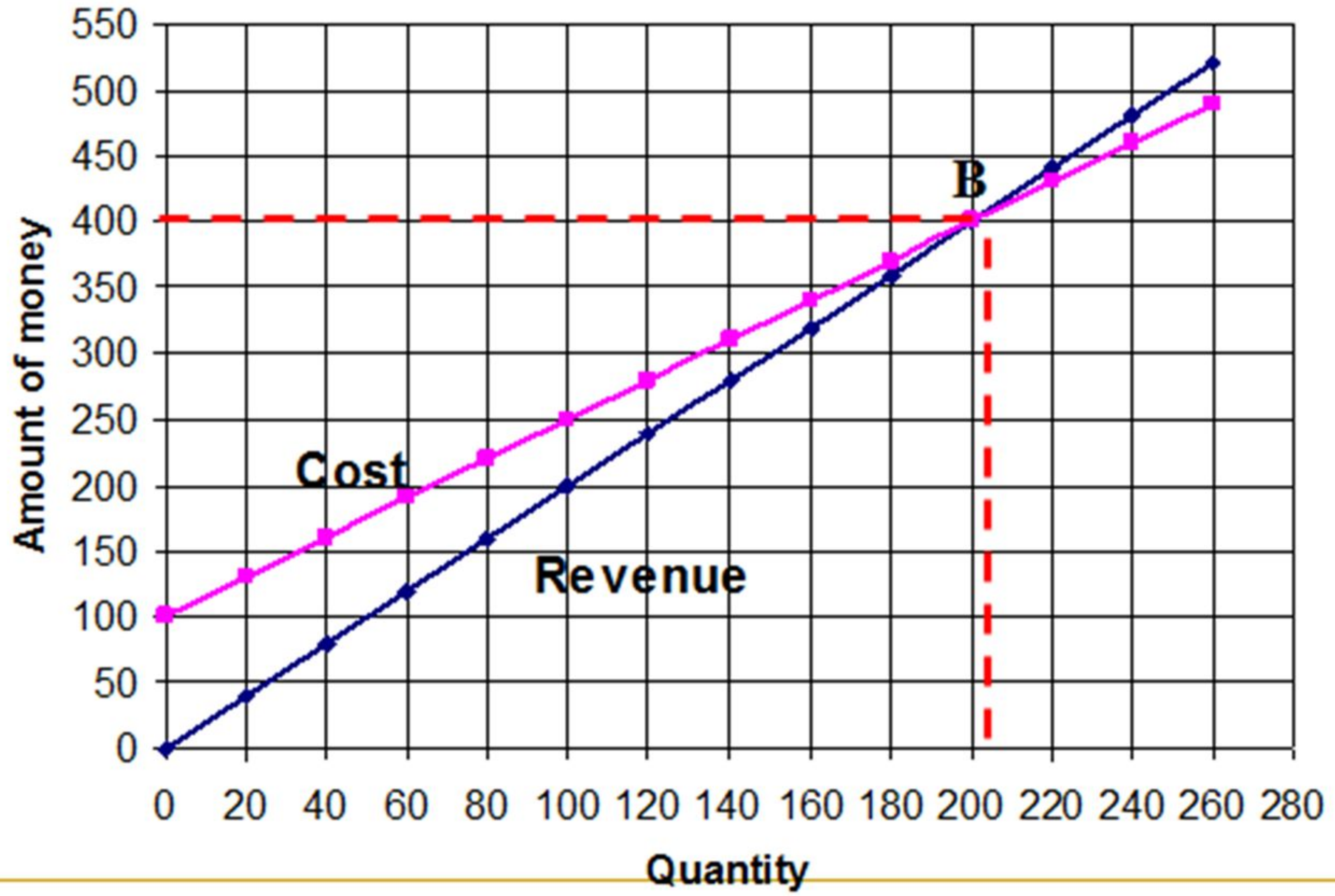
where, **R** is the revenue,

Q is the quantity,

C is the total cost

$(Q;\ R)=(Q;\ C)=(200;400)$ *is* the breakeven point

Breakeven point in graph



Three methods of solving SLE

- Gauss's method
- Matrix inverse's method
- Cramer's method

Let's solve the following Simultaneous linear equations using each of the three methods above:

$$\begin{cases} 2x + 5y = 19 \\ x + 2y = 8 \end{cases}$$

Gauss' method (elimination)

- eliminate x or y in the SLEs

$$\begin{cases} 2x+5y=19 & |\cdot 1 \\ x+2y=8 & |\cdot(-2) \end{cases} \Rightarrow \left\{ \begin{array}{rl} 2x+5y= & 19 \\ -2x-4y= & -16 \end{array} \right| (+) \Rightarrow$$

$$\Rightarrow \begin{cases} 2x+5y=19 \\ y=3 \end{cases} \Rightarrow \begin{cases} 2x+5\cdot 3=19 \\ y=3 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=3 \end{cases}$$

Thus, $(x;\ y)=(2;\ 3)$ is the solution of the SLEs

Matrix inverse method

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

$$if \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \; X = \begin{pmatrix} x \\ y \end{pmatrix}; \; B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\mathbf{X = A^{-1}B}$$

Matrix inverse method

$$\begin{cases} 2x+5y=19 \\ x+2y=8 \end{cases} \quad A=\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}; \; X=\begin{pmatrix} x \\ y \end{pmatrix}; \; B=\begin{pmatrix} 19 \\ 8 \end{pmatrix}$$

$$A^{-1}=\frac{1}{(a_{11}\cdot a_{22})-(a_{12}\cdot a_{21})}\cdot\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}=$$

$$=\frac{1}{4-5}\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 19 \\ 8 \end{pmatrix}=\begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 19 \\ 8 \end{pmatrix}=\begin{pmatrix} -38+40 \\ 19-16 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Cramer's method

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};\ A_x = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix};\ A_y = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$$

$$x = \frac{|A_x|}{|A|} \qquad\qquad y = \frac{|A_y|}{|A|}$$

Cramer's method

Example:

$$\begin{cases} 2x+5y=19 \\ x+2y=8 \end{cases} \quad A=\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}; \quad A_1=\begin{pmatrix} 19 & 5 \\ 8 & 2 \end{pmatrix}; \quad A_2=\begin{pmatrix} 2 & 19 \\ 1 & 8 \end{pmatrix}$$

$$|A|=-1 \qquad |A_x|=-2 \qquad |A_y|=-3$$

$$x=\frac{|A_x|}{|A|}=\frac{-2}{-1}=2 \qquad y=\frac{|A_y|}{|A|}=\frac{-3}{-1}=3$$