Lecture 6.

Techniques of Integration (part 1)

BECAUSE OF THE FUNDAMENTAL THEOREM of Calculus, we can integrate a function if we know an antiderivative, that is, an indefinite integral. We summarize here the most important integrals that we have learned so far.

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \ a > 0$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

In this chapter we develop techniques for using these basic integration formulas to obtain indefinite integrals of more complicated functions. We learned the most important method of integration, the Substitution Rule, in Section 5.5. The other general technique, integration by parts, is presented in Section 7.1. Then we learn methods that are special to particular classes of functions, such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore we discuss a strategy for integration in Section 7.5.

7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The integration rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

Integration by Parts: Indefinite Integrals

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

1
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

2
$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Find $\int x \sin x \, dx$.

SOLUTION USING FORMULA 1 Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\int x \sin x \, dx = f(x)g(x) - \int g(x)f'(x) \, dx$$

$$= x(-\cos x) - \int (-\cos x) \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

It is helpful to use the pattern:

$u = \square \qquad dv = \square$

$du = \square \qquad v = \square$

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

SOLUTION USING FORMULA 2 Let

$$u = x \qquad dv = \sin x \, dx$$

Then $\qquad du = dx \qquad v = -\cos x$

and so

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

■

NOTE Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x\, dx$ and expressed it in terms of the simpler integral $\int \cos x\, dx$. If we had instead chosen $u = \sin x$ and $dv = x\, dx$, then $du = \cos x\, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x\, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\, dx$$

Although this is true, $\int x^2 \cos x\, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\, dx$ can be readily integrated to give v.

EXAMPLE 2 Evaluate $\int \ln x\, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\, dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx$$

It's customary to write $\int 1\, dx$ as $\int dx$.

$$= x \ln x - \int dx$$

Check the answer by differentiating it.

$$= x \ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

EXAMPLE 3 Find $\int t^2 e^t\, dt$.

SOLUTION Notice that e^t is unchanged when differentiated or integrated whereas t^2 becomes simpler when differentiated, so we choose

$$u = t^2 \qquad dv = e^t\, dt$$

Then

$$du = 2t\, dt \qquad v = e^t$$

Integration by parts gives

$$\int t^2 e^t\, dt = t^2 e^t - 2\int t e^t\, dt \tag{3}$$

The integral that we obtained, $\int t e^t\, dt$, is simpler than the original integral but is still not obvious. Therefore we use integration by parts a second time, this time with $u = t$ and $dv = e^t\, dt$. Then $du = dt$, $v = e^t$, and

$$\int t e^t\, dt = t e^t - \int e^t\, dt$$

$$= t e^t - e^t + C$$

Putting this in Equation 3, we get

$$\int t^2 e^t\, dt = t^2 e^t - 2\int t e^t\, dt$$

$$= t^2 e^t - 2(t e^t - e^t + C)$$

$$= t^2 e^t - 2t e^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C$$

■

EXAMPLE 4 Evaluate $\int e^x \sin x \, dx$.

An easier method, using complex numbers, is given in Exercise 50 in Appendix H.

SOLUTION Neither e^x nor $\sin x$ becomes simpler when differentiated, so let's try choosing $u = e^x$ and $dv = \sin x \, dx$. (It turns out that, in this example, choosing $u = \sin x$, $dv = e^x \, dx$ also works.) Then $du = e^x \, dx$ and $v = -\cos x$, so integration by parts gives

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \tag{4}$$

The integral that we have obtained, $\int e^x \cos x \, dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. It is important that we again choose $u = e^x$, so $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \tag{5}$$

Figure 1 illustrates Example 4 by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2}e^x(\sin x - \cos x)$. As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.

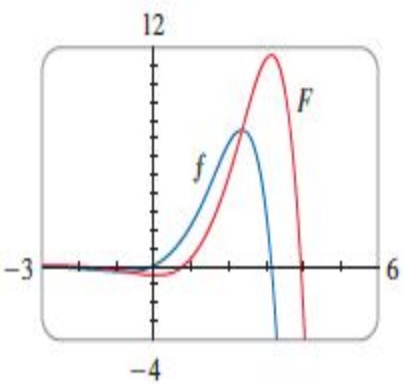

FIGURE 1

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x \, dx$, which is where we started. However, if we put the expression for $\int e^x \cos x \, dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x \, dx$ to both sides, we obtain

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x \, dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$

Since $\tan^{-1}x \geqslant 0$ for $x \geqslant 0$, the integral in Example 5 can be interpreted as the area of the region shown in Figure 2.

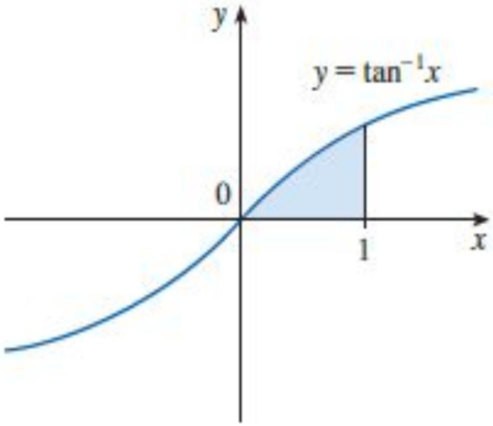

FIGURE 2

Integration by Parts: Definite Integrals

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

6
$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

EXAMPLE 5 Calculate $\int_0^1 \tan^{-1}x\,dx$.

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then
$$du = \frac{dx}{1+x^2} \qquad v = x$$

So Formula 6 gives

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 1 \cdot \tan^{-1}1 - 0 \cdot \tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\int_0^1 \frac{x}{1 + x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{dt}{t} = \tfrac{1}{2}\ln|t|\Big]_1^2$$

$$= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2$$

Therefore

$$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$

■

Reduction Formulas

The preceding examples show that integration by parts often allows us to express one integral in terms of a simpler one. If the integrand contains a power of a function, we can sometimes use integration by parts to reduce the power. In this way we can find a *reduction formula* as in the next example.

Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n-1$ and $n-2$.

EXAMPLE 6 Prove the reduction formula

$$\boxed{7} \qquad \int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let

$$u = \sin^{n-1}x \qquad dv = \sin x\,dx$$

Then

$$du = (n-1)\sin^{n-2}x\cos x\,dx \qquad v = -\cos x$$

and integration by parts gives

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \qquad \blacksquare$$

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

7.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

Integrals of Powers of Sine and Cosine

We begin by considering integrals in which the integrand is a power of sine, a power of cosine, or a product of these.

EXAMPLE 1 Evaluate $\int \cos^3 x \, dx$.

SOLUTION Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x \, dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. Thus here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x)\cos x$$

We can then evaluate the integral by substituting $u = \sin x$, so $du = \cos x \, dx$ and

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x)\cos x \, dx \\ &= \int (1 - u^2)\, du = u - \tfrac{1}{3}u^3 + C \\ &= \sin x - \tfrac{1}{3}\sin^3 x + C \end{aligned}$$

■

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine). The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

EXAMPLE 2 Find $\int \sin^5 x \cos^2 x\, dx$.

SOLUTION We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor. Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x\, dx$ and so

$$\int \sin^5 x \cos^2 x\, dx = \int (\sin^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6)\, du$$

$$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

■

Figure 1 shows the graphs of the integrand $\sin^5 x \cos^2 x$ in Example 2 and its indefinite integral (with $C = 0$). Which is which?

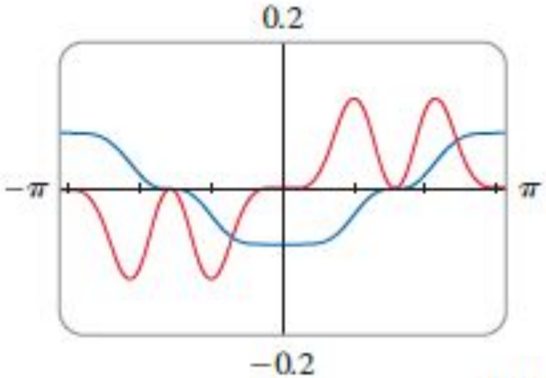

FIGURE 1

In the preceding examples, an odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power. If the integrand contains even powers of both sine and cosine, this strategy fails. In this case, we can take advantage of the following half-angle identities (see Equations 18b and 18a in Appendix D):

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

EXAMPLE 3 Evaluate $\int_0^\pi \sin^2 x \, dx$.

SOLUTION If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\int_0^\pi \sin^2 x \, dx = \tfrac{1}{2} \int_0^\pi (1 - \cos 2x) \, dx$$

$$= \left[\tfrac{1}{2}\left(x - \tfrac{1}{2} \sin 2x\right)\right]_0^\pi$$

$$= \tfrac{1}{2}\left(\pi - \tfrac{1}{2} \sin 2\pi\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2} \sin 0\right) = \tfrac{1}{2}\pi$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$. Another method for evaluating this integral was given in Exercise 7.1.53. ■

Example 3 shows that the area of the region shown in Figure 2 is $\pi/2$.

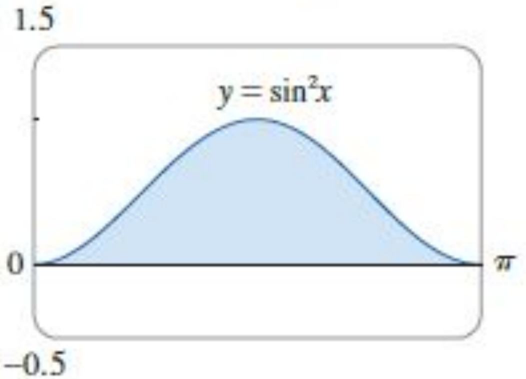

FIGURE 2

EXAMPLE 4 Find $\int \sin^4 x\, dx$.

SOLUTION We could evaluate this integral using the reduction formula for $\int \sin^n x\, dx$ (Equation 7.1.7) together with Example 3 (as in Exercise 7.1.53), but a better method is to write $\sin^4 x = (\sin^2 x)^2$ and use a half-angle formula:

$$\begin{aligned} \int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left[\tfrac{1}{2}(1 - \cos 2x)\right]^2 dx \\ &= \tfrac{1}{4} \int \left[1 - 2\cos 2x + \cos^2(2x)\right] dx \end{aligned}$$

Since $\cos^2(2x)$ occurs, we use the half-angle formula for cosine to write

$$\cos^2(2x) = \tfrac{1}{2}[1 + \cos(2 \cdot 2x)] = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\begin{aligned} \int \sin^4 x\, dx &= \tfrac{1}{4} \int \left[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\right] dx \\ &= \tfrac{1}{4} \int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{1}{4}\left(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right) + C \end{aligned}$$

■

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$, where $m \geqslant 0$ and $n \geqslant 0$ are integers.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$. See Example 1.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. See Example 2.

[Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

See Examples 3 and 4.

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Integrals of Powers of Secant and Tangent

We use similar reasoning to evaluate integrals of the form $\int \tan^m x \sec^n x\, dx$. Because $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$. Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

EXAMPLE 5 Evaluate $\int \tan^6 x \sec^4 x\, dx$.

SOLUTION If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$. We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x\, dx$:

$$\begin{aligned}\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x \sec^2 x \sec^2 x\, dx \\ &= \int \tan^6 x\, (1 + \tan^2 x) \sec^2 x\, dx \\ &= \int u^6 (1 + u^2)\, du = \int (u^6 + u^8)\, du \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C = \tfrac{1}{7} \tan^7 x + \tfrac{1}{9} \tan^9 x + C\end{aligned}$$

■

EXAMPLE 6 Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

SOLUTION If we separate a $\sec^2\theta$ factor, as in the preceding example, we are left with a $\sec^5\theta$ factor, which isn't easily converted to tangent. However, if we separate a $\sec\theta \tan\theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$. We can then evaluate the integral by substituting $u = \sec\theta$, so $du = \sec\theta \tan\theta \, d\theta$:

$$
\begin{aligned}
\int \tan^5\theta \sec^7\theta \, d\theta &= \int \tan^4\theta \sec^6\theta \sec\theta \tan\theta \, d\theta \\
&= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \, d\theta \\
&= \int (u^2 - 1)^2 u^6 \, du \\
&= \int (u^{10} - 2u^8 + u^6) \, du \\
&= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \\
&= \tfrac{1}{11}\sec^{11}\theta - \tfrac{2}{9}\sec^9\theta + \tfrac{1}{7}\sec^7\theta + C
\end{aligned}
$$

■

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(a) If the power of secant is even ($n = 2k,\ k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x\, (\sec^2 x)^{k-1} \sec^2 x\, dx$$

$$= \int \tan^m x\, (1 + \tan^2 x)^{k-1} \sec^2 x\, dx$$

Then substitute $u = \tan x$. See Example 5.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x\ \sec x\ \tan x\, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x\ \sec x\ \tan x\, dx$$

Then substitute $u = \sec x$. See Example 6.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity. We will sometimes need to be able to integrate $\tan x$ by using the formula established in (5.5.5):

$$\int \tan x\, dx = \ln|\sec x| + C$$

We will also need the indefinite integral of secant:

1

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by $\sec x + \tan x$:

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)\, dx$, so the integral becomes $\int (1/u)\, du = \ln|u| + C$. Thus we have

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Formula 1 was discovered by James Gregory in 1668. (See his biography in Section 3.4.) Gregory used this formula to solve a problem in constructing nautical tables.

EXAMPLE 7 Find $\int \tan^3 x \, dx$.

SOLUTION Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \tfrac{1}{2} \tan^2 x - \ln |\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x \, dx$. ■

If an even power of tangent appears with an odd power of secant, it is helpful to express the integrand completely in terms of $\sec x$. Powers of $\sec x$ may require integration by parts, as shown in the following example.

EXAMPLE 8 Find $\int \sec^3 x\, dx$.

SOLUTION Here we integrate by parts with

$$u = \sec x \qquad dv = \sec^2 x\, dx$$
$$du = \sec x \tan x\, dx \qquad v = \tan x$$

Then
$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\, dx = \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$ ■

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 8.

Finally, integrals of the form

$$\int \cot^m x \csc^n x\, dx$$

can be found in a similar way by using the identity $1 + \cot^2 x = \csc^2 x$.

Using Product Identities

The following product identities are useful in evaluating certain trigonometric integrals.

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x \, dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x \, dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx$$

$$= \tfrac{1}{2} \int (-\sin x + \sin 9x) \, dx$$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{9}\cos 9x\right) + C$$

These product identities are discussed in Appendix D.