Mathematical analysis (3 term)

Lectures – 26 hours

Practice -24 hours

Test (зачет)

Auto-test conditions –

Practical works and 2 control tests

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Structure of course

- Multivariable functions (recall)
- Multiple integrals
- Differential equations
- Line integrals

Multivariable functions.

Let number y is assigned to every point $M(x_1, x_2, ..., x_n) \in V \subset R^n$. So function of n variables is defined – $y = y(M)$ or $y = f(x_1, x_2, ..., x_n)$. Sometimes we denote it as $f : V \to R$.

Let investigate partial case – function of two variables. Function of two variables is a rule that assigns to every couple of independent variables x, y (arguments) the value of depended variable z (function).

We shall denote this function as $z = f(x, y)$ or $z = z(x, y)$ (or we shall use any other letter instead of z - $u = f(x, y)$).

Because ordered couple of values x, y defines the point on the plane, then we also denote function as $z = z(M)$ where M is the point of the plane XOY with coordinates (x, y).

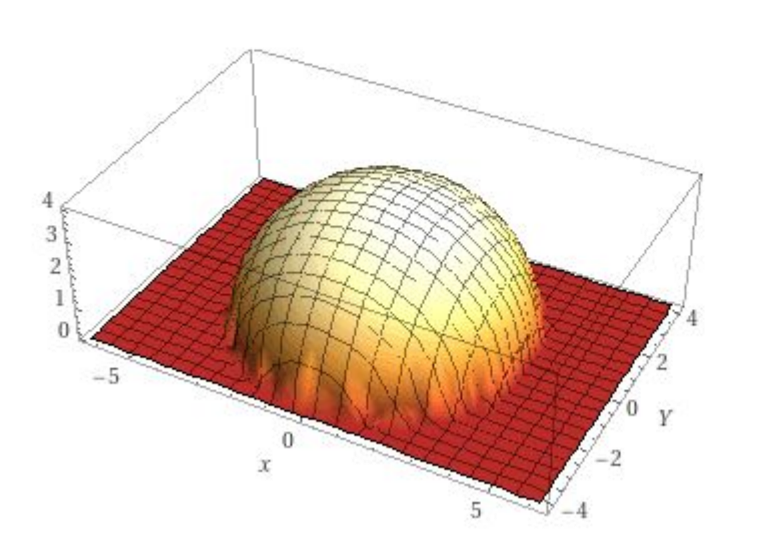
Geometrically the graph of the function of two variables is a **surface** in three-dimensional **space.** Usually we learn just a space, but sometimes the graph may be, for example, a space curve or line (lines) or even a single point.

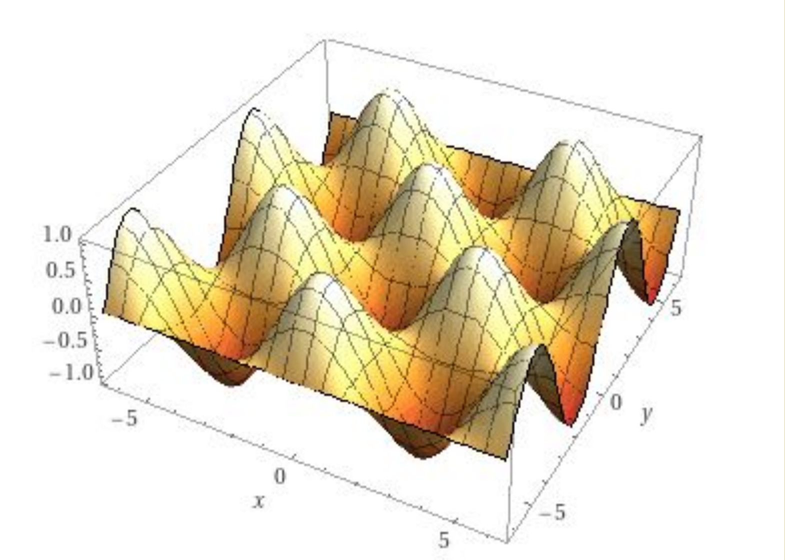
The **domain of function** $z = f(x, y)$ is the set of all possible values of x and y. (Set of inputs)

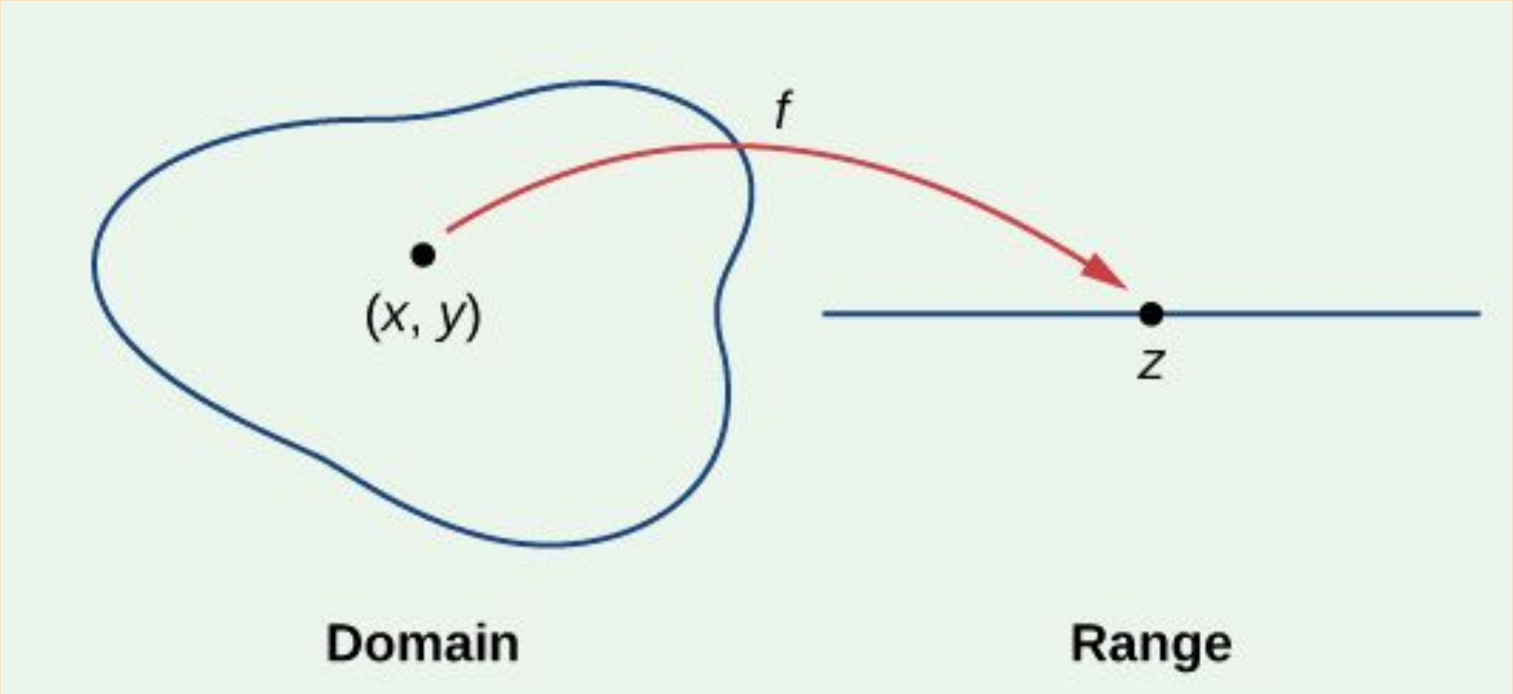
The **range** is a set of all possible values of z from the domain. (Set of outputs)

$$z = \sqrt{16 - x^2 - y^2}$$

$$z = \cos x \cdot \sin y$$







f
(x, y)
z
Domain
Range

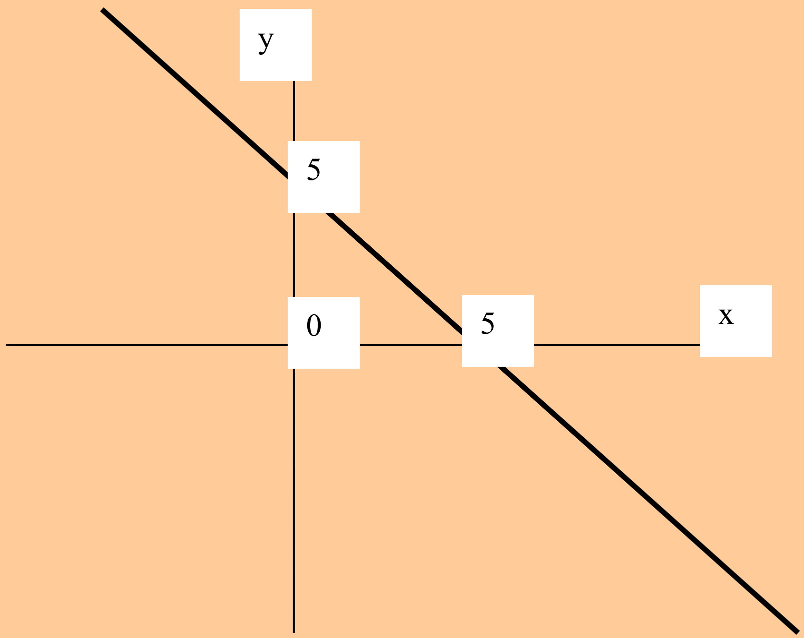
Example 1. Find the domain of $z = \dfrac{x^2 + 4xy - 3}{x + y - 5}$.

Solution. Denominator mustn't be equal to zero:

$x + y - 5 \neq 0$

$y \neq 5 - x$

Answer. All coordinate plane except points belonged to line $y = 5 - x$



Ex.2. Find the domain of $f(x,y)=\sqrt{3y+2}$.

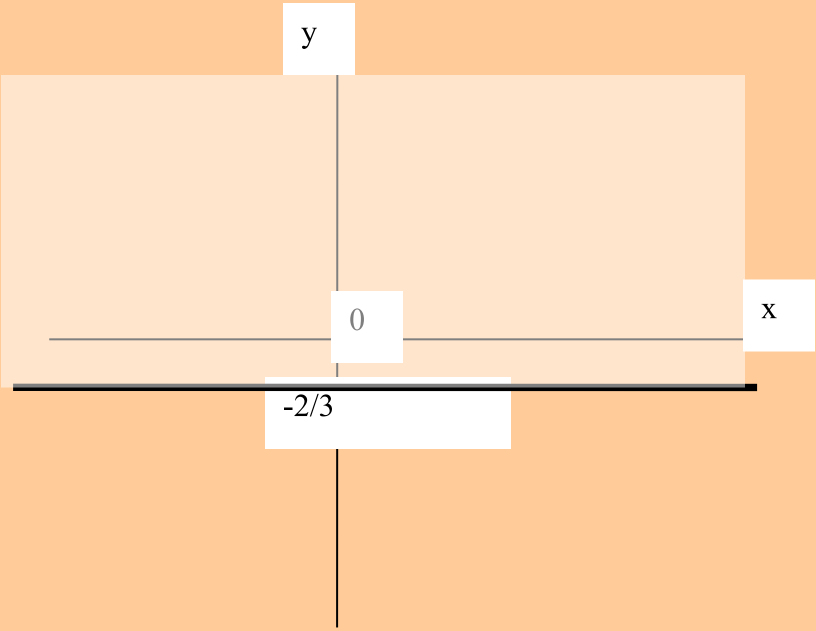
Solution. Root expression must be non-negative

$$3y+2\geq 0$$

$$3y\geq -2$$

$$y\geq -\frac{2}{3}$$

Answer. Half plane $y\geq -\frac{2}{3}$



Example 3. Find the domain of $f(x, y) = -\frac{2y}{\sqrt{x-1}}$.

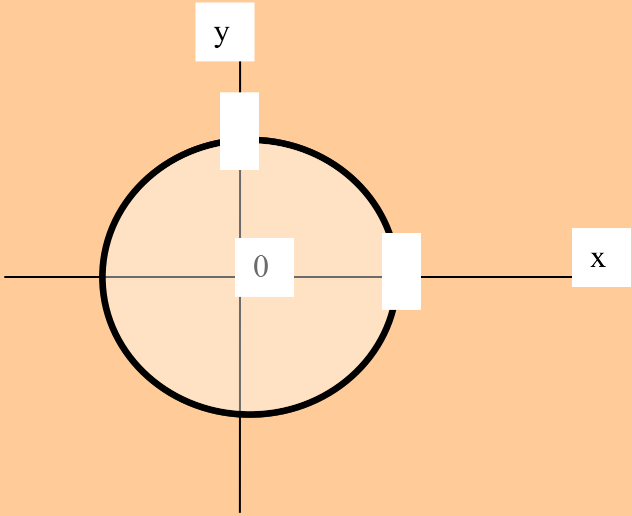
Solve it by yourselves!

Ex.4. Find the domain of $f(x, y) = \frac{xy}{\sqrt{5 - x^2 - y^2}}$.

Solution. Root expression must be non-negative and mustn't turn to zero, because it is denominator. So root expression must be positive:

$5 - x^2 - y^2 > 0$

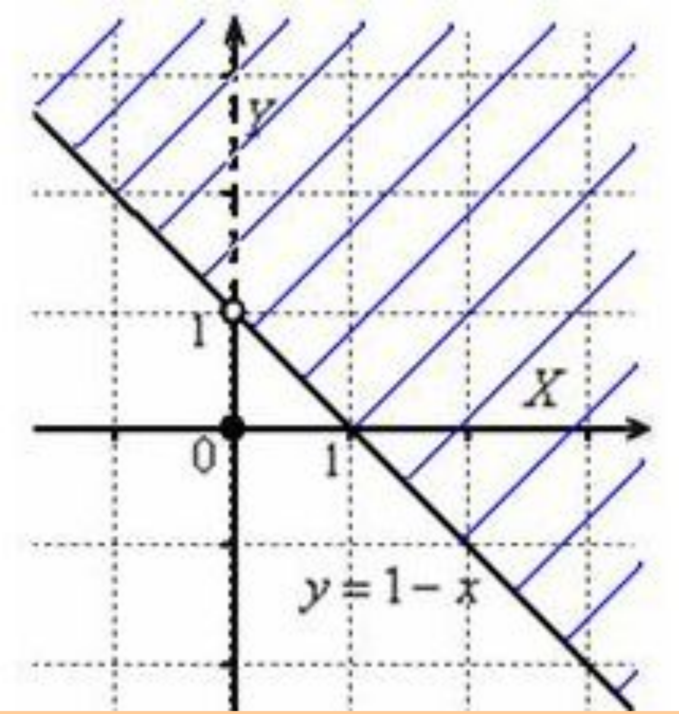
$x^2 + y^2 < 5$



Ex.5. Find the domain of $f(x,y)=\dfrac{\sqrt{x+y-1}}{x}$.

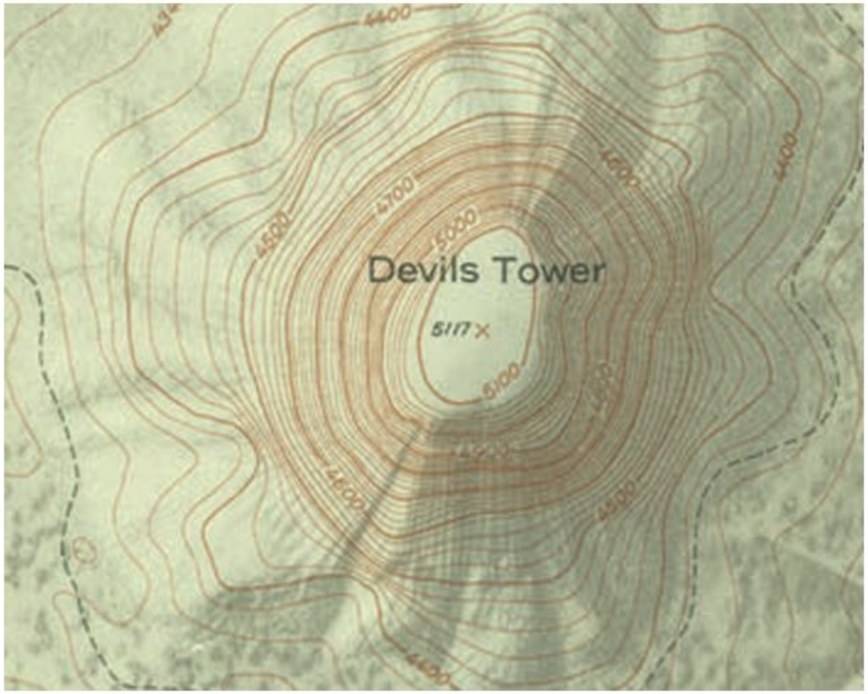
Solution. Root expression must be non-negative and denominator must be non-zero. So, domain is defined by the system of inequalities:

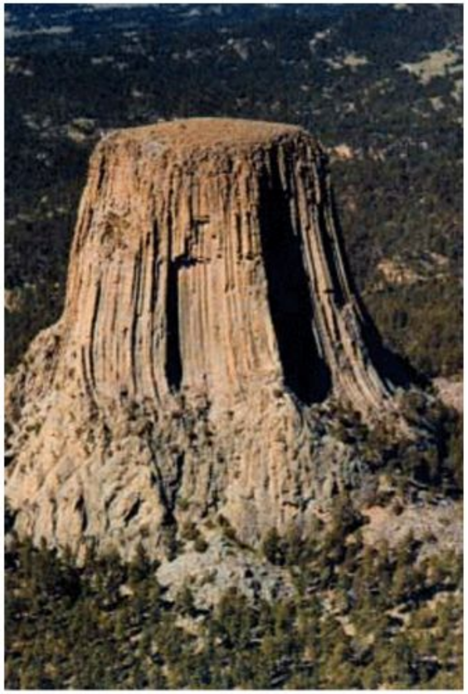
$$\begin{cases} x+y-1\geq 0 \\ x\neq 0 \end{cases}$$



Level curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation. A level curve of a function of two variables f(x,y) is completely analogous to a contour line on a topographical map.





Definition. Given a function $f(x,y)$ and a number c in the range of f, a **level curve** of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x,y)=c$.

So, level curves are horizontal "cuts" of surface taken on different heights. These cuts or cross sections are drown by planes $z=C=const$ after that they are projected on plane XOY

Example 6. Find some level curves for the function $z(x,y)=\sqrt{9-x^2-y^2}$.

Solution. The range of z is closed interval [0; 3]. First we choose any number in this interval – say, c=2. The level curve corresponding to c=2 is described by equation:

$$\sqrt{9-x^2-y^2}=2$$

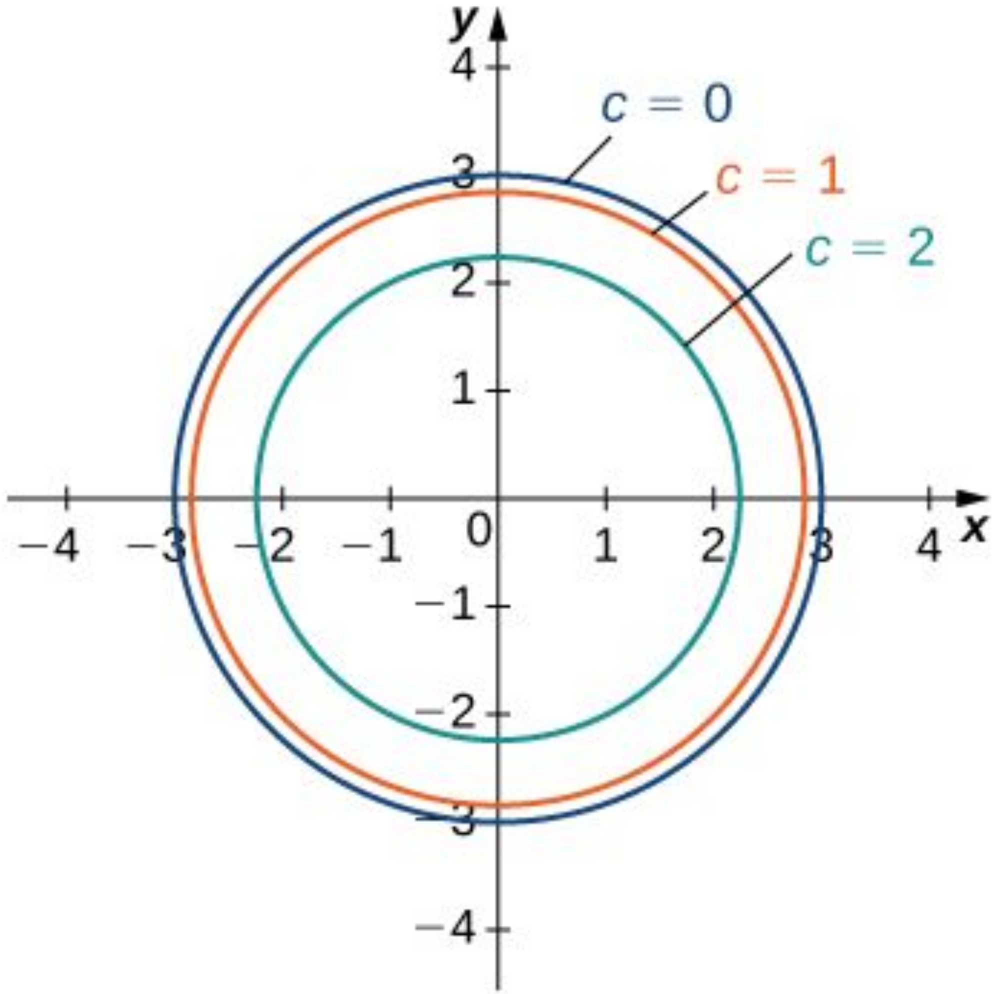
To simplify, square both sides of this equation:

$$9-x^2-y^2=4$$

Then we can rewrite this as

$$x^2+y^2=5$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If c=3, then the circle has radius 0, so it consists solely of the origin. At the bottom you can see curves of this function corresponding to c=0, 1, 2, and 3.



y
x
c = 0
c = 1
c = 2
4
3
2
1
0
−1
−2
−3
−4

Multivariable functions.
Partial derivatives and partial differentials

To simplify the recording and presentation, we will now limit ourselves by the case of a function of three variables. However, all of the following will be true for functions of any number of variables.

So, let function $u = f(x, y, z)$ is given in some domain M, we take point $M_0(x_0, y_0, z_0)$ in this domain. If we fixed constant values $y = y_0$ and $z = z_0$ will change variable x, then our function $u = f(x, y_0, z_0)$ will be function of **single** value x. And now we can investigate the calculating of its derivative in some point $x = x_0$. Let's give this value x_0 an increment Δx, and then the function will get an increment

$$\Delta_x u = \Delta_x f(x_0, y_0, z_0) = f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)$$

This function increment is called partial increment (in relation to x), because it is caused by changing the value of only one variable. By the derivative definition it is the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}.$$

This derivative is called the partial derivative of function $u = f(x, y, z)$ in relation to variable x at the point (x_0, y_0, z_0).

Partial derivatives

$$\frac{\partial u}{\partial x} \quad \frac{\partial f(x_0, y_0, z_0)}{\partial x}$$

$$u'_x \quad f'_x(x_0, y_0, z_0)$$

$$D_x u \quad D_x f(x_0, y_0, z_0)$$

Notations of partial derivatives (in relation to x)

Analogically

$$\lim_{\Delta y \to 0} \frac{\Delta_y u}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y, z_0) - f(x_0, y_0, z_0)}{\Delta y}.$$

partial derivatives (in relation to y)

$$\frac{\partial u}{\partial y}, \ \frac{\partial f(x_0, y_0, z_0)}{\partial y}; \quad u'_y, \ f'_y(x_0, y_0, z_0); \quad D_y u, \ D_y f(x_0, y_0, z_0).$$

Notations of partial derivatives (in relation to y)

Examples

Example 7 Find partial derivatives of function $f(x,y)=2x^3y^2+2x+4y$

$$\frac{\partial f}{\partial x}=6x^2y^2+2 \quad , \quad \frac{\partial f}{\partial y}=4x^3y+4$$

Example 8 Find partial derivatives of function $f(x,y)=x^y$

$$\frac{\partial f}{\partial x}=y\cdot x^{y-1} \quad , \quad \frac{\partial f}{\partial y}=x^y\cdot \ln x$$

Example 9 Write a function that expresses the side of a triangle through the other two sides and the angle. Find all partial derivatives of this function.

Product of partial derivative $\frac{\partial u}{\partial x}$ by the arbitrary increment Δx is called **partial differential** in relation to x of function $u = f(x, y, z)$; it is denoted as

$$d_x u = \frac{\partial u}{\partial x} \cdot \Delta x .$$

Analogically,

$$d_y u = \frac{\partial u}{\partial y} \cdot \Delta y ,\ d_z u = \frac{\partial u}{\partial z} \cdot \Delta z .$$

So, we can see that partial derivatives may be represented as **fractions**:

$$\frac{\partial u}{\partial x} = \frac{d_x u}{dx}, \quad \frac{\partial u}{\partial y} = \frac{d_y u}{dy}, \quad \frac{\partial u}{\partial z} = \frac{d_z u}{dz}$$

(I hope you don't forget that <u>differential of independent variable is equal to its increment</u>)

Total increment

If we fix some values $x = x_0, y = y_0, z = z_0$ of independent variables and after that give them some increments $\Delta x, \Delta y, \Delta z$, then function $u = f(x, y, z)$ obtains increment

$$\Delta u = \Delta f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0),$$

that is called **total increment of function.**

For one variable function $y = f(x)$ the function increment have been represented as $\Delta y = f'(x_0) \cdot \Delta x + \alpha \cdot \Delta x$,

where α depends on Δx and $\alpha \to 0$ when $\Delta x \to 0$. But $f'(x_0) \cdot$ **does not depend on** Δx !

Analogical expression for three variables function will be:

$$\Delta u = \Delta f(x_0, y_0, z_0) =$$
$$= f'_x(x_0, y_0, z_0) \cdot \Delta x + f'_y(x_0, y_0, z_0) \cdot \Delta y +$$
$$+ f'_z(x_0, y_0, z_0) \cdot \Delta z + \alpha \cdot \Delta x + \beta \cdot \Delta y + \gamma \cdot \Delta z,$$

$\alpha \to 0$ when $\Delta x \to 0$, $\chi \to 0$ when $\Delta y \to 0$, $\gamma \to 0$ when $\Delta z \to 0$,

If partial derivatives in some point exist and are continuous then function is continuous in this point.
The opposite is wrong!

Definition. Function $f(x,y,z)$ is called differentiable in point (x,y,z), if its total increment has the form

$$\Delta f(x,y,z) = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) = A\Delta x + B\Delta y + C\Delta z + \alpha\Delta x + \beta\Delta y + \gamma\Delta z \tag{4}$$

where A,B,C ***does not depend on*** $\Delta x, \Delta y, \Delta z$,

and $\lim_{\Delta x\to 0,\, \Delta y\to 0,\, \Delta z\to 0} \alpha =$

$$= \lim_{\Delta x\to 0,\, \Delta y\to 0,\, \Delta z\to 0} \beta =$$

$$= \lim_{\Delta x\to 0,\, \Delta y\to 0,\, \Delta z\to 0} \gamma = 0$$

Another form – function is differentiable, if its total increment has the form

$$\Delta f(x,y,z) = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) = A\Delta x + B\Delta y + C\Delta z + o(\rho)$$

где $\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ (5)

Theorem. If function is differentiable in point M(x,y,z), then partial derivatives exist in this point and

$$A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial y}, \quad C = \frac{\partial f}{\partial z}$$

Definition. The linear part of formulas (4) and (5) is called **total differential** and denotes as

$$df(x_0,y_0,z_0) = f'_x(x_0,y_0,z_0)\cdot\Delta x + f'_y(x_0,y_0,z_0)\cdot\Delta y + f'_z(x_0,y_0,z_0)\cdot\Delta z =$$
$$= f'_x(x_0,y_0,z_0)\cdot dx + f'_y(x_0,y_0,z_0)\cdot dy + f'_z(x_0,y_0,z_0)\cdot dz \qquad \textbf{(1.6)}$$

or

$$du = u'_x\cdot dx + u'_y\cdot dy + u'_z\cdot dz. \qquad \textbf{(1.6*)}$$

Each of addends in formula 6 is called partial differential. So, the total differential is the sum of partial differentials. And independent increments are equal to its differentials (as for function od one variable).

Example 11. Calculate partial differentials in relation to every independent variable and a total differential of function

$$u = \sin(x + y) - \cos(z - x)$$

Solution.

$$d_x u = \frac{\partial u}{\partial x} dx, \quad d_y u = \frac{\partial u}{\partial y} dy, \quad d_z u = \frac{\partial u}{\partial z} dz$$

Calculate partial derivatives:

$$\frac{\partial u}{\partial x} = \cos(x + y) - \sin(z - x),$$

$$\frac{\partial u}{\partial y} = \cos(x + y),$$

$$\frac{\partial u}{\partial z} = \sin(z - x),$$

Then partial differentials will be

$$d_x u = \left(\cos(x + y) - \sin(z - x)\right) dx,$$

$$d_y u = \cos(x + y) dy,$$

$$d_z u = \sin(z - x) dz$$

And total differential

$$du = \left(\cos(x + y) - \sin(z - x)\right) dx + \cos(x + y) dy + \sin(z - x) dz$$

Approximation using total differential

$$f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) = f(x_0, y_0, z_0) + \Delta f$$
$$\Delta f \approx df = f'_x \cdot \Delta x + f'_y \cdot \Delta y + f'_z \cdot \Delta z$$

Example 12. Approximate $1{,}02^{0{,}97}$

Solution. Let's take function of two variables

$f(x,y) = x^y$

Let $x_0 = 1, \Delta x = 0{,}02$ and $y_0 = 1, \Delta y = -0{,}03$

Calculate $f(x_0, y_0) = 1^1 = 1$

Calculate partial derivatives (see example 8) in point (x_0, y_0);

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = y \cdot x^{y-1} = 1 \cdot 1^{1-1} = 1 \quad , \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = x^y \cdot \ln x = 1^1 \cdot \ln 1 = 1 \cdot 0 = 0$$

Then

$$1{,}02^{0{,}97} \approx f(x_0, y_0, z_0) + f'_x \cdot \Delta x + f'_y \cdot \Delta y = 1 + 1 \cdot 0{,}02 + 0 \cdot (-0.03) = 1{,}02$$