



WESTMINSTER

INTERNATIONAL UNIVERSITY IN TASHKENT

An Accredited Institution of the University of Westminster (UK)

# LECTURE 6

## Probability

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- The meaning of probability and relevant concepts
- The basic operations of probability
- Sets, combination, and permutation
- Mathematical expectation

# Probability

**Probability** is a numerical measure of the chance (or likelihood) that a particular event will occur.



**Probability is always between 0 and 1**

Probability values are always assigned on a scale of **0** to **1**:

**0** indicates that an event is very unlikely to occur

**1** indicates that an event is almost certain to occur

# Experiment vs Sample Space

- The **probability** is a chance or likelihood of an event to happen
- An **experiment** is an activity with an observable result
- The **trials** – repetition of an experiment
- The **outcomes** – results of each trial
- A **sample space** is the set of all possible outcomes
- A **sample point** is an element of the sample space
- An **Event** is a subset of the sample space

# Experiment vs Sample Space

**Experiment:** rolling a die

**Sample space:** {1, 2, 3, 4, 5, 6}

**Event:** rolling a 1; rolling a number less than 5; rolling an even number

- Probability of event occurring =

$$\frac{\textit{Number of ways the event can occur}}{\textit{Total number of outcomes}}$$

- Events can have 1 element: {1}
- Or many elements {1, 2, 3, 4}, {2, 4, 6}

# Experiment vs Sample Space

- Case 1: Probability of rolling a 1:

$$\frac{\textit{Number of ways the event can occur}}{\textit{Total number of outcomes}} = \frac{\{1\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{1}{6}$$

- Case 2: Probability of rolling a number less than 5: =

$$= \frac{\{1, 2, 3, 4\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{4}{6}$$

- Case 3: Probability of rolling an even number:

$$= \frac{\{2, 4, 6\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{3}{6}$$

# Calculation of probability

There are 6 blue, 3 red, 2 yellow, and 1 green marbles in the box.

What is the probability of picking a red marble?





# Calculation of probability

There are 6 blue, 3 red, 2 yellow, and 1 green marbles in the box.

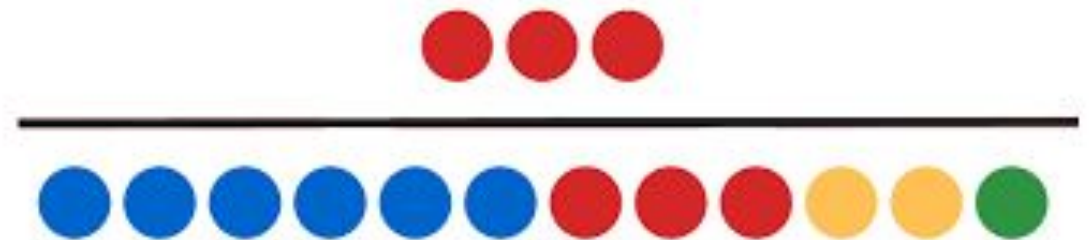
What is the probability of picking a red marble?

**Probability of an event =**

$$\frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

Thus, the probability of picking a red marbles is:  $3/12=0.25$  or 25%

Probability of Red



# Calculation of probability

- If there are  $n$  experimental outcomes, the sum of the probabilities for all the experimental outcomes must be equal to 1
- In the marble scenario,

$$\begin{aligned} P(\text{blue}) + P(\text{red}) + P(\text{yellow}) + P(\text{green}) &= \\ &= 0.5 + 0.25 + 1/6 + 1/12 = 1 \end{aligned}$$



# Examples...

<b>Experiment</b>	<b>Experimental outcomes</b>	<b>Sample space</b>
Toss a coin	Head, tail	$S = \{\text{Head, Tail}\}$
Apply for a job	Hired, not hired	$S = \{\text{Hired, Not Hired}\}$
Roll a die	1, 2, 3, 4, 5, 6	$S = \{1, 2, 3, 4, 5, 6\}$
Play a game	Win, lose, draw	$S = \{\text{Win, Lose, Draw}\}$
Run a business	Profit, loss, even	$S = \{\text{Profit, Loss, Even}\}$

# Types of counting rules

- Multiple step
- Combination
- Permutation

# Multiple step experiment

Experiment of a sequence of  $k$  steps

Total number of experimental outcomes is the product of number of outcomes in each step

$$(n_1)(n_2)(n_3)\dots(n_{k-1})(n_k)$$

*Example:* let's toss the coin *twice*



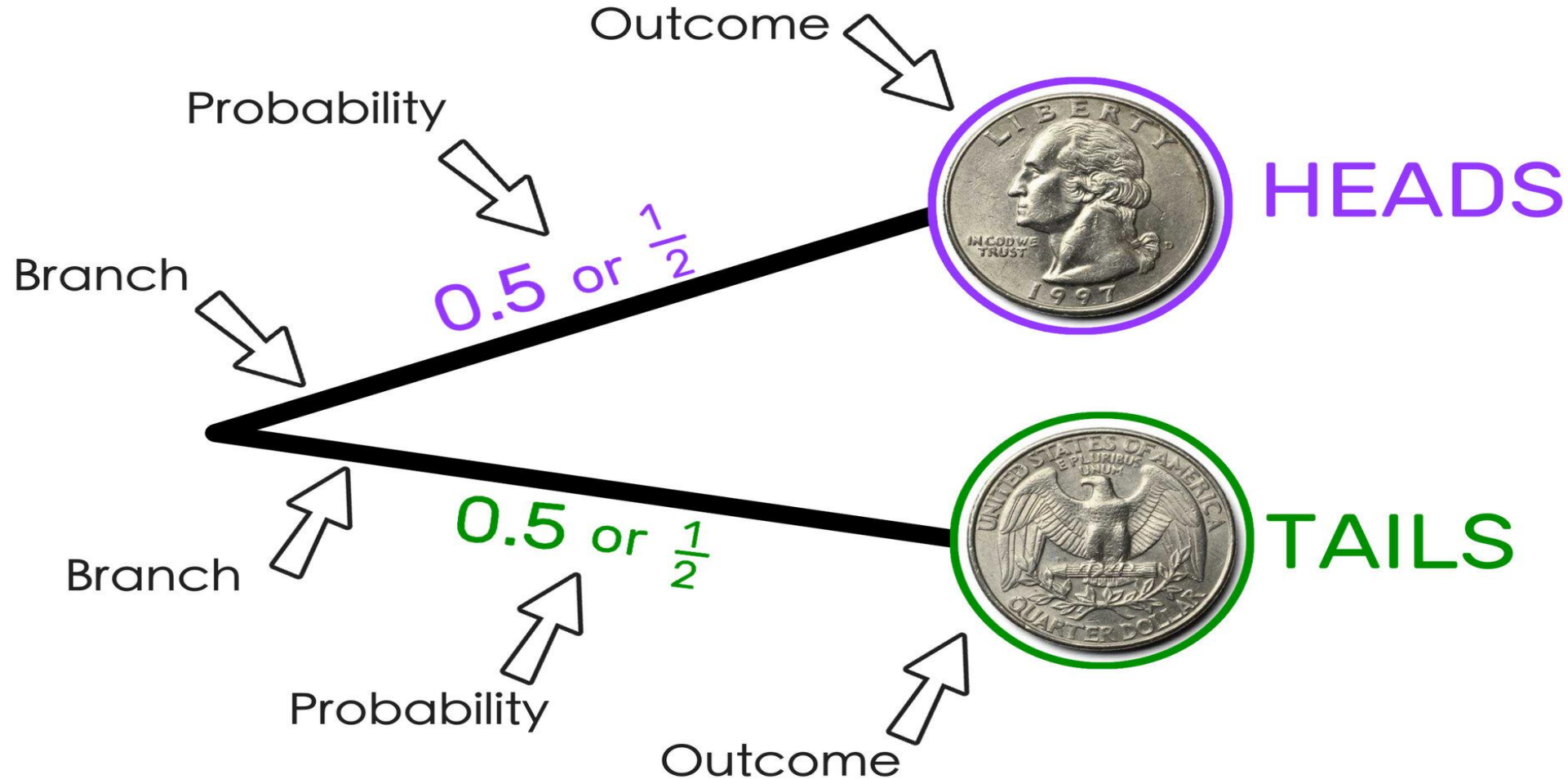
Total number of outcomes =  $(n_1)(n_2) = 2*2=4$

Thus, **sample space** is

$$S = \{(\text{Head, Head}), (\text{Head, Tail}), (\text{Tail, Head}), (\text{Tail, Tail})\}$$

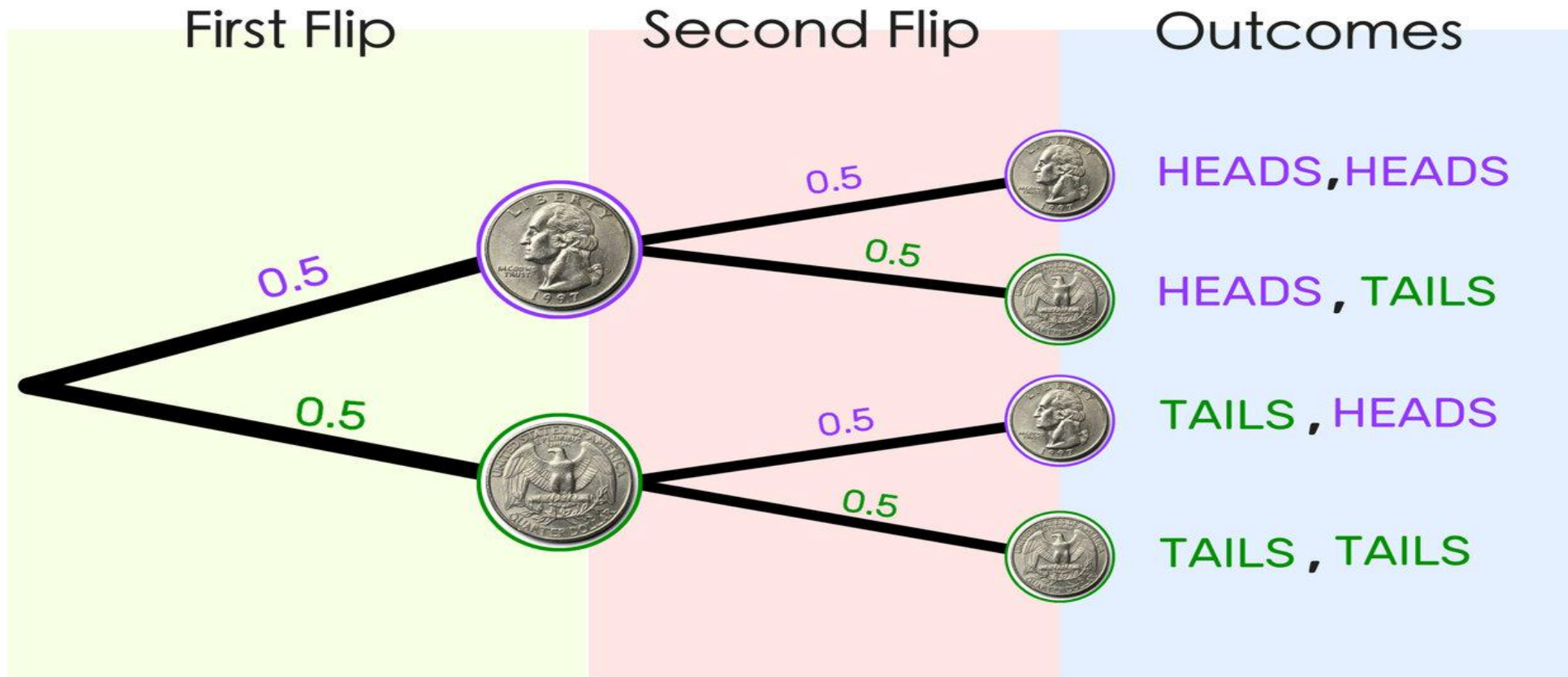
# Probability Tree Diagram

## TOSSING A COIN



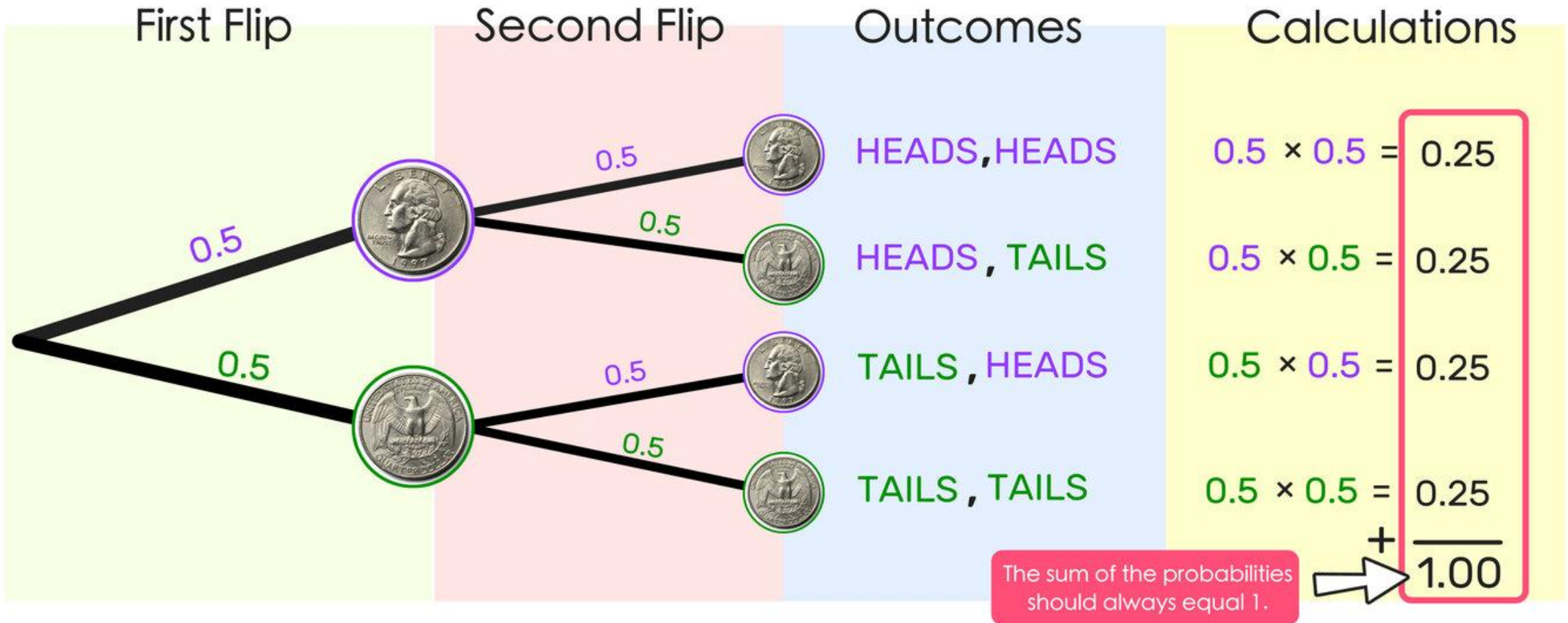
# Probability Tree Diagram

## TOSSING A COIN TWICE!



# Probability calculation

**PROBABILITY RULE** To find the probability of an outcome, multiply the probabilities of the branches.





## ***Self exam-training task:***

Find the number of all possible outcomes for rolling a die three times



## *Example:*

In this classroom, a lecturer randomly picks two of five students (let's say *Shakhzoda*, *Imana*, *Azamat*, *Zakariyo* and *Artyom*) to test their knowledge of probability. In a group of five smart students, **how many combinations of two students may be selected?**

(sequence of selection does not matter)

*Verbal solution:* a lecturer may have 10 picks

Shakhzoda with Imana

Shakhzoda with Azamat

Shakhzoda with Zakariyo

Shakhzoda with Artyom

Imana with Azamat

Imana with Zakariyo

Imana with Artyom

Azamat with Zakariyo

Azamat with Artyom

Zakariyo with Artyom

**Combination formula:**  $n$  objects are to be selected from a set of  $N$  objects, where the **order of selection** is not important.

$$C_N^n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

Where,  $n! = n \cdot (n-1) \cdot (n-2) \dots 3 \cdot 2 \cdot 1$

**Example:**  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

**Solution:**

$N = 5$  and  $n = 2$ .

Total number of outcomes =

$$C_5^2 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$$

## *Example:*

In this classroom, out of these five students, let's assume their names are *Aziza*, *Bekzod*, *Charos*, *Daler* and *Erkin*, **how many ways do we have in order to have one interviewer and one interviewee?**

*Verbal solution:* a lecturer may have 20 picks

1. AB	6. BD	11. BA	16. DB
2. AC	7. BE	12. CA	17. EB
3. AD	8. CD	13. DA	18. DC
4. AE	9. CE	14. EA	19. EC
5. BC	10. DE	15. CB	20. ED

**Permutation formula:**  $n$  objects are to be selected from a set of  $N$  objects, where the **order of selection** is important.

$$P_N^n = \frac{N!}{(N - n)!}$$

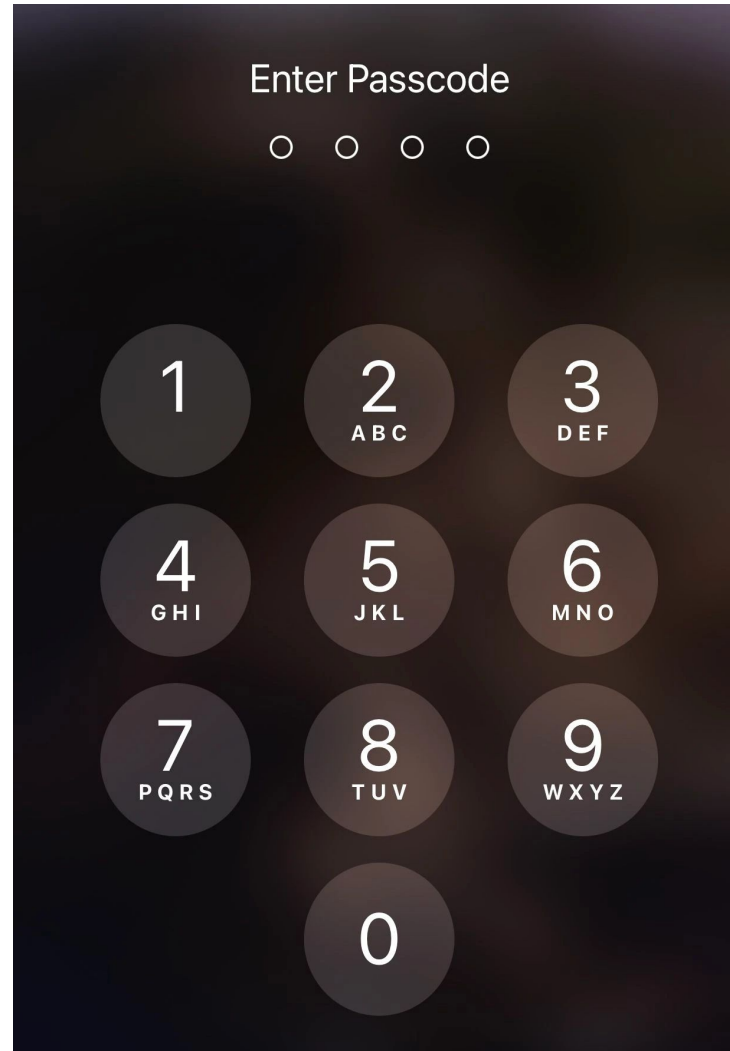
**Solution:**

$$N = 5 \quad \text{and} \quad n = 2$$

Total number of outcomes:

$$P_5^2 = \frac{5!}{(5 - 2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20$$

# What Permutation can tell us ?

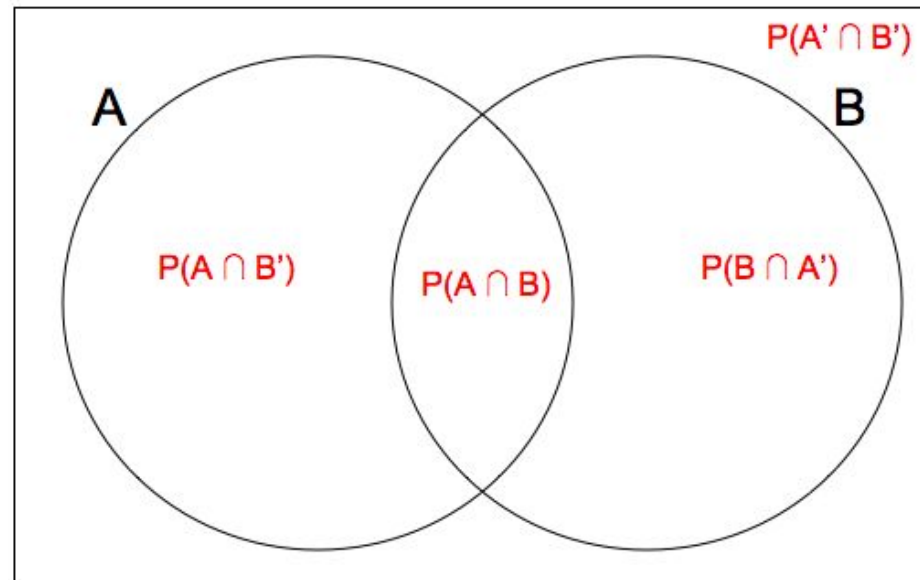


# Operations with events

$A \cup B$  The **union** of events A and B is the event containing all experimental outcomes belonging to A or B or both.

$A \cap B$  The **intersection** of A and B is the event containing the experimental outcomes belonging to both A and B.

$A^c$  The **complement** of an event A is an event consisting of all experimental outcomes that are not in A.



# Operations with events

- Toss a die and observe the number that appears on top
- $S = \{1, 2, 3, 4, 5, 6\}$  (*sample space*)
- $A = \{2, 4, 6\}$  (*even numbers*)
- $B = \{1, 3, 5\}$  (*odd numbers*)
- $C = \{2, 3, 5\}$  (*prime numbers*)

$A \cup C = \{2, 3, 4, 5, 6\}$  - the event that an even number or a prime number is observed

$B \cap C = \{3, 5\}$  - the event that an odd prime number is observed

$C^c = \{1, 4, 6\}$  - the event that a non prime number is observed

Exercises:

Find 1)  $B \cup C$ ;    2)  $A \cap C$ ;    3)  $(B \cup C)^c$



# Relationship of events

- Mutually exclusive events -

$P(A \cap B) = 0$  Events **A** and **B** do not have any experimental outcomes in common

- Dependent events

$P(A|B) = \frac{P(A \cap B)}{P(B)}$  Event **A** has an influence on the event

- Independent events

$P(A|B) = P(A)$  Event **A** has no influence on the event **B**


# Addition rule for union

**Question:** On Quantitative Methods module for 600 CIFS students at WIUT, 480 passed the in-class test and 450 passed the final exam, 390 students passed both exams.

Due to high failure rate, the module leader decides to give a passing grade to any student who passed at least one of the two exams.

**What is the probability of passing this module?**

- The **addition rule** is used to compute the probability of the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$


$$1) \quad P(I) = \frac{480}{600} = 0.80$$

$$2) \quad P(F) = \frac{450}{600} = 0.75$$

$$3) \quad P(I \cap F) = \frac{390}{600} = 0.65$$

$$\begin{aligned} P(I \cup F) &= P(I) + P(F) - P(I \cap F) \\ &= 0.8 + 0.75 - 0.65 = 0.90 \end{aligned}$$

# Multiplication rule for intersection

**The multiplication rule is used to compute the probability of the intersection of two events**

$$P(A \cap B) = P(B)P(A|B)$$

$$\left( \text{derives from } P(A|B) = \frac{P(A \cap B)}{P(B)} \right)$$

**The multiplication rule (simplified) for independent events**

$$P(A \cap B) = P(B)P(A)$$

# Multiplication Rule

A bowl contains 7 red marbles and 3 black marbles. Two marbles are drawn *without replacement* from the bowl. What is the probability that both of the marbles are black?



Let A = the event that the first marble is black;  
let B = the event that the second marble is black.

We know the following:

- ✓ In the beginning, there are 10 marbles in the bowl, 3 of which are black. Therefore,  $P(A) = 3/10$ .
- ✓ After the first selection, there are 9 marbles in the bowl, 2 of which are black. Therefore,  $P(B|A) = 2/9$ .

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A) P(B|A)$$

$$P(A \cap B) = (3/10) * (2/9) = 6/90 = 2/30 = 0.067 \text{ or } 6.67\%$$

# Mathematical expectation

**365bet.com** sent you following offers for coming El-Classico game depending on your betting:

- If Barcelona wins, they will triple your money
- If Real Madrid wins, they will double your money
- If game results in a draw, they will quadruple your money

If you would like to bet for \$100, assuming the possibilities of outcomes are equally likely, what will be the expected sum of your money?

Your possible earnings:

- \$300 or \$0 if you bet on Barcelona's victory
- \$200 or \$0 if you bet on Real Madrid's victory
- \$400 or \$0 if you bet on a draw

Hence, the expected sum of your income will be

$$E(i) = \$200 \times \frac{1}{3} + \$300 \times \frac{1}{3} + \$400 \times \frac{1}{3} = \$300$$

Today, you learned:

- Basic concepts within probability theory
- Basic operations of calculating the sample space and number of probable events (combinations, permutations)
- Operations with events
- Relationship of events
- Multiplication and Addition Rule
- Expected Value

- Jon Curwin, “Quantitative methods.” Ch-9 (p.249).