WESTMINSTER

INTERNATIONAL UNIVERSITY IN TASHKENT

An Accredited Institution of the University of Westminster (UK)

LECTURE 6
Probability

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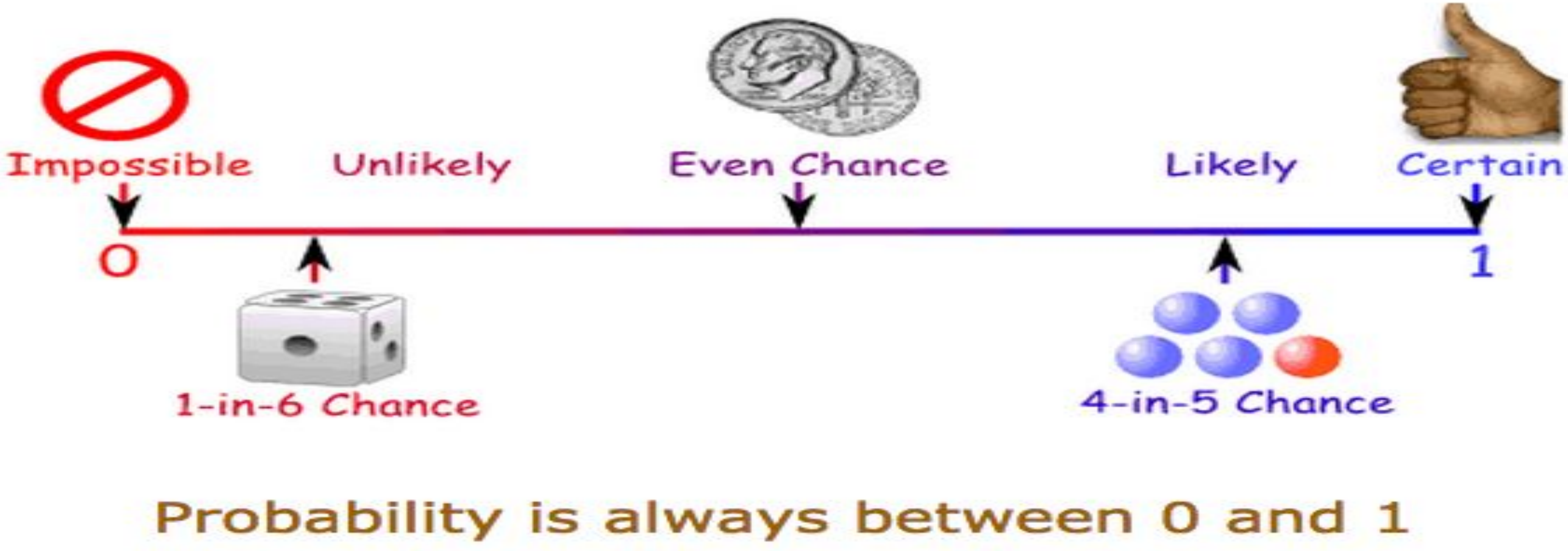
Lecture outline

- The meaning of probability and relevant concepts
- The basic operations of probability
- Sets, combination, and permutation
- Mathematical expectation

Probability

Probability is a numerical measure of the chance (or likelihood) that a particular event will occur.



Probability values are always assigned on a scale of **0** to **1**:

0 *indicates that an event is very unlikely to occur*

1 *indicates that an even is almost certain to occur*

Experiment vs Sample Space

- The **probability** is a chance or likelihood of an event to happen
- An **experiment** is an activity with an observable result
- The **trials** – repetition of an experiment
- The **outcomes** – results of each trial
- A **sample space** is the set of all possible outcomes
- A **sample point** is an element of the sample space
- An **Event** is a subset of the sample space

Experiment vs Sample Space

- **Experiment:** rolling a die

Sample space: {1, 2, 3, 4, 5, 6}

Event: rolling a 1; rolling a number less than 5; rolling an even number

- Probability of event occurring =

$$\frac{Number\ of\ ways\ the\ event\ can\ occur}{Total\ number\ of\ outcomes}$$

- Events can have 1 element: {1}
- Or many elements {1, 2, 3, 4}, {2, 4, 6}

Experiment vs Sample Space

- Case 1: Probability of rolling a 1:

$$\frac{Number\ of\ ways\ the\ event\ can\ occur}{Total\ number\ of\ outcomes} = \frac{\{1\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{1}{6}$$

- Case 2: Probability of rolling a number less than 5: =

$$= \frac{\{1, 2, 3, 4\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{4}{6}$$

- Case 3: Probability of rolling an even number:

$$= \frac{\{2, 4,6\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{3}{6}$$

Calculation of probability

There are 6 blue, 3 red, 2 yellow, and 1 green marbles in the box.

What is the probability of picking a red marble?

Calculation of probability

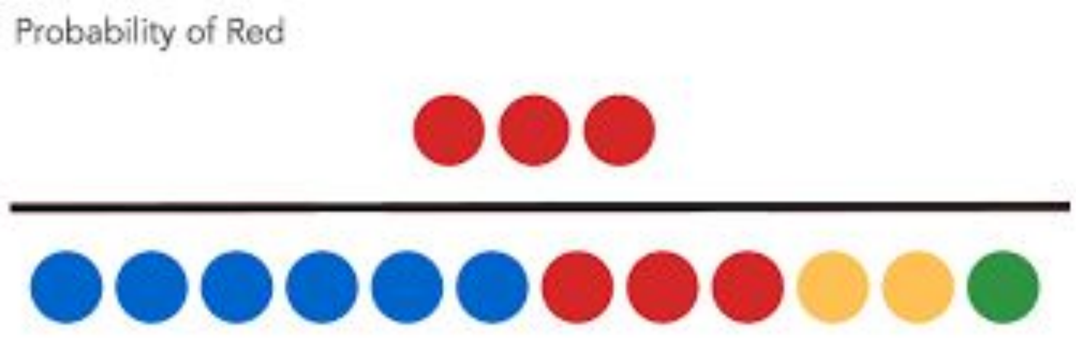
There are 6 blue, 3 red, 2 yellow, and 1 green marbles in the box.

Probability of an event =

$$\frac{\textit{Number of ways the event can occur}}{\textit{Total number of possible outcomes}}$$

What is the probability of picking a red marble?

Thus, the probability of picking a red marbles is: 3/12=0.25 or 25%



Calculation of probability

- If there are ***n*** experimental outcomes, the sum of the probabilities for all the experimental outcomes must be equal to 1
- In the marble scenario,

$$P(\text{blue}) + P(\text{red}) + P(\text{yellow}) + P(\text{green}) =$$
$$= 0.5 + 0.25 + 1/6 + 1/12 = 1$$

Examples…

Experiment	Experimental outcomes	Sample space
Toss a coin	Head, tail	S = {Head, Tail}
Apply for a job	Hired, not hired	S = {Hired, Not Hired}
Roll a die	1, 2, 3, 4, 5, 6	S = {1, 2, 3, 4, 5, 6}
Play a game	Win, lose, draw	S = {Win, Lose, Draw}
Run a business	Profit, loss, even	S = {Profit, Loss, Even}

Types of counting rules

- Multiple step
- Combination
- Permutation

Multiple step experiment

Experiment of a sequence of ***k*** steps

Total number of experimental outcomes is the product of number of outcomes in each step

$(n_1)(n_2)(n_3)\ldots(n_{k-1})(n_k)$

Example: let's toss the coin twice

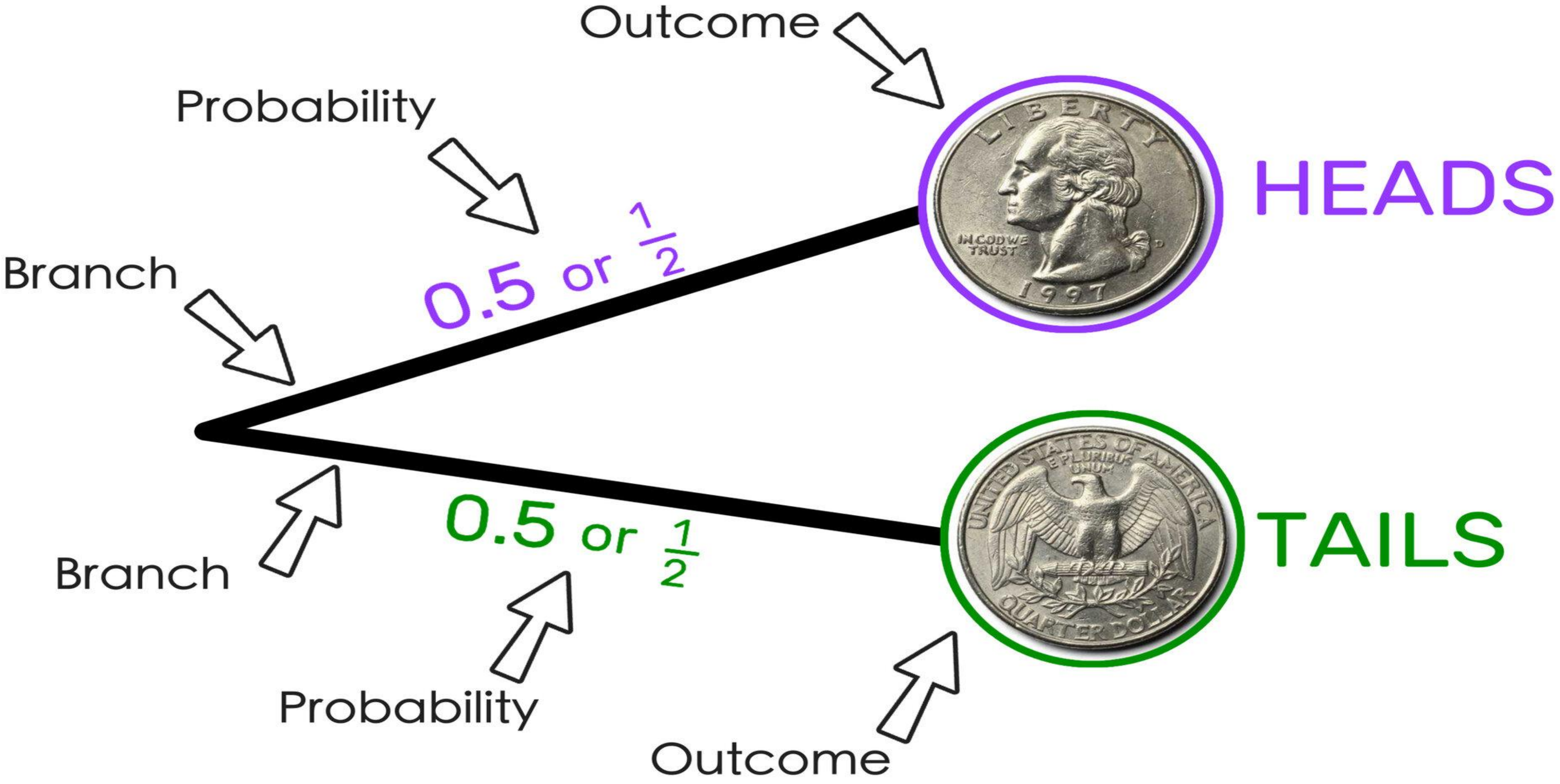
Total number of outcomes = $(n_1)(n_2) = 2*2=4$

Thus, sample space is

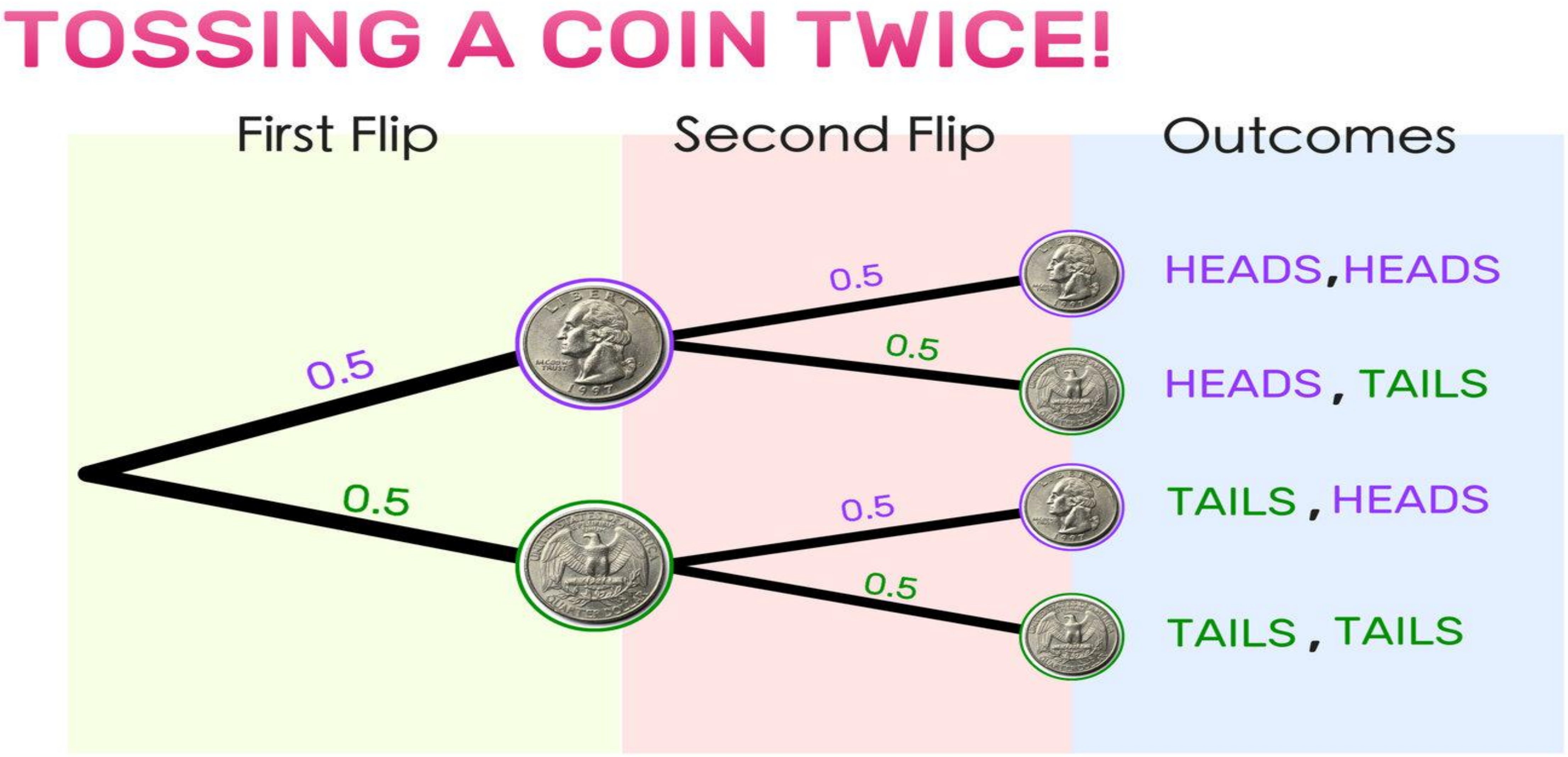
$$S = \{(\text{Head, Head}), (\text{Head, Tail}), (\text{Tail, Head}), (\text{Tail, Tail})\}$$

Probability Tree Diagram

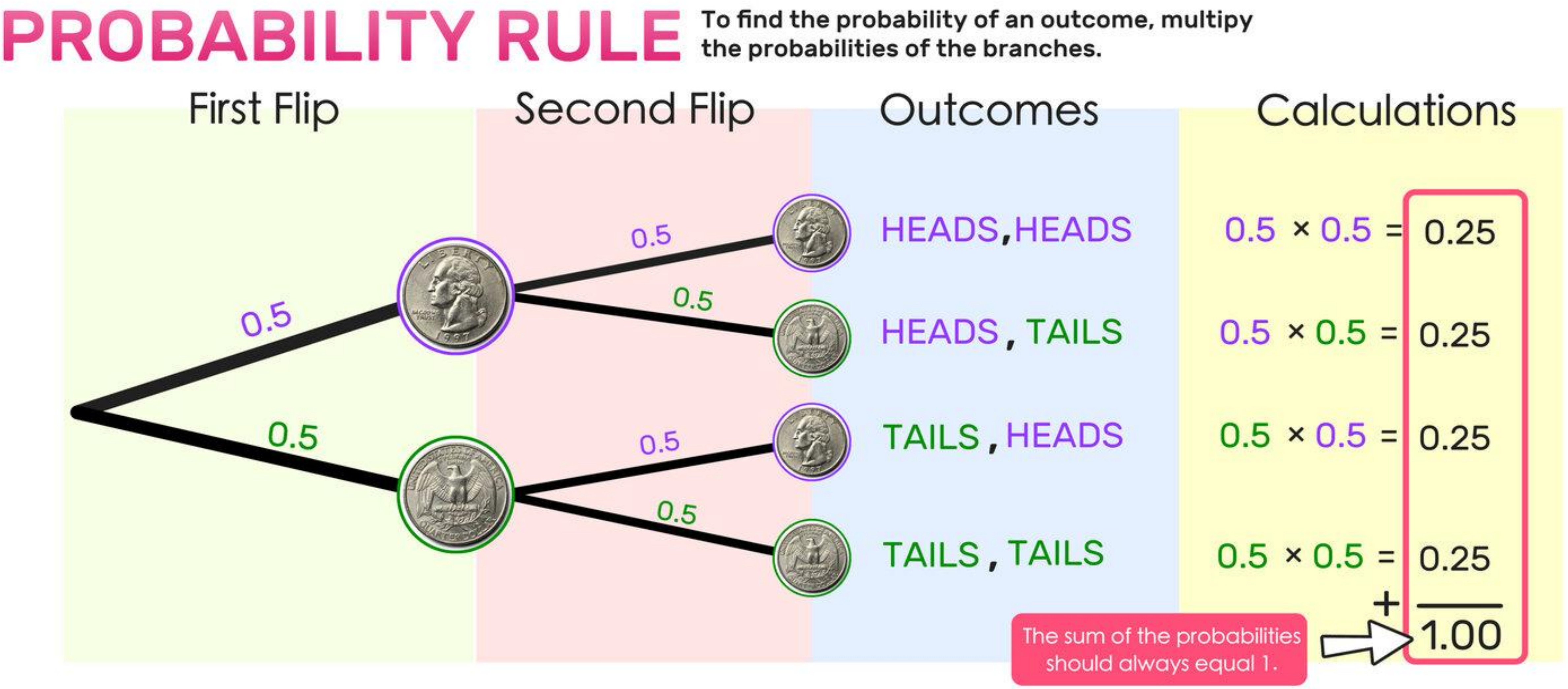
TOSSING A COIN



Probability Tree Diagram



Probability calculation



Self exam-training task:

Find the number of all possible outcomes for rolling a die three times

Combination

Example:

In this classroom, a lecturer randomly picks two of five students (let's say *Shakhzoda*, *Imana*, *Azamat*, *Zakariyo* and *Artyom*) to test their knowledge of probability. In a group of five smart students, **how many combinations of two students may be selected?**

(sequence of selection does not matter)

Verbal solution: a lecturer may have 10 picks

Shakhzoda with Imana
Shakhzoda with Azamat
Shakhzoda with Zakariyo
Shakhzoda with Artyom
Imana with Azamat
Imana with Zakariyo
Imana with Artyom
Azamat with Zakariyo
Azamat with Artyom
Zakariyo with Artyom

Combination

Combination formula: ***n*** objects are to be selected from a set of ***N*** objects, where the order of selection is not important.

$$C_N^n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

Where, $n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$

Example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Solution:

N = 5 and $n = 2$.

Total number of outcomes =

$$C_5^2 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$$

Permutation

Example:

In this classroom, out of these five students, let's assume their names are *Aziza*, *Bekzod*, *Charos*, *Daler* and *Erkin*, **how may ways do we have in order to have one interviewer and one interviewee?**

Verbal solution: a lecturer may have 20 picks

1. AB	6. BD	11. BA	16. DB
2. AC	7. BE	12. CA	17. EB
3. AD	8. CD	13. DA	18. DC
4. AE	9. CE	14. EA	19. EC
5. BC	10. DE	15. CB	20. ED

Permutation

Permutation formula: ***n*** objects are to be selected from a set of ***N*** objects, where the order of selection is important.

$$P_N^n = \frac{N!}{(N-n)!}$$

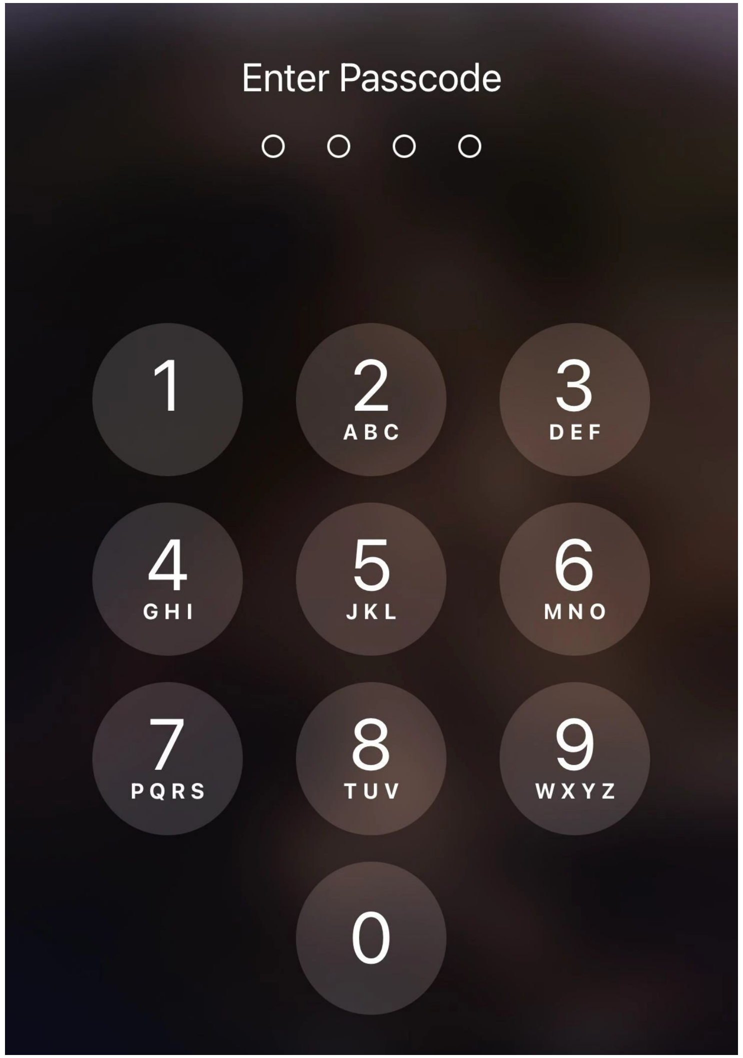
Solution:

$\mathrm{N} = 5$ and $n = 2$

Total number of outcomes:

$$P_5^2 = \frac{5!}{(5-2)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1} = 20$$

What Permutation can tell us ?

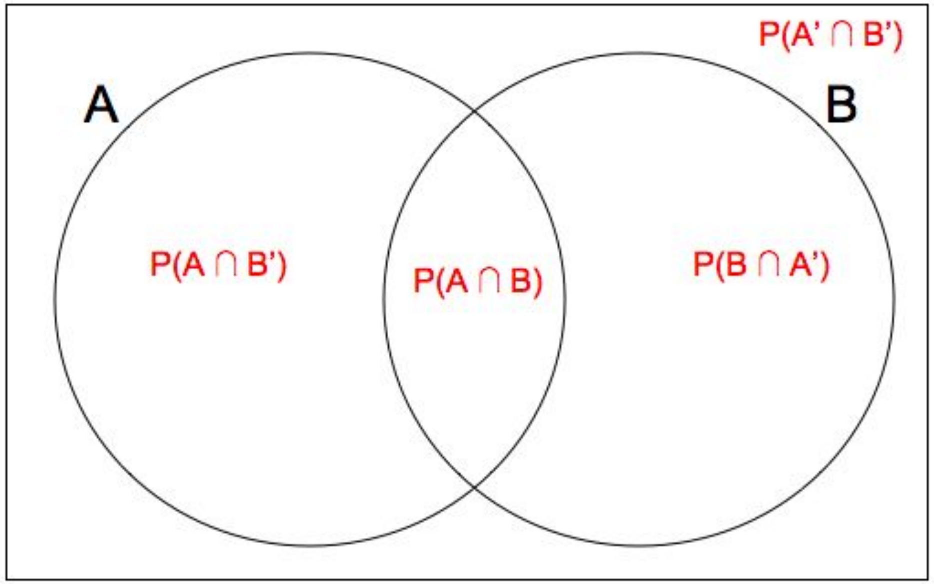


Operations with events

$A \cup B$ The ***union*** of events A and B is the event containing all experimental outcomes belonging to A or B or both.

$A \cap B$ The ***intersection*** of A and B is the event containing the experimental outcomes belonging to both A and B.

A^C The ***complement*** of an event A is an event consisting of all experimental outcomes that are not in A.



Operations with events

- Toss a die and observe the number that appears on top
- S = {1, 2, 3, 4, 5, 6} *(sample space)*
- A = {2, 4, 6} *(even numbers)*
- B = {1, 3, 5} *(odd numbers)*
- C = {2, 3, 5} *(prime numbers)*

$A \cup C = \{2, 3, 4, 5, 6)$ - the event that an even number or a prime number is observed

$B \cap C = \{3, 5\}$ - the event that an odd prime number is observed

$C^C = \{1, 4, 6\}$ - the event that a non prime number is observed

Exercises:

Find 1) $B \cup C$; 2) $A \cap C$; 3) $(B \cup C)^C$

Relationship of events

- Mutually exclusive events -

  $$P(A \cap B) = 0$$

  The events **A** and **B** do not have any experimental outcomes in common

- Dependent events

  $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  The event **A** has an influence on the event

- Independent events

  $$P(A|B) = P(A)$$

  The event A has no influence on the event B

Addition rule for union

Question: On Quantitative Methods module for 600 CIFS students at WIUT, 480 passed the in-class test and 450 passed the final exam, 390 students passed both exams.

Due to high failure rate, the module leader decides to give a passing grade to any student who passed at least one of the two exams.

What is the probability of passing this module?

- The **addition rule** is used to compute the probability of the union of two events:

?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

1) $P(I) = \frac{480}{600} = 0.80$

2) $P(F) = \frac{450}{600} = 0.75$

3) $P(I \cap F) = \frac{390}{600} = 0.65$

$$P(I \cup F) = P(I) + P(F) - P(I \cap F)$$

$$= 0.8 + 0.75 - 0.65 = 0.90$$

Multiplication rule for intersection

The multiplication rule is used to compute the probability of the intersection of two events

$$P(A \cap B) = P(B)P(A|B)$$

$$\left(\textit{derives} \quad \textit{from} \quad P(A|B) = \frac{P(A \cap B)}{P(B)} \right)$$

The multiplication rule (simplified) for independent events

$$P(A \cap B) = P(B)P(A)$$

Multiplication Rule

A bowl contains 7 red marbles and 3 black marbles. Two marbles are drawn *without replacement* from the bowl. What is the probability that both of the marbles are black?

Solution

Let A = the event that the first marble is black;
let B = the event that the second marble is black.
We know the following:

- ✔ In the beginning, there are 10 marbles in the bowl, 3 of which are black. Therefore, $P(A) = 3/10$.
- ✔ After the first selection, there are 9 marbles in the bowl, 2 of which are black. Therefore, $P(B|A) = 2/9$.

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A)\, P(B|A)$$

$$P(A \cap B) = (3/10) * (2/9) = 6/90 = 2/30 = 0.067 \text{ or } 6.67\%$$

Mathematical expectation

365bet.com sent you following offers for coming El-Classico game depending on your betting:

a) If Barcelona wins, they will triple your money
b) If Real Madrid wins, they will double your money
c) If game results in a draw, they will quadruple your money

If you would like to bet for \$100, assuming the possibilities of outcomes are equally likely, what will be the expected sum of your money?

Your possible earnings:

- \$300 or \$0 if you bet on Barcelona's victory
- \$200 or \$0 if you bet on Real Madrid's victory
- \$400 or \$0 if you bet on a draw

Hence, the expected sum of your income will be

$$E(i) = \$200 \times \frac{1}{3} + \$300 \times \frac{1}{3} + \$400 \times \frac{1}{3} = \$300$$

Concluding remarks

Today, you learned:

- Basic concepts within probability theory
- Basic operations of calculating the sample space and number of probable events (combinations, permutations)
- Operations with events
- Relationship of events
- Multiplication and Addition Rule
- Expected Value

Essential reading

- Jon Curwin, “Quantitative methods.” Ch-9 (p.249).