# Lesson11.

# Tangent and Derivative.

# The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

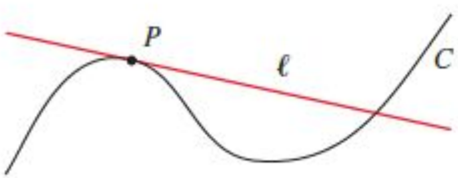
## The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line $\ell$ that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows a line $\ell$ that appears to be a tangent to the curve $C$ at point $P$, but it intersects $C$ twice.

To be specific, let's look at the problem of trying to find a tangent line $\ell$ to the parabola $y = x^2$ in the following example.


(a)




(b)

**FIGURE 1**

**EXAMPLE 1** Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

**SOLUTION** We will be able to find an equation of the tangent line $\ell$ as soon as we know its slope $m$. The difficulty is that we know only one point, $P$, on $\ell$, whereas we need two points to compute the slope. But observe that we can compute an approximation to $m$ by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope $m_{PQ}$ of the secant line $PQ$. (A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts [intersects] a curve more than once.)

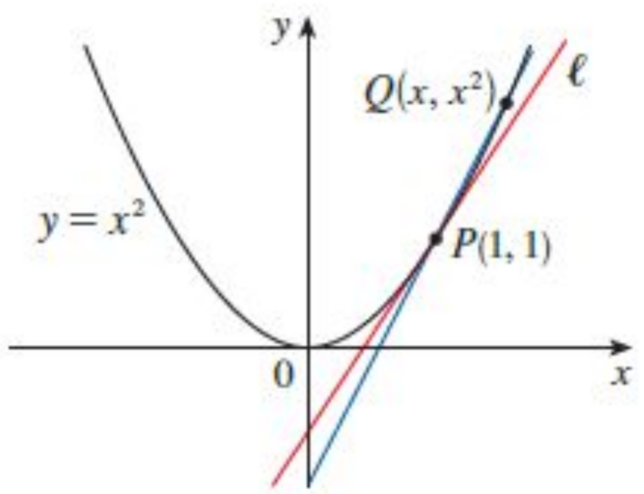



**FIGURE 2**

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of $m_{PQ}$ for several values of $x$ close to 1. The closer $Q$ is to $P$, the closer $x$ is to 1 and, it appears from the tables, the closer $m_{PQ}$ is to 2. This suggests that the slope of the tangent line $\ell$ should be $m = 2$.

| $x$ | $m_{PQ}$ |
|---|---|
| 2 | 3 |
| 1.5 | 2.5 |
| 1.1 | 2.1 |
| 1.01 | 2.01 |
| 1.001 | 2.001 |

| $x$ | $m_{PQ}$ |
|---|---|
| 0 | 1 |
| 0.5 | 1.5 |
| 0.9 | 1.9 |
| 0.99 | 1.99 |
| 0.999 | 1.999 |

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

■

## The Velocity Problem

If you watch the speedometer of a car as you drive in city traffic, you see that the speed doesn't stay the same for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined?

Let's consider the *velocity problem:* Find the instantaneous velocity of an object moving along a straight path at a specific time if the position of the object at any time is known. In the next example, we investigate the velocity of a falling ball. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after $t$ seconds is denoted by $s(t)$ and measured in meters, then (at the earth's surface) Galileo's observation is expressed by the equation

$$s(t) = 4.9t^2$$

**EXAMPLE 3** Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

**SOLUTION** The difficulty in finding the instantaneous velocity at 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}\end{aligned}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

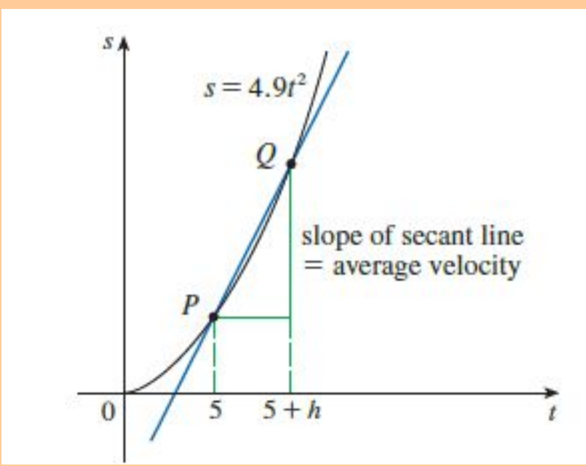
| Time interval | Average velocity (m/s) |
|---|---|
| $5 \leq t \leq 5.1$ | 49.49 |
| $5 \leq t \leq 5.05$ | 49.245 |
| $5 \leq t \leq 5.01$ | 49.049 |
| $5 \leq t \leq 5.001$ | 49.0049 |

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the *limiting value* of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is 49 m/s. ■

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the velocity problem. If we draw the graph of the distance function of the ball (as in Figure 6) and we consider the points $P(5, 4.9(5)^2)$ and $Q(5 + h, 4.9(5 + h)^2)$ on the graph, then the slope of the secant line $PQ$ is

$$m_{PQ} = \frac{4.9(5 + h)^2 - 4.9(5)^2}{(5 + h) - 5}$$

which is the same as the average velocity over the time interval $[5, 5 + h]$. Therefore the velocity at time $t = 5$ (the limit of these average velocities as $h$ approaches 0) must be equal to the slope of the tangent line at $P$ (the limit of the slopes of the secant lines).



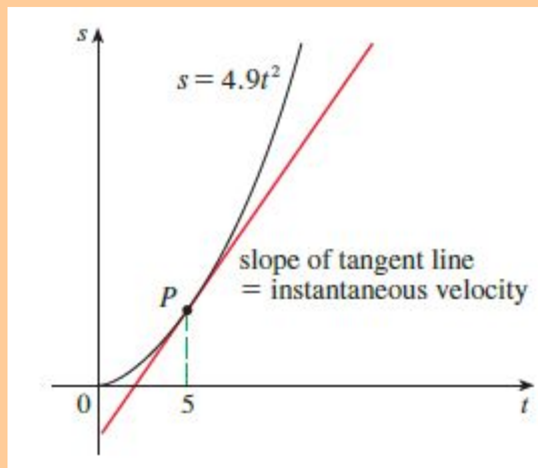




**FIGURE 6**

# 2.7 Derivatives and Rates of Change

Now that we have defined limits and have learned techniques for computing them, we revisit the problems of finding tangent lines and velocities from Section 2.1. The special type of limit that occurs in both of these problems is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

## Tangents

If a curve $C$ has equation $y = f(x)$ and we want to find the tangent line to $C$ at the point $P(a, f(a))$, then we consider (as we did in Section 2.1) a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line $PQ$:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

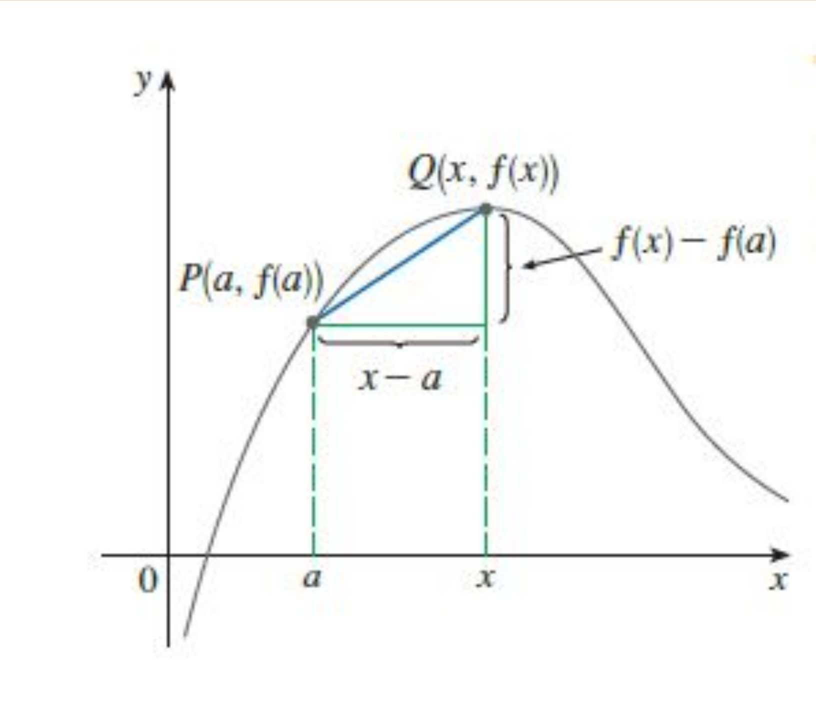
Then we let $Q$ approach $P$ along the curve $C$ by letting $x$ approach $a$. If $m_{PQ}$ approaches a number $m$, then we define the *tangent line* $\ell$ to be the line through $P$ with slope $m$. (This amounts to saying that the tangent line is the limiting position of the secant line $PQ$ as $Q$ approaches $P$. See Figure 1.)

**1 Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 2.1.1.



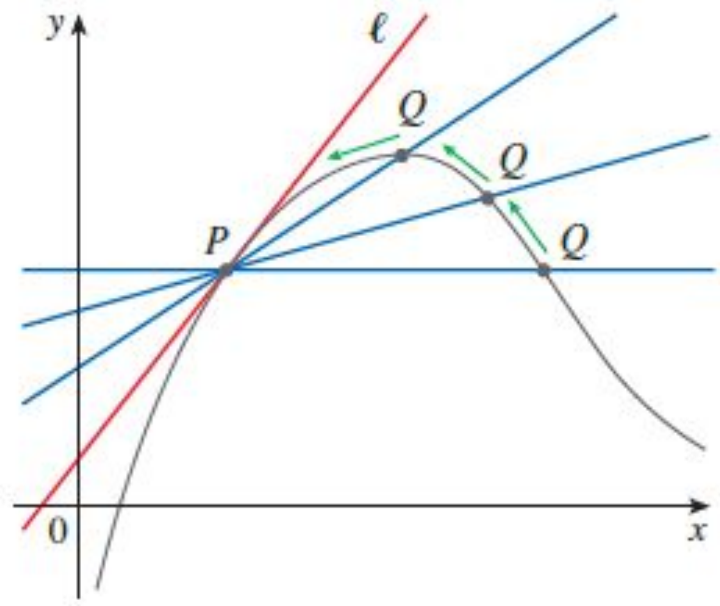




**FIGURE 1**

**EXAMPLE 1** Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

**SOLUTION** Here we have $a = 1$ and $f(x) = x^2$, so the slope is

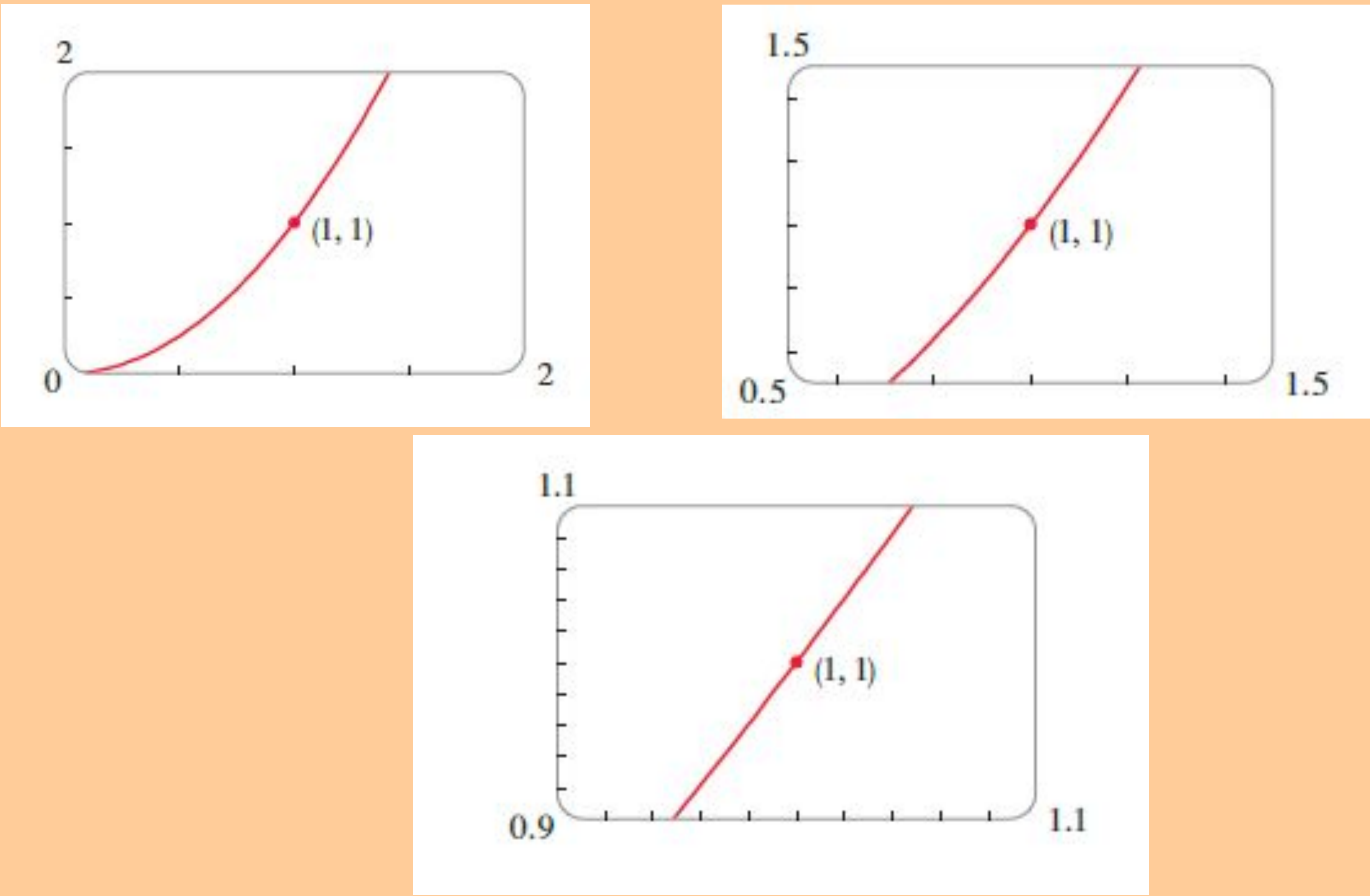
$$
\begin{aligned}
m &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\
&= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} \\
&= \lim_{x \to 1} (x + 1) = 1 + 1 = 2
\end{aligned}
$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

■

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.



There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line $PQ$ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and $Q$ is located to the right of $P$. If it happened that $h < 0$, however, $Q$ would be to the left of $P$.)

Notice that as $x$ approaches $a$, $h$ approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2
$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

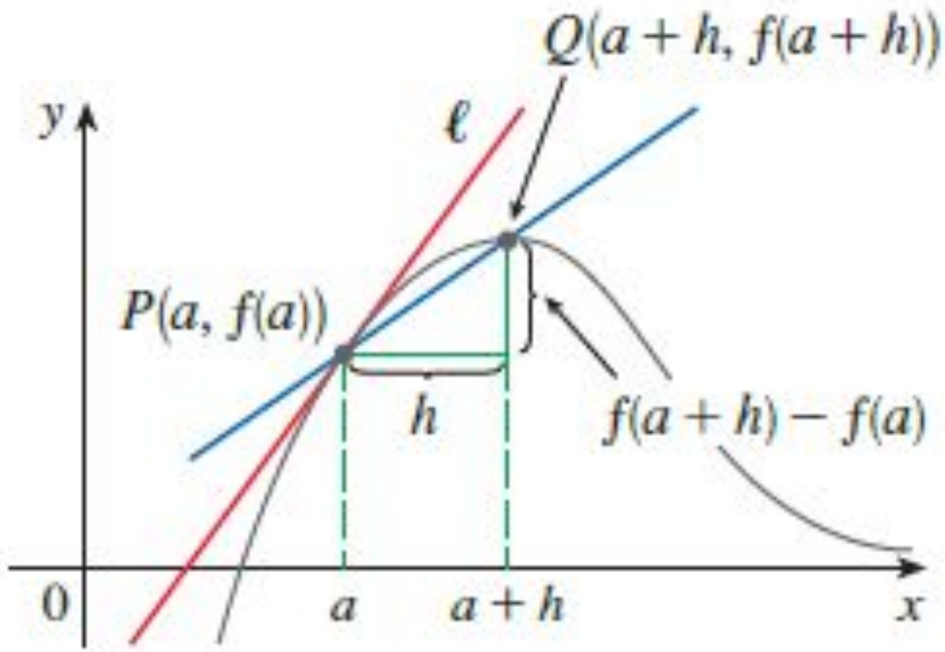



**FIGURE 3**

**EXAMPLE 2** Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

**SOLUTION** Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $$x + 3y - 6 = 0$$

The hyperbola and its tangent are shown in Figure 4. ■

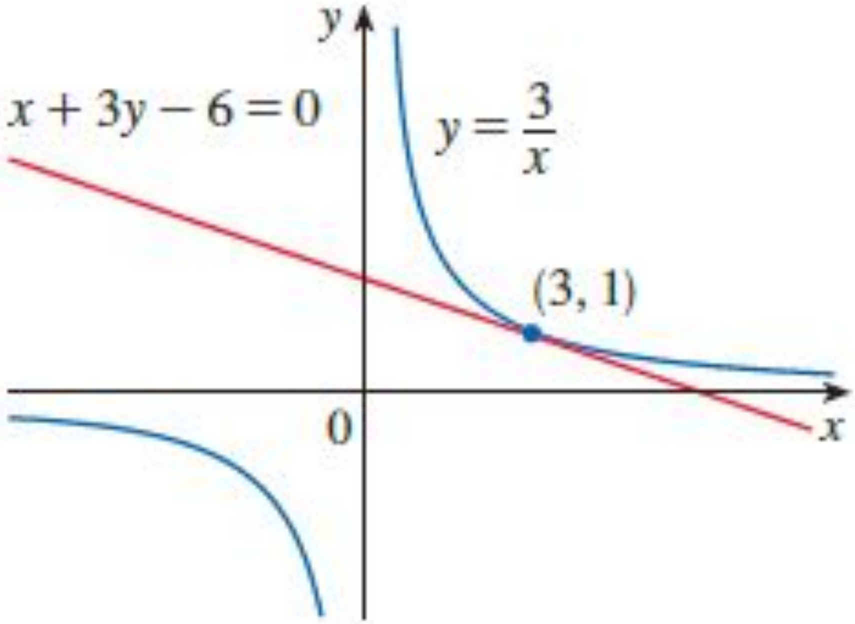



**FIGURE 4**

## Velocities

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where $s$ is the displacement (directed distance) of the object from the origin at time $t$. The function $f$ that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$, the change in position is $f(a + h) - f(a)$. (See Figure 5.)

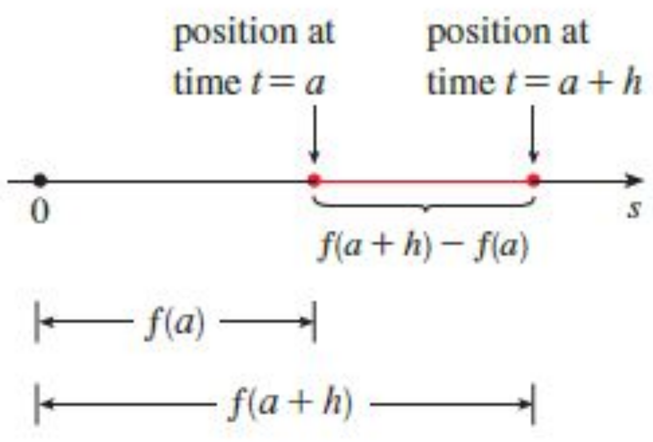



**FIGURE 5**

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line $PQ$ in Figure 6.

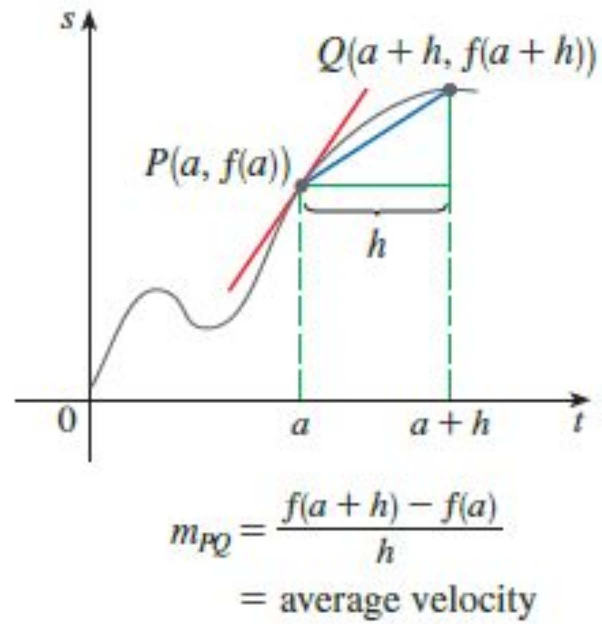



**FIGURE 6**

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let $h$ approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities.

$\boxed{3}$ **Definition** The **instantaneous velocity** of an object with position function $f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

provided that this limit exists.

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at $P$ (compare Equation 2 and the expression in Definition 3).

**EXAMPLE 3** Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

**SOLUTION** Since two different velocities are requested, it's efficient to start by finding the velocity at a general time $t = a$. Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$\begin{aligned} v(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a + h)^2 - 4.9a^2}{h} \\ &= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a \end{aligned}$$

(a) The velocity after 5 seconds is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time $t$ when $s(t) = 450$, that is,

$$4.9t^2 = 450$$

This gives

$$t^2 = \frac{450}{4.9} \qquad \text{and} \qquad t = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

## Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Definition 3). In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

**4 Definition** The **derivative of a function $f$ at a number $a$**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and $h$ approaches 0 if and only if $x$ approaches $a$. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines (see Definition 1), is

**5**

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

**EXAMPLE 4** Use Definition 4 to find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) $a$.

**SOLUTION**

(a) From Definition 4 we have

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{(2 + h)^2 - 8(2 + h) + 9 - (-3)}{h} \\
&= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h} \\
&= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h - 4)}{h} = \lim_{h \to 0} (h - 4) = -4
\end{aligned}
$$

(b)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} (2a + h - 8) = 2a - 8$$

As a check on our work in part (a), notice that if we let $a = 2$, then $f'(2) = 2(2) - 8 = -4$. ■

**19–20** Use Definition 4 to find $f'(a)$ at the given number $a$.

**19.** $f(x) = \sqrt{4x + 1}, \quad a = 6$

**20.** $f(x) = 5x^4, \quad a = -1$

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**21–22** Use Equation 5 to find $f'(a)$ at the given number $a$.

**21.** $f(x) = \dfrac{x^2}{x + 6}, \quad a = 3$

**22.** $f(x) = \dfrac{1}{\sqrt{2x + 2}}, \quad a = 1$

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**23–26** Find $f'(a)$.

**23.** $f(x) = 2x^2 - 5x + 3$

**24.** $f(t) = t^3 - 3t$

**25.** $f(t) = \dfrac{1}{t^2 + 1}$

**26.** $f(x) = \dfrac{x}{1 - 4x}$

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