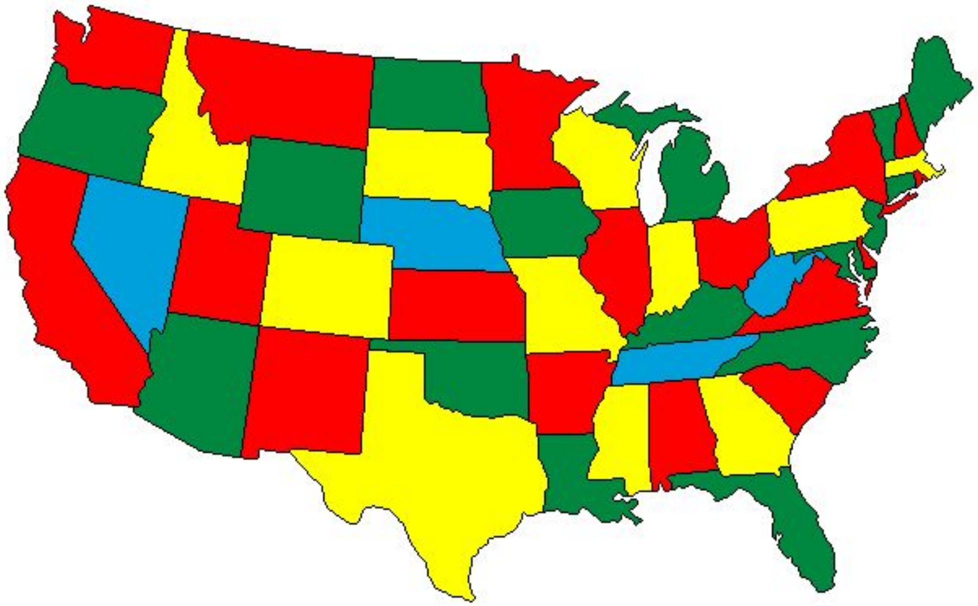
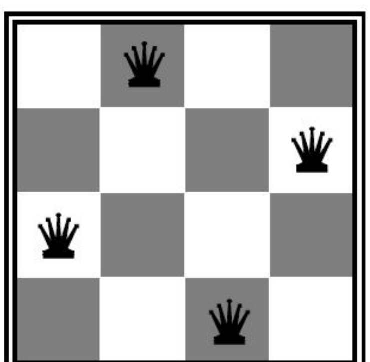
Constraint Satisfaction Problems (Chapter 6)

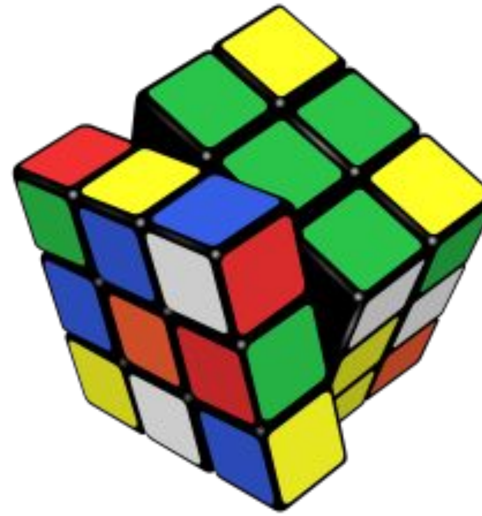




8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

What is search for?

- Assumptions: single agent, deterministic, fully observable, discrete environment
- **Search for *planning***
  - The path to the goal is the important thing
  - Paths have various costs, depths
- **Search for *assignment***
  - Assign values to variables while respecting certain constraints
  - The goal (complete, consistent assignment) is the important thing



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				7				
	4	8		2		1		3
	5	2					9	
		1						
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Constraint satisfaction problems (CSPs)

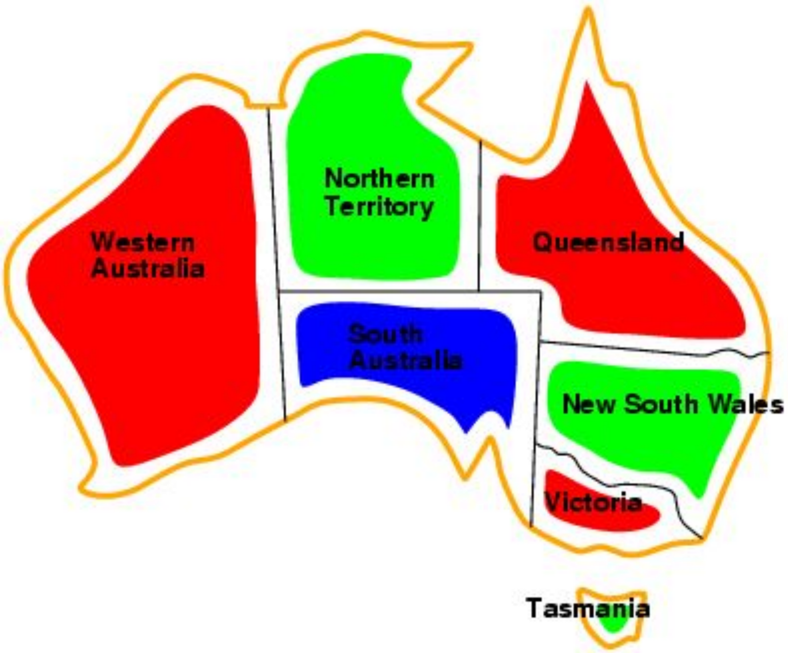
- Definition:
  - **State** is defined by variables X_i with values from domain D_i
  - **Goal test** is a set of constraints specifying allowable combinations of values for subsets of variables
  - **Solution** is a complete, consistent assignment

- How does this compare to the "generic" tree search formulation?
  - A more structured representation for states, expressed in a formal representation language
  - Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map Coloring

- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g., WA ≠ NT, or (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

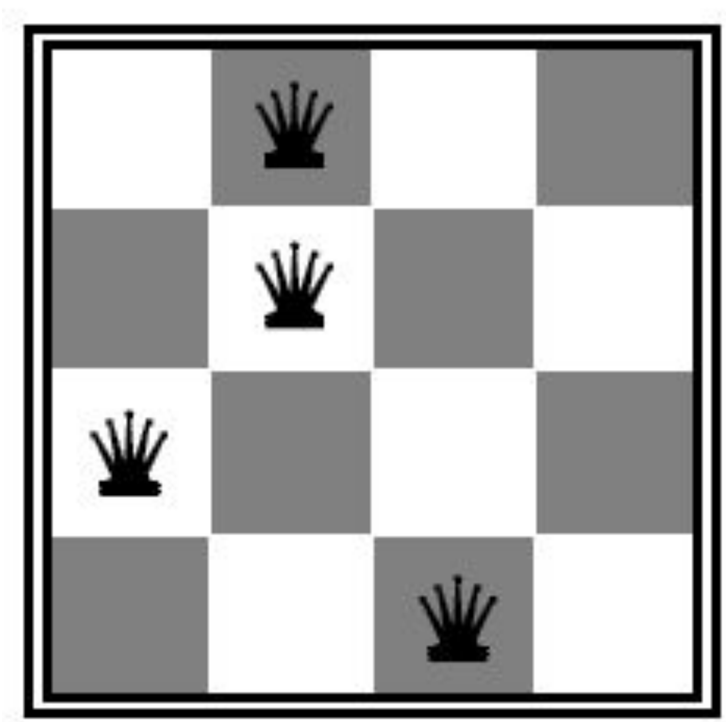
Example: Map Coloring

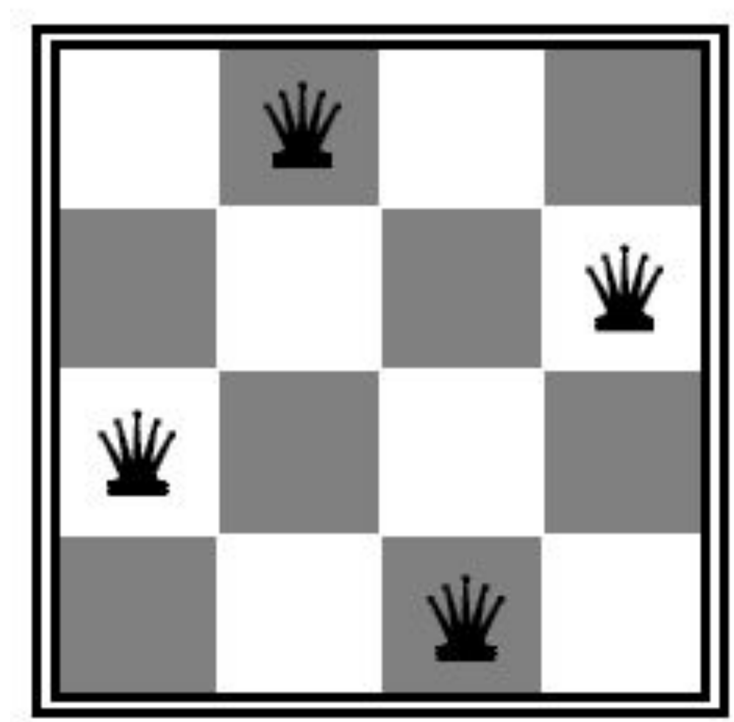


- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Example: *n*-queens problem

- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal





Example: N-Queens

- **Variables:** X_{ij}
- **Domains:** {0, 1}
- **Constraints:**

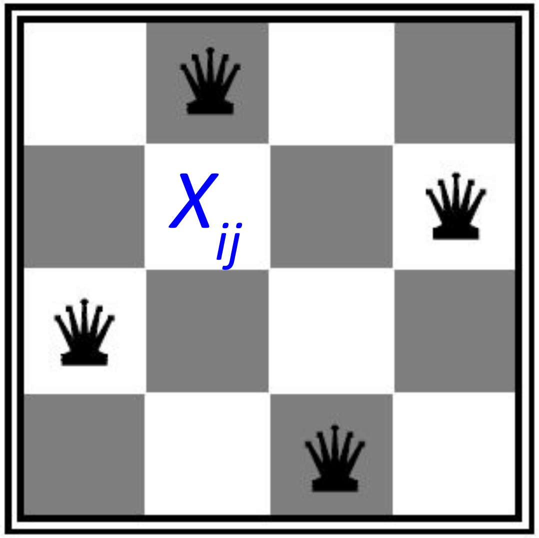
$$\Sigma_{i,j} X_{ij} = N$$

$$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$(X_{ij}, X_{i+k,\, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$$

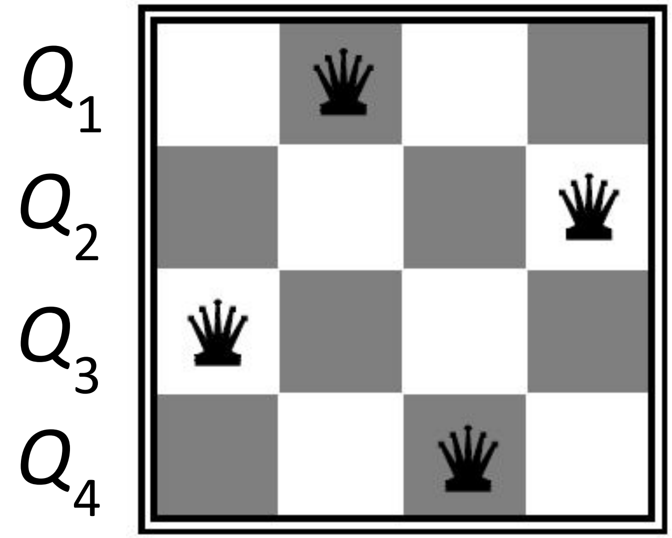
$$(X_{ij}, X_{i+k,\, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$$



N-Queens: Alternative formulation

- **Variables:** Q_i
- **Domains:** $\{1, \ldots, N\}$
- **Constraints:**

  $\forall\ i, j$ non-threatening (Q_i, Q_j)



Example: Cryptarithmetic

- **Variables:** T, W, O, F, U, R

  X_1, X_2
- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  Alldiff(T, W, O, F, U, R)

  $T \neq 0, F \neq 0$

$$
\begin{array}{cccc}
 & X_2 & X_1 & \\
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R
\end{array}
$$

Example: Sudoku

- **Variables:** X_{ij}
- **Domains:** {1, 2, …, 9}
- **Constraints:**

  Alldiff(X_{ij} in the same *unit*)

					8			4
	8	4		1	6			
			5			1		
1		3	8			9		
6		8		X_{ij}		4		3
		2			9	5		1
		7			2			
			7	8		2	6	
2			3					

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- More examples of CSPs: http://www.csplib.org/

Standard search formulation (incremental)

- **States:**
  - Variables and values assigned so far
- **Initial state:**
  - The empty assignment
- **Action:**
  - Choose any unassigned variable and assign to it a value that does not violate any constraints
    - Fail if no legal assignments
- **Goal test:**
  - The current assignment is complete and satisfies all constraints

Standard search formulation (incremental)

- What is the depth of any solution (assuming ***n*** variables)?

  n (this is good)

- Given that there are ***m*** possible values for any variable, how many paths are there in the search tree?

  $n! \cdot m^n$ (this is bad)

- How can we reduce the branching factor?

Backtracking search

- In CSP's, variable assignments are **commutative**
  - For example, *[WA = red then NT = green]* is the same as *[NT = green then WA = red]*
- We only need to consider assignments to a single variable at each level (i.e., we fix the order of assignments)
  - Then there are only m^n leaves
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

Example

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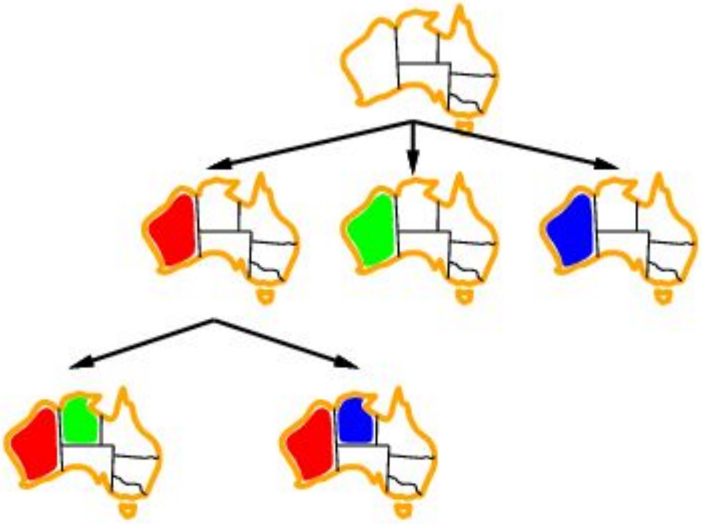
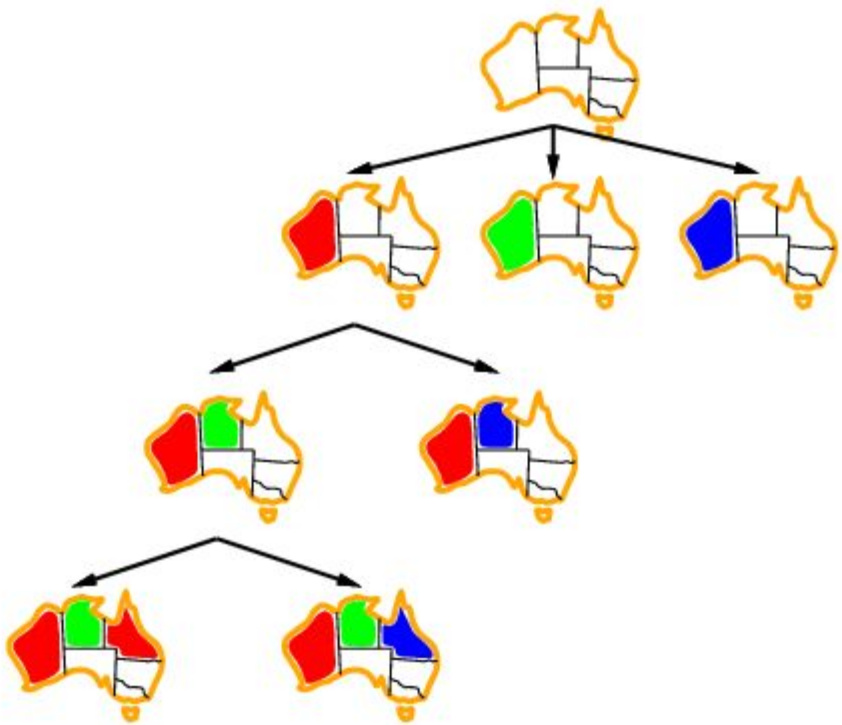


Example



Backtracking search algorithm

```
function RECURSIVE-BACKTRACKING(assignment, csp)
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
       if value is consistent with assignment given CONSTRAINTS[csp]
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Which variable should be assigned next?

- **Most constrained variable:**
  - Choose the variable with the fewest legal values
  - A.k.a. **minimum remaining values** (MRV) heuristic

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Given a variable, in which order should its values be tried?

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

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Which assignment for Q should we choose?

Early detection of failure

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Apply *inference* to reduce the space of possible assignments and detect failure early

Early detection of failure

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Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

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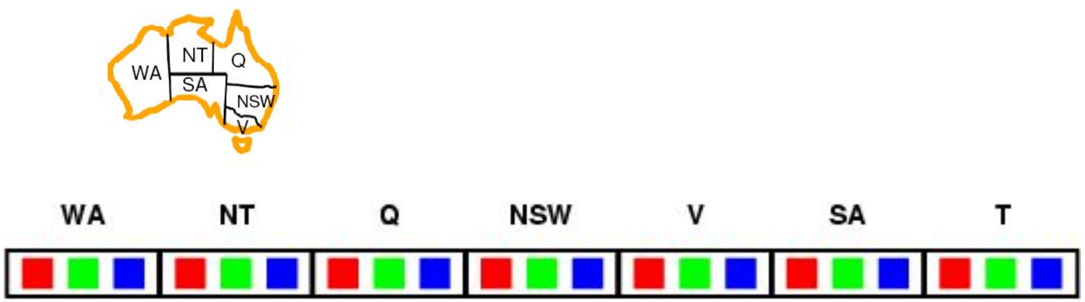

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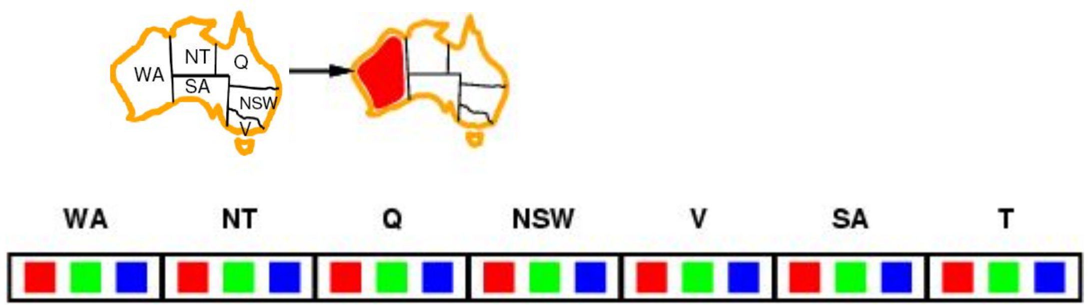

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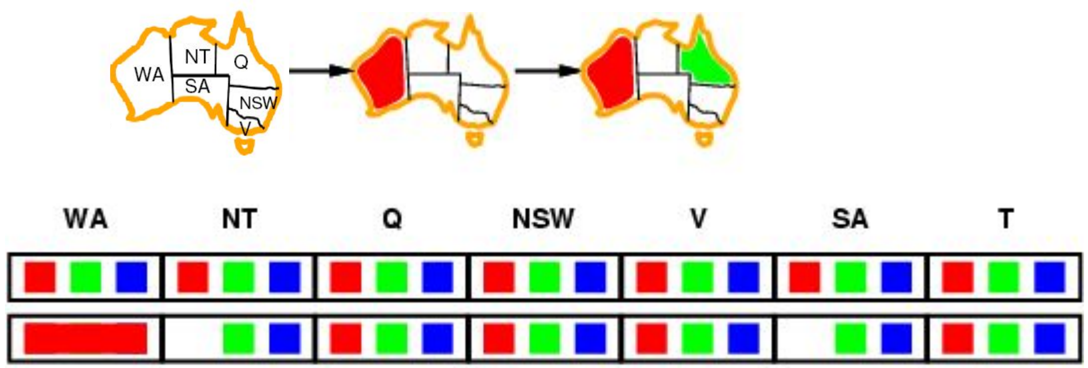

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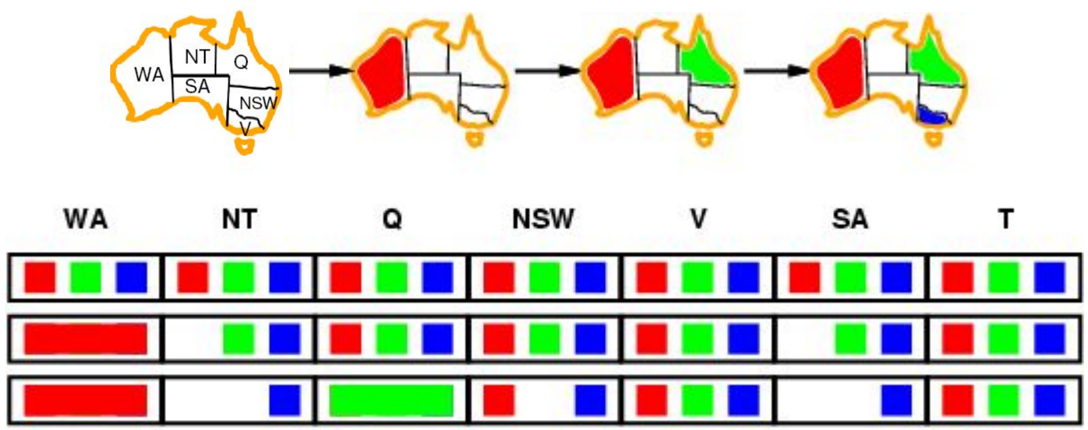
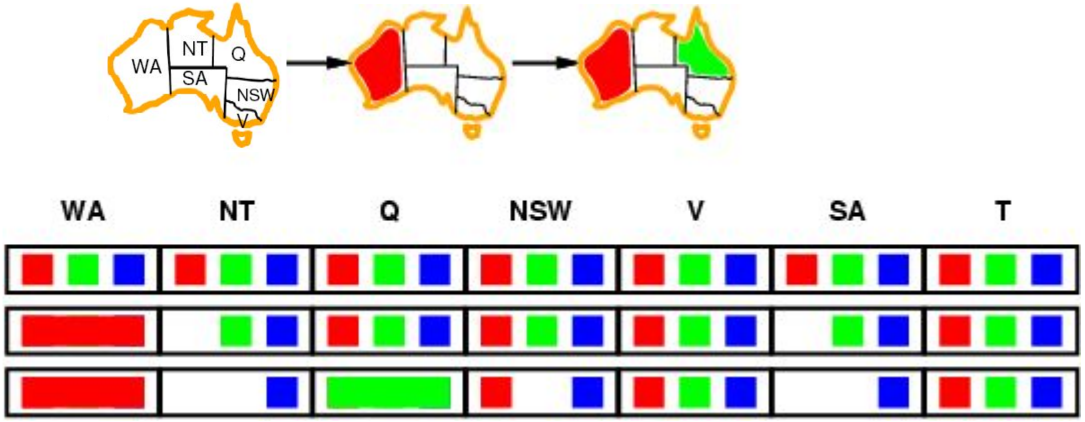


Constraint propagation

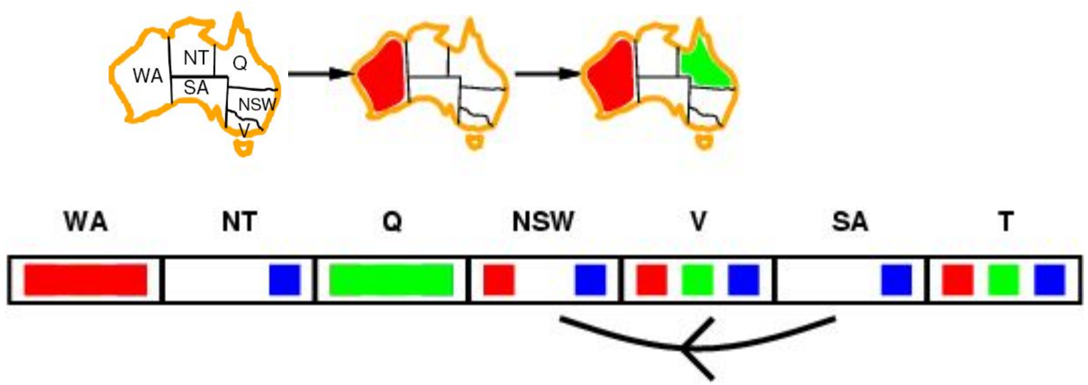
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures



- NT and SA cannot both be blue!
- **Constraint propagation** repeatedly enforces constraints *locally*

Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* □ *Y* is consistent iff for every value of *X* there is some allowed value of *Y*



Consistent!

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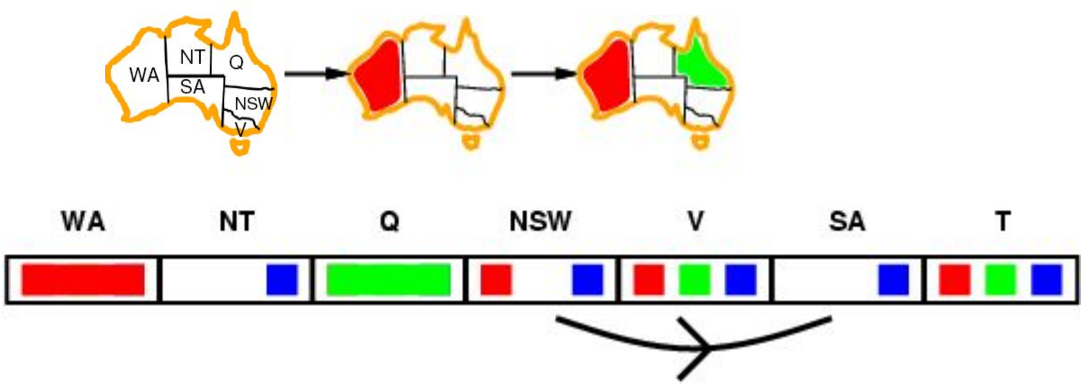

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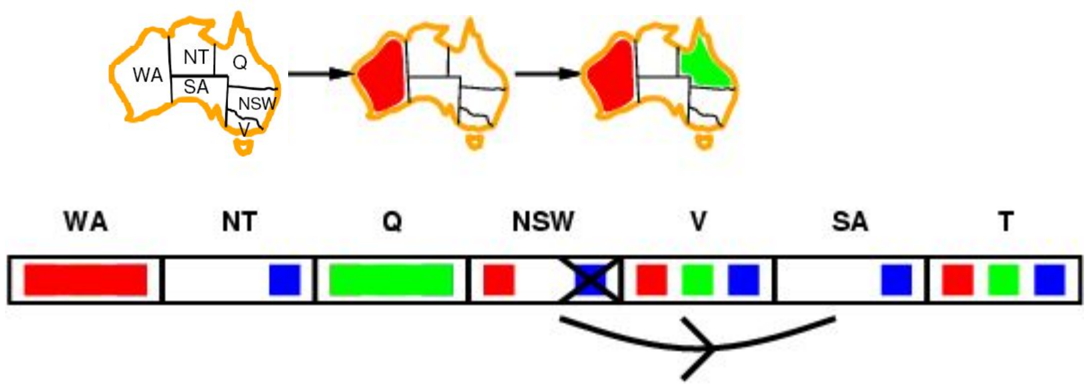

- If *X* loses a value, all pairs *Z* □ *X* need to be rechecked

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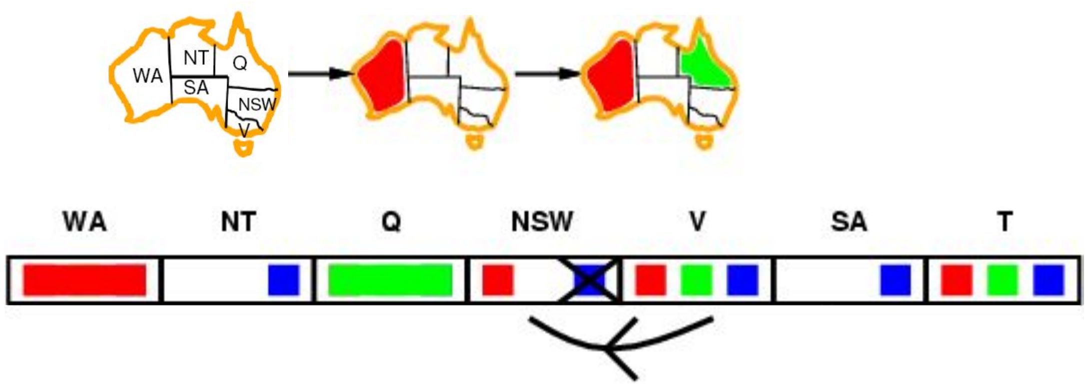

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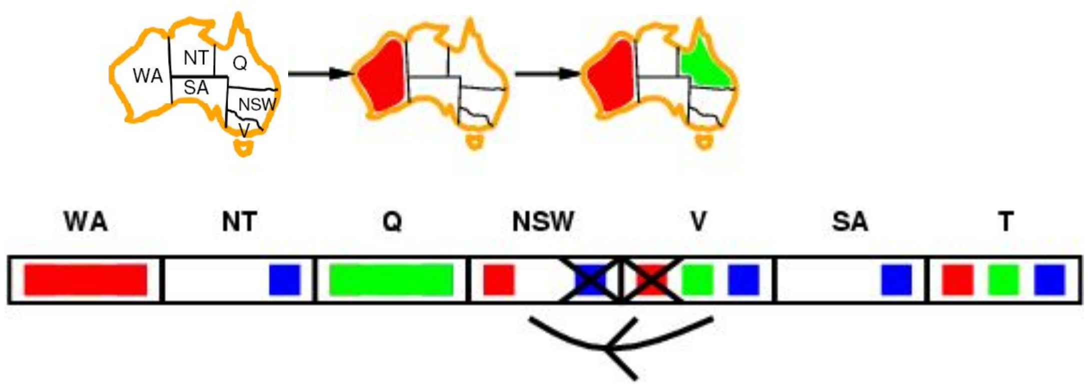

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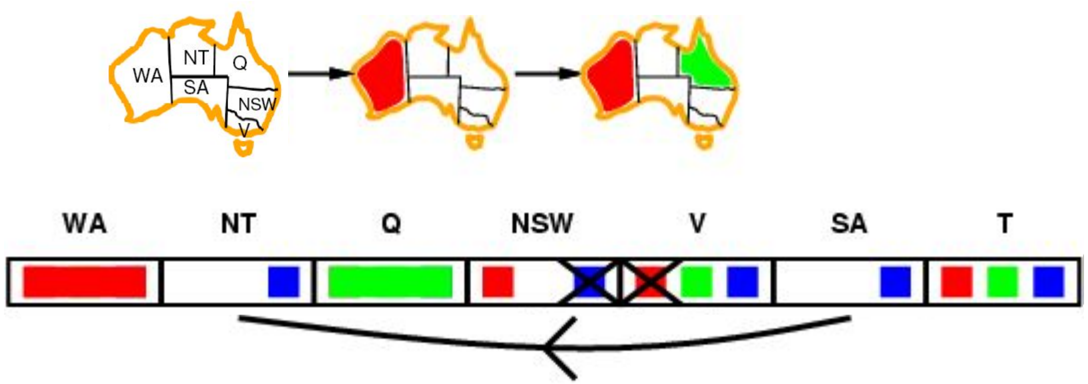

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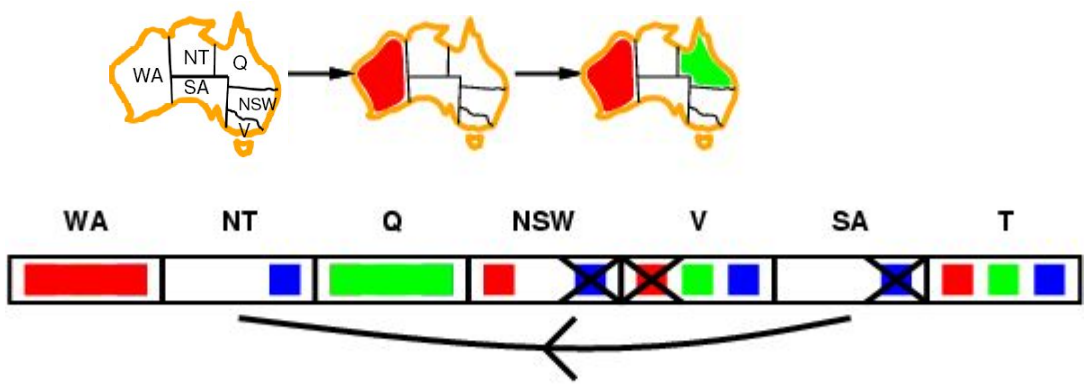


- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

Arc consistency algorithm AC-3

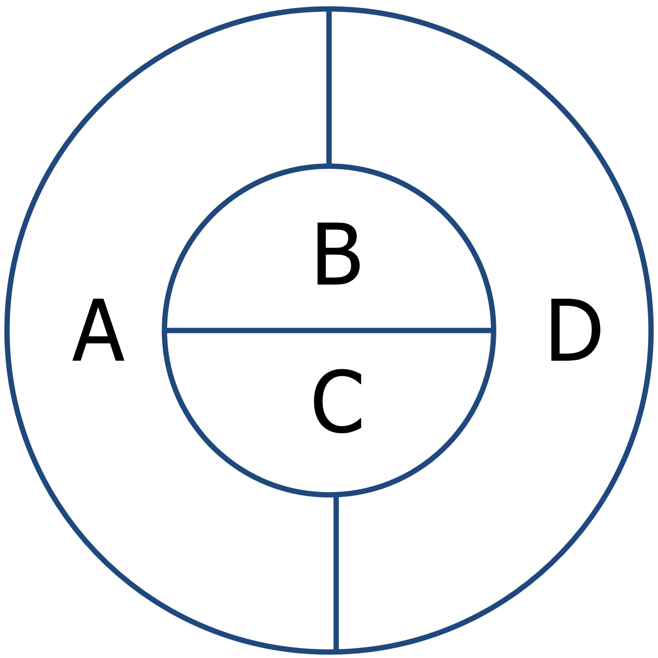
function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

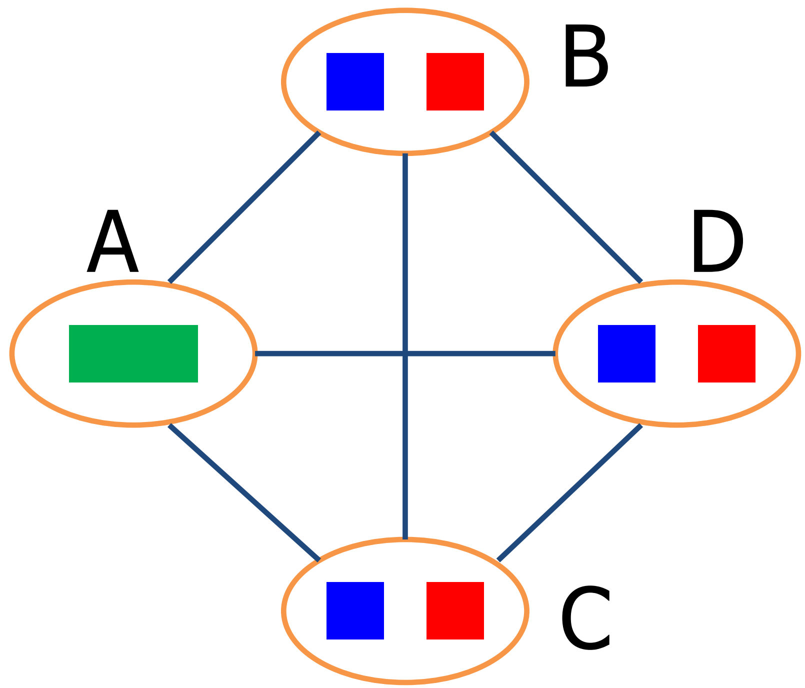
  while *queue* is not empty
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN[X_i]
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*
  return *removed*

Does arc consistency always detect the lack of a solution?



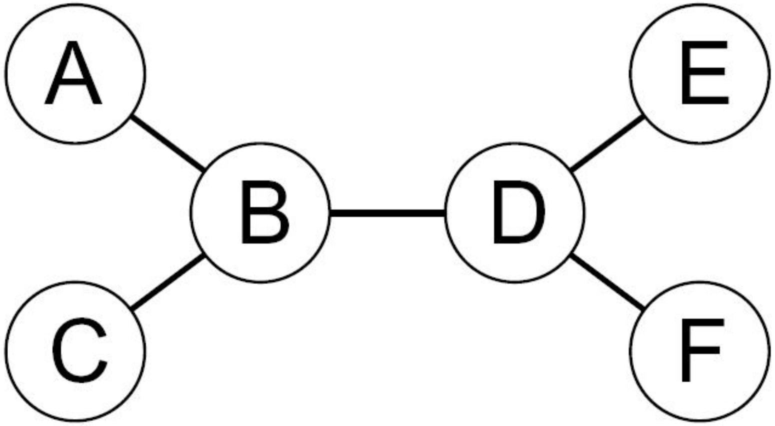


- There exist stronger notions of consistency (path consistency, k-consistency), but we won't worry about them

Tree-structured CSPs

- Certain kinds of CSPs can be solved without resorting to backtracking search!

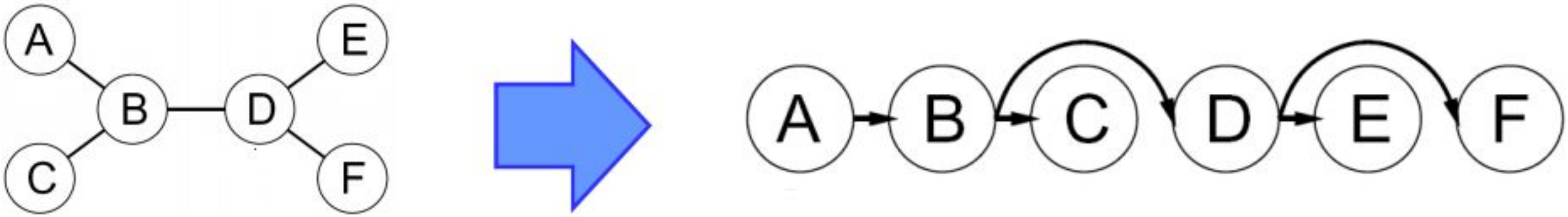
- *Tree-structured CSP*: constraint graph does not have any loops



Source: P. Abbeel, D. Klein, L. Zettlemoyer

Algorithm for tree-structured CSPs

- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



http://cs188ai.wikia.com/wiki/Tree_Structure_CSPs

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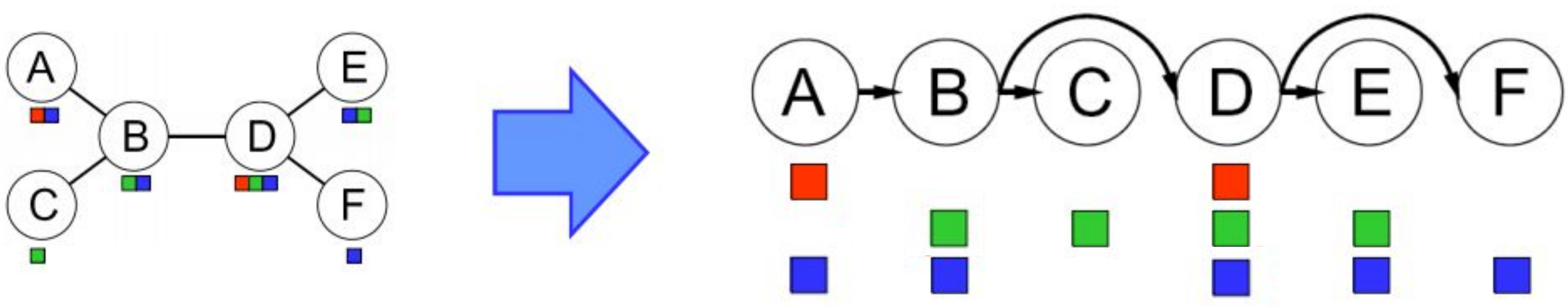
- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering
- Backward removal phase: check arc consistency starting from the rightmost node and going backwards



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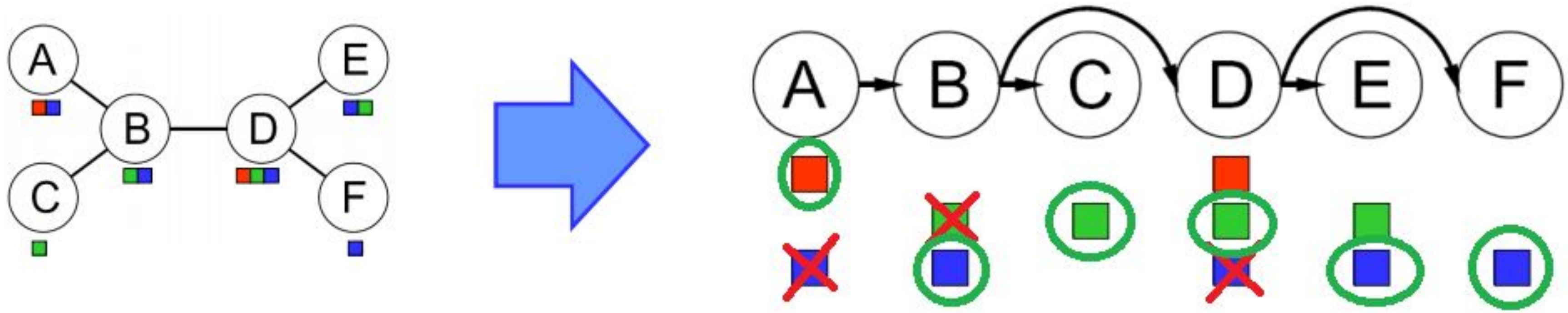
Algorithm for tree-structured CSPs

- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering
- Backward removal phase: check arc consistency starting from the rightmost node and going backwards
- Forward assignment phase: select an element from the domain of each variable going left to right. We are guaranteed that there will be a valid assignment because each arc is consistent



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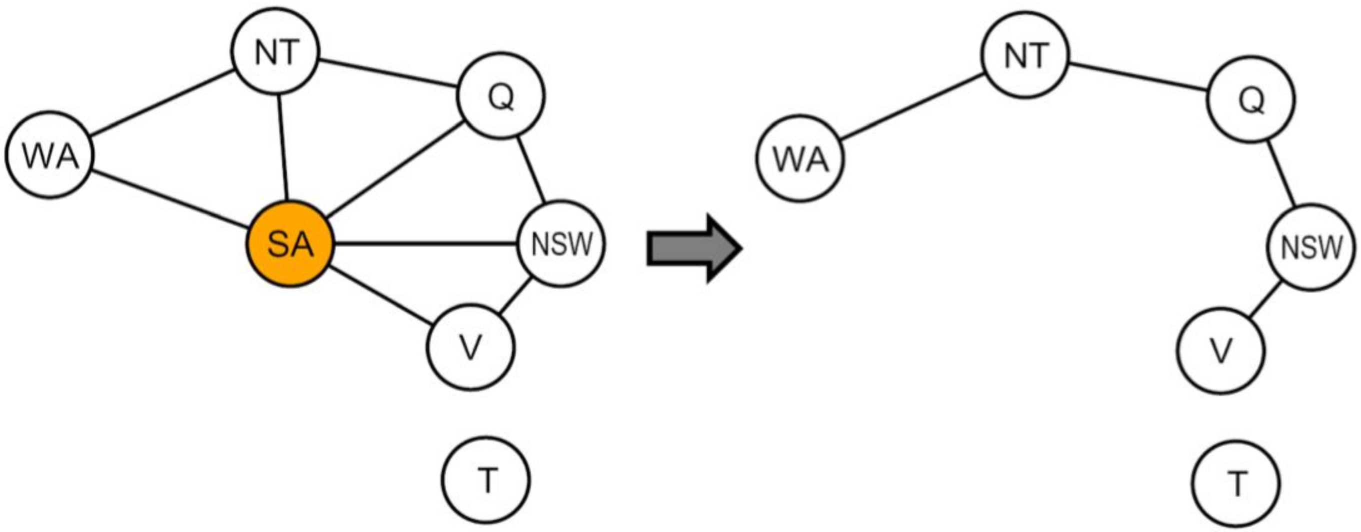
Algorithm for tree-structured CSPs

- If n is the numebr of variables and m is the domain size, what is the running time of this algorithm?
  - O(nm^2): we have to check arc consistency once for every node in the graph (every node has one parent), which involves looking at pairs of domain values

Nearly tree-structured CSPs



- **Cutset conditioning:** find a subset of variables whose removal makes the graph a tree, instantiate that set in all possible ways, prune the domains of the remaining variables and try to solve the resulting tree-structured CSP
- Cutset size c gives runtime $O(m^c\,(n - c)m^2)$

Source: P. Abbeel, D. Klein, L. Zettlemoyer

Algorithm for tree-structured CSPs

- Running time is O(nm^2)
  (n is the number of variables, m is the domain size)
  - We have to check arc consistency once for every node in the graph (every node has one parent), which involves looking at pairs of domain values
- What about backtracking search for general CSPs?
  - Worst case O(m^n)
- Can we do better?

Computational complexity of CSPs

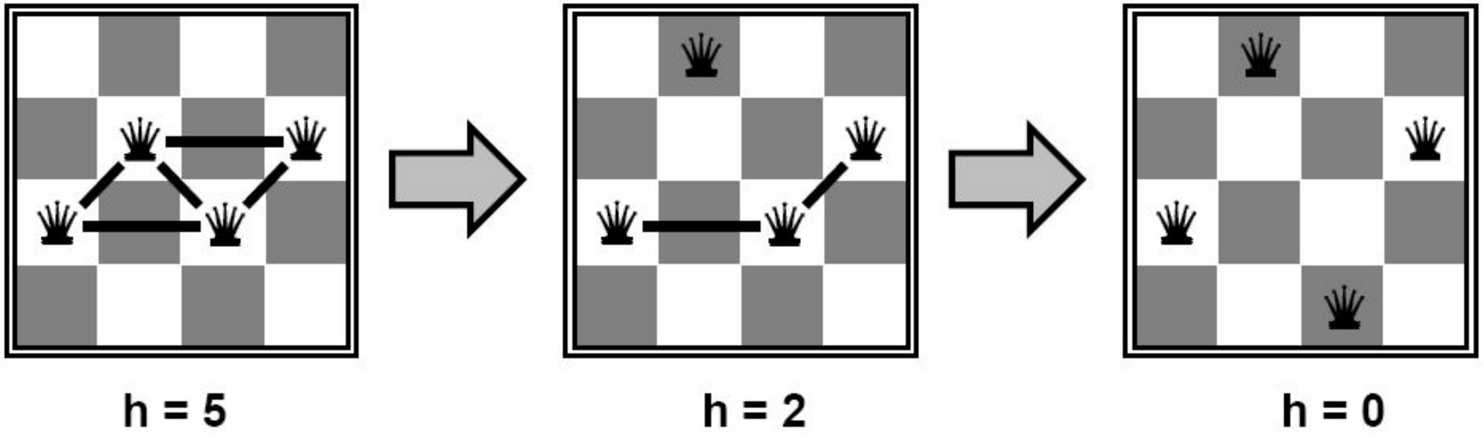
- The satisfiability (SAT) problem:
  - Given a Boolean formula, is there an assignment of the variables that makes it evaluate to true?

$$(X_1 \vee \overline{X}_7 \vee X_{13}) \wedge (\overline{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

- SAT is *NP-complete*
  - NP: class of decision problems for which the "yes" answer can be verified in polynomial time
  - An NP-complete problem is in NP and every other problem in NP can be efficiently reduced to it (Cook, 1971)
  - Other NP-complete problems: graph coloring, n-puzzle, generalized sudoku
  - It is not known whether P = NP, i.e., no efficient algorithms for solving SAT in general are known

Local search for CSPs

- Start with "complete" states, i.e., all variables assigned
- Allow states with unsatisfied constraints
- Attempt to improve states by reassigning variable values
- Hill-climbing search:
  - In each iteration, randomly select any conflicted variable and choose value that violates the fewest constraints
  - I.e., attempt to greedily minimize total number of violated constraints



h = number of conflicts

Local search for CSPs

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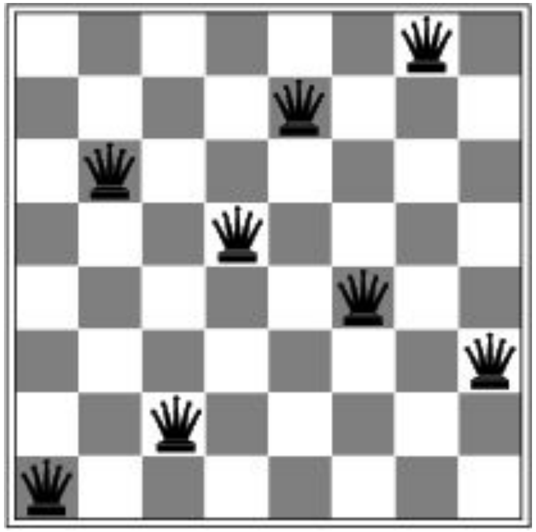

$h = 1$

Local search for CSPs

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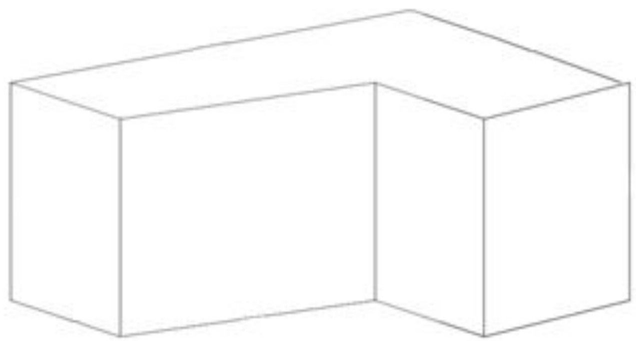
  - Problem: *local minima*
- For more on local search, see ch. 4

CSP in computer vision: Line drawing interpretation

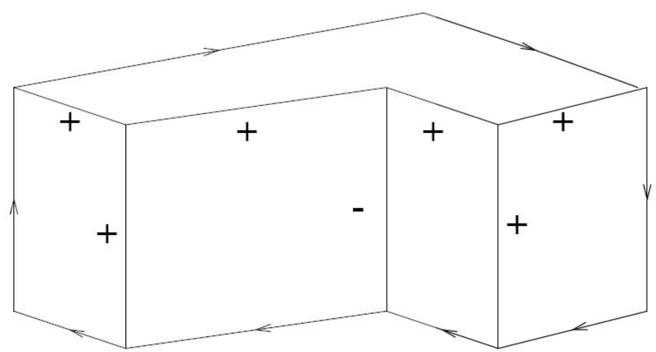
An example polyhedron:



Variables: edges

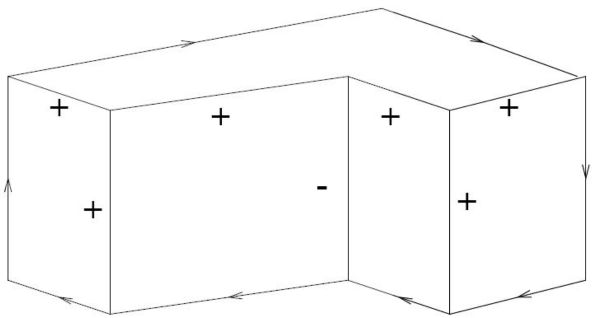
Domains: +, −, →, ←

Desired output:

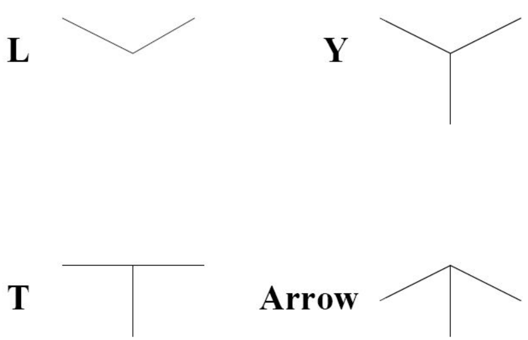


David Waltz, 1975

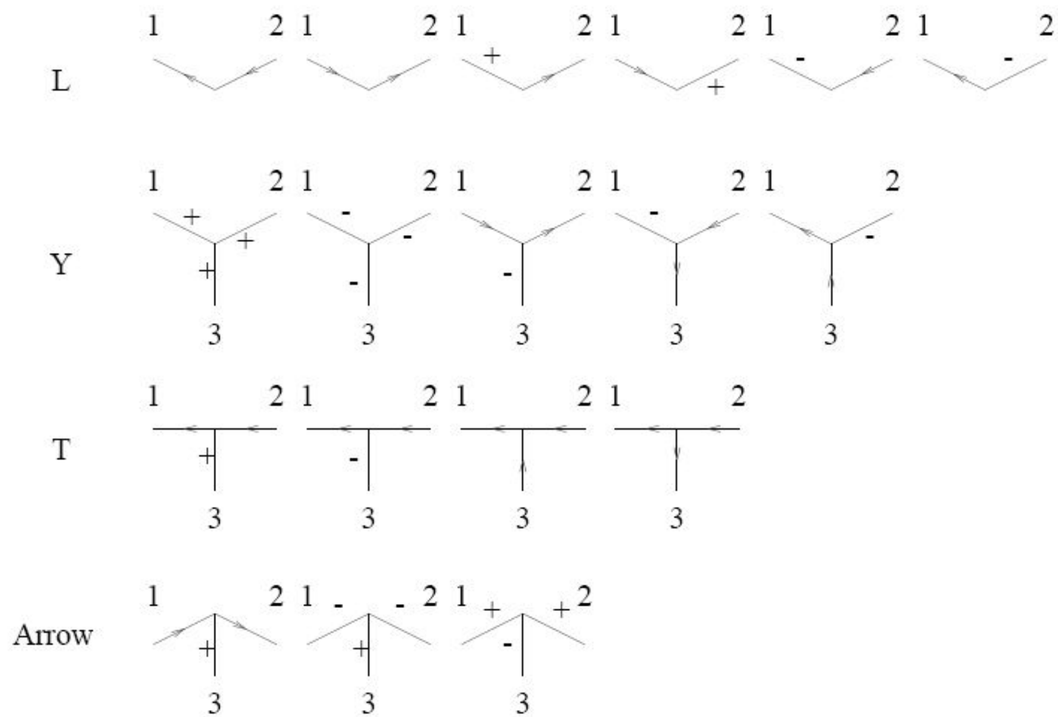
CSP in computer vision: Line drawing interpretation



Four vertex types:



Constraints imposed by each vertex type:



[David Waltz](), 1975

CSP in computer vision: 4D Cities

1. When was each photograph taken?
2. When did each building first appear?
3. When was each building removed?

Set of Photographs:

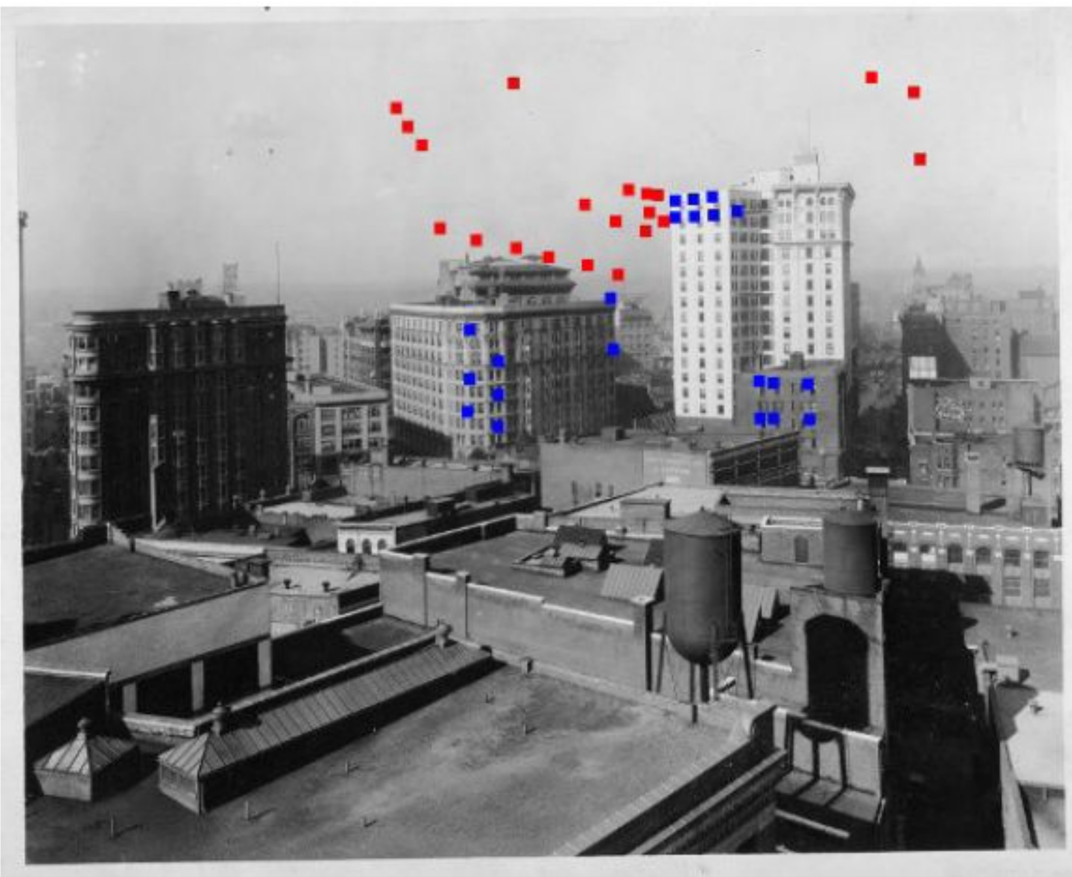
Set of Objects: Buildings

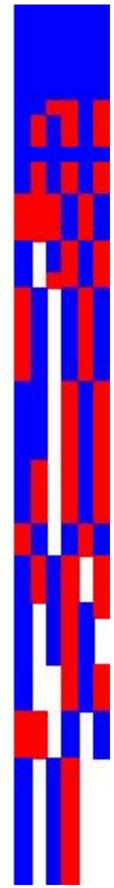
G. Schindler, F. Dellaert, and S.B. Kang, [Inferring Temporal Order of Images From 3D Structure](), IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR), 2007.

http://www.cc.gatech.edu/~phlosoft/

CSP in computer vision: 4D Cities

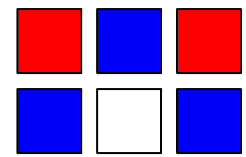
observed missing occluded



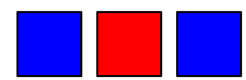


Columns: images
Rows: points

Satisfies constraints:



Violates constraints:

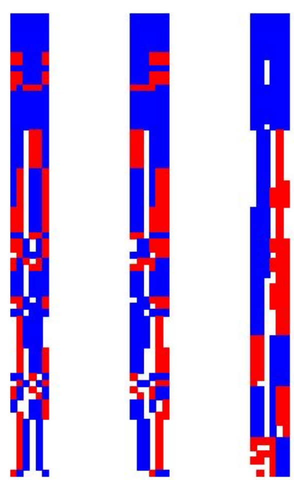


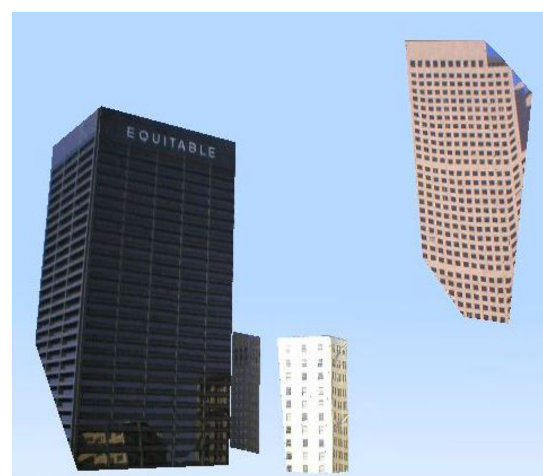
- Goal: reorder images (columns) to have as few violations as possible

CSP in computer vision: 4D Cities

- **Goal:** reorder images (columns) to have as few violations as possible
- **Local search:** start with random ordering of columns, swap columns or groups of columns to reduce the number of conflicts

- Can also reorder the rows to group together points that appear and disappear at the same time – that gives you buildings





Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- **Backtracking** = depth-first search where successor states are generated by considering assignments to a single variable
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure
  - **Constraint propagation** (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Complexity of CSPs
  - NP-complete in general (exponential worst-case running time)
  - Efficient solutions possible for special cases (e.g., tree-structured CSPs)
- Alternatives to backtracking search: local search