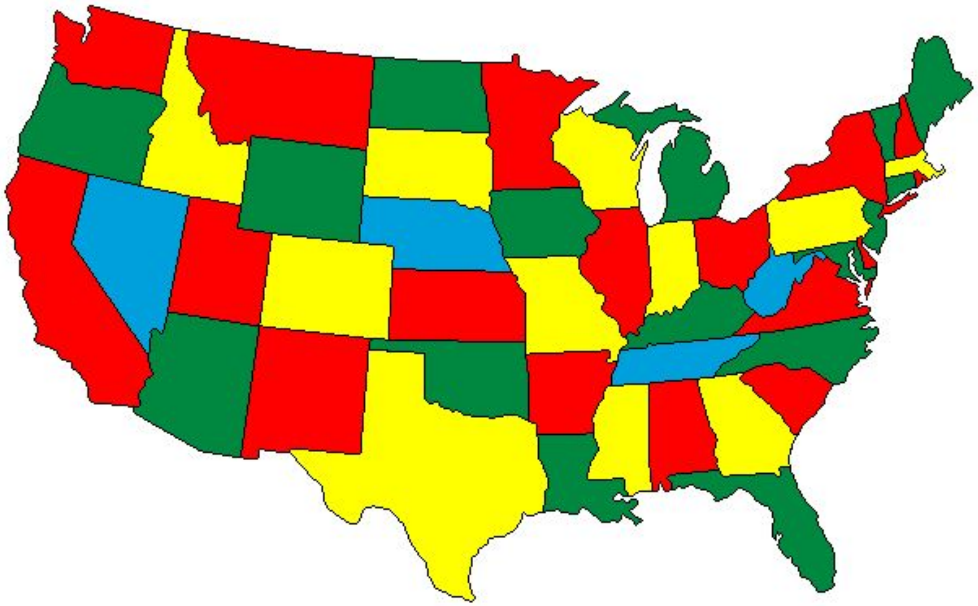
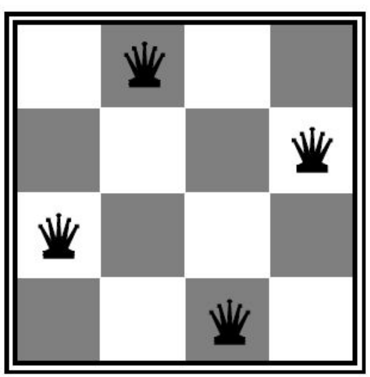
# Constraint Satisfaction Problems (Chapter 6)





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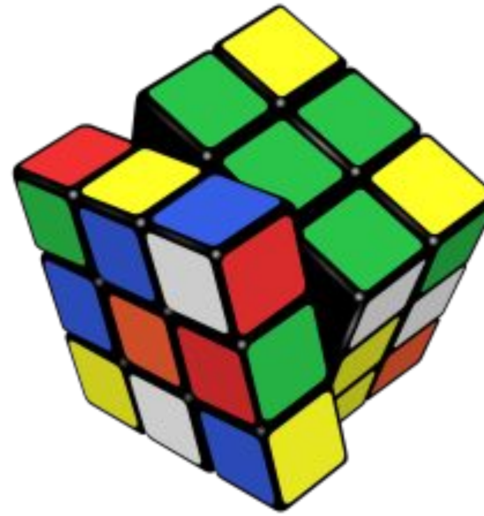
| | | | | | | 4 | | |
| | 1 | | | | | 6 | 5 | |
| 5 | | 9 | | 3 | | 7 | 8 | |
| | | | | 7 | | | | |
| | 4 | 8 | | 2 | | 1 | | 3 |
| | 5 | 2 | | | | | 9 | |
| | | 1 | | | | | | |
| 3 | | | 9 | | 2 | | | 5 |

# What is search for?

- Assumptions: single agent, deterministic, fully observable, discrete environment
- **Search for *planning***
  - The path to the goal is the important thing
  - Paths have various costs, depths
- **Search for *assignment***
  - Assign values to variables while respecting certain constraints
  - The goal (complete, consistent assignment) is the important thing



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| | | | | 7 | | | | |
| | 4 | 8 | | 2 | | 1 | | 3 |
| | 5 | 2 | | | | | 9 | |
| | | 1 | | | | | | |
| 3 | | | 9 | | 2 | | | 5 |

# Constraint satisfaction problems (CSPs)

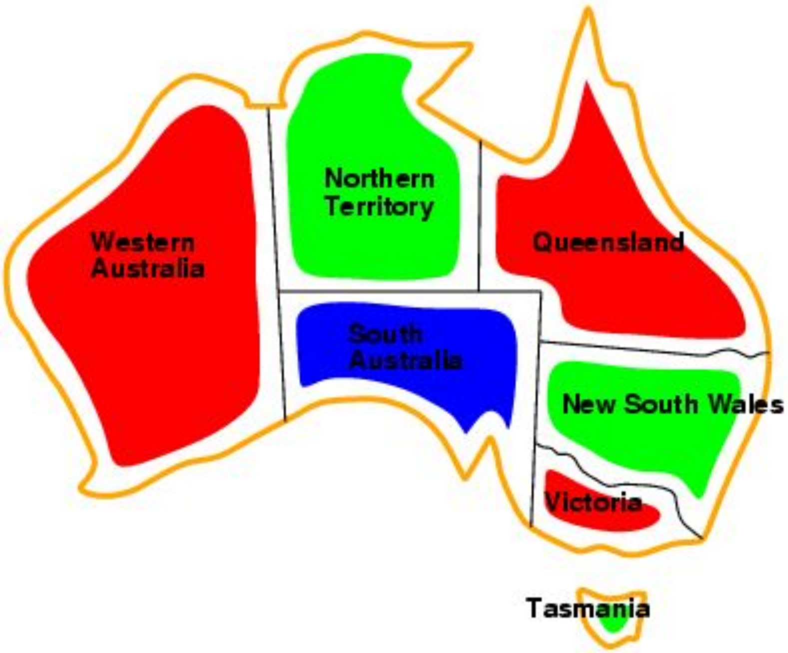
- Definition:
  - **State** is defined by variables $X_i$ with values from domain $D_i$
  - **Goal test** is a set of constraints specifying allowable combinations of values for subsets of variables
  - **Solution** is a complete, consistent assignment

- How does this compare to the "generic" tree search formulation?
  - A more structured representation for states, expressed in a formal representation language
  - Allows useful general-purpose algorithms with more power than standard search algorithms

# Example: Map Coloring


- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g., WA ≠ NT, or (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

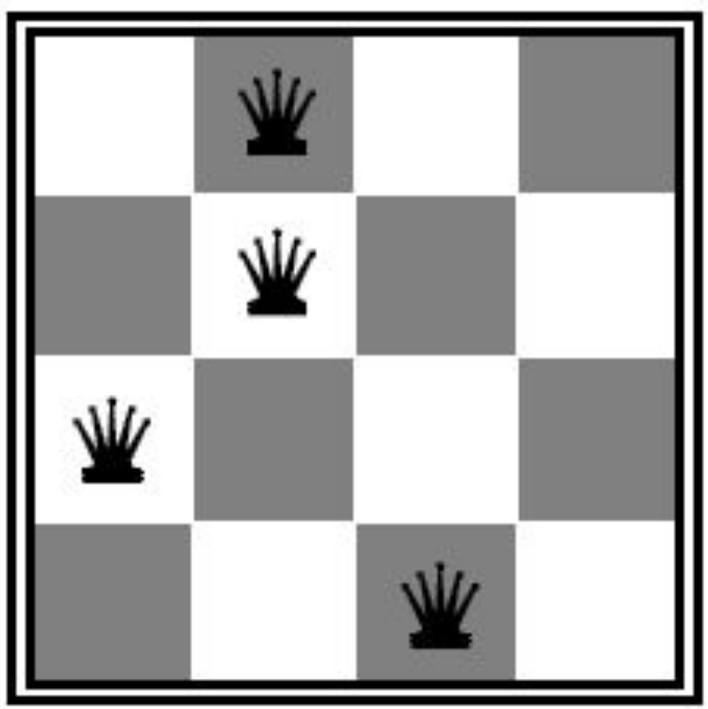
# Example: Map Coloring

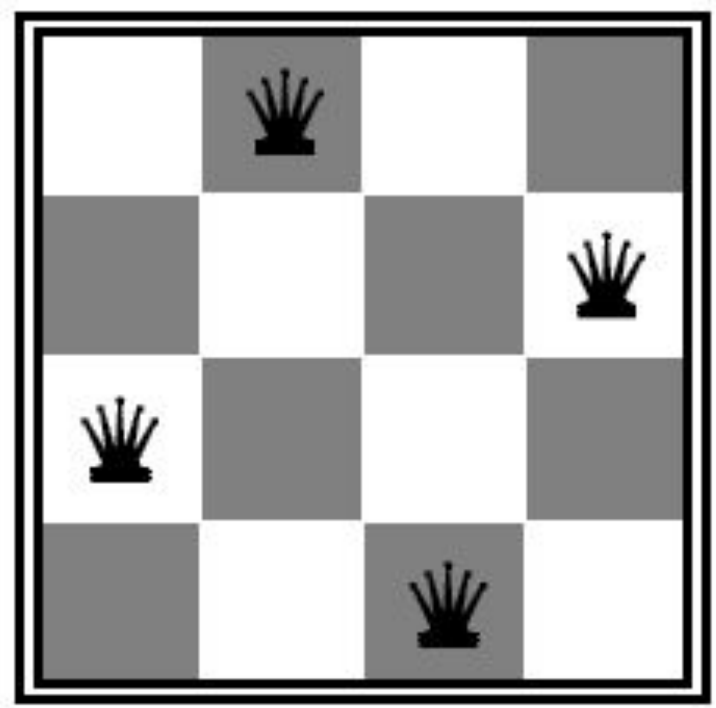



- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

# Example: *n*-queens problem

- Put *n* queens on an *n* × *n* board with no two queens on the same row, column, or diagonal





# Example: N-Queens

- **Variables:** $X_{ij}$
- **Domains:** {0, 1}
- **Constraints:**

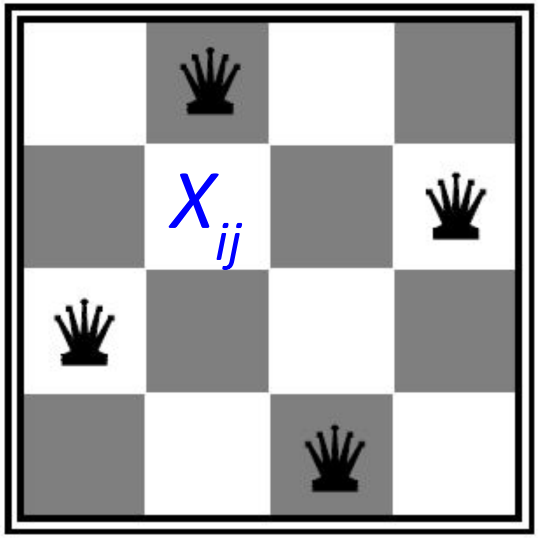
$\sum_{i,j} X_{ij} = N$

$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{i+k,\, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$

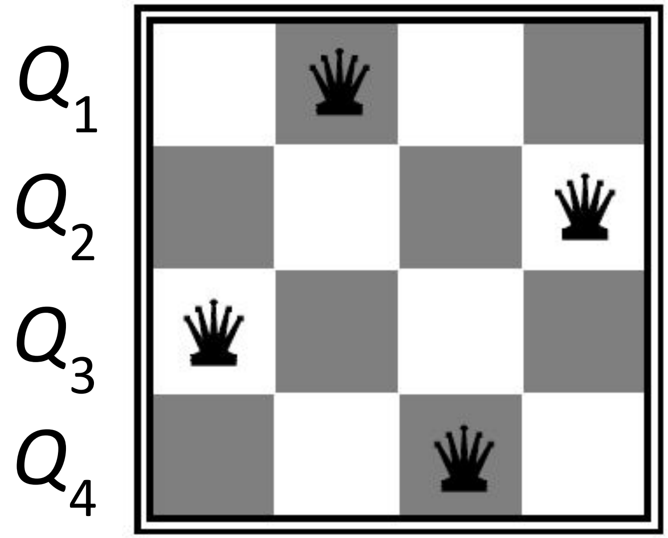
$(X_{ij}, X_{i+k,\, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$



# N-Queens: Alternative formulation

- **Variables:** $Q_i$
- **Domains:** $\{1, \ldots, N\}$
- **Constraints:**

  $\forall$ *i*, *j* non-threatening $(Q_i, Q_j)$



# Example: Cryptarithmetic

- **Variables:** T, W, O, F, U, R

  $X_1$, $X_2$
- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  Alldiff(T, W, O, F, U, R)

  $T \neq 0$, $F \neq 0$

$$
\begin{array}{cccc}
 & X_2 & X_1 & \\
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R
\end{array}
$$

# Example: Sudoku

- **Variables:** $X_{ij}$
- **Domains:** {1, 2, …, 9}
- **Constraints:**

  Alldiff($X_{ij}$ in the same *unit*)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | 8 | | | 4 |
| | 8 | 4 | | 1 | 6 | | | |
| | | | 5 | | | 1 | | |
| 1 | | 3 | 8 | | | 9 | | |
| 6 | | 8 | | $X_{ij}$ | | 4 | | 3 |
| | | 2 | | | 9 | 5 | | 1 |
| | | 7 | | | 2 | | | |
| | | | 7 | 8 | | 2 | 6 | |
| 2 | | | 3 | | | | | |

# Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- More examples of CSPs: http://www.csplib.org/

# Standard search formulation (incremental)

- **States:**
  – Variables and values assigned so far
- **Initial state:**
  – The empty assignment
- **Action:**
  – Choose any unassigned variable and assign to it a value that does not violate any constraints
    - Fail if no legal assignments
- **Goal test:**
  – The current assignment is complete and satisfies all constraints

# Standard search formulation (incremental)

- What is the depth of any solution (assuming $n$ variables)?
  $n$ (this is good)

- Given that there are $m$ possible values for any variable, how many paths are there in the search tree?
  $n! \cdot m^n$ (this is bad)

- How can we reduce the branching factor?

# Backtracking search

- In CSP's, variable assignments are **commutative**
  - For example, *[WA = red then NT = green]* is the same as *[NT = green then WA = red]*
- We only need to consider assignments to a single variable at each level (i.e., we fix the order of assignments)
  - Then there are only $m^n$ leaves
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

# Example

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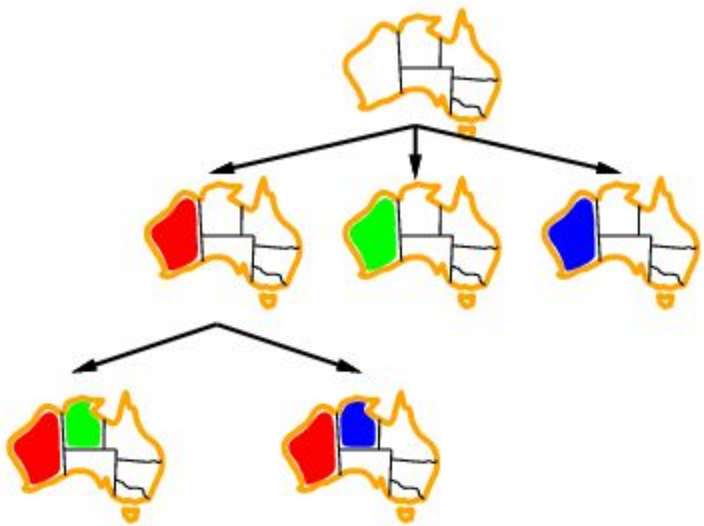
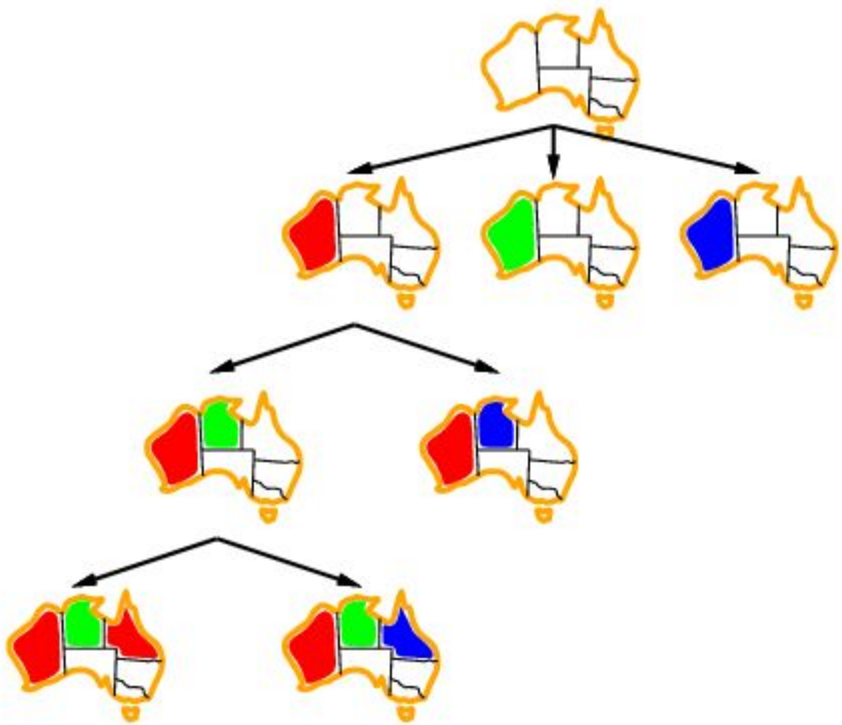


# Example



# Backtracking search algorithm

```
function RECURSIVE-BACKTRACKING(assignment, csp)
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
      if value is consistent with assignment given CONSTRAINTS[csp]
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

# Which variable should be assigned next?

- **Most constrained variable:**
  - Choose the variable with the fewest legal values
  - A.k.a. **minimum remaining values** (MRV) heuristic

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  - Tie-breaker among most constrained variables

# Which variable should be assigned next?

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# Given a variable, in which order should its values be tried?

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

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Which assignment for Q should we choose?

# Early detection of failure

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Apply *inference* to reduce the space of possible assignments and detect failure early

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# Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

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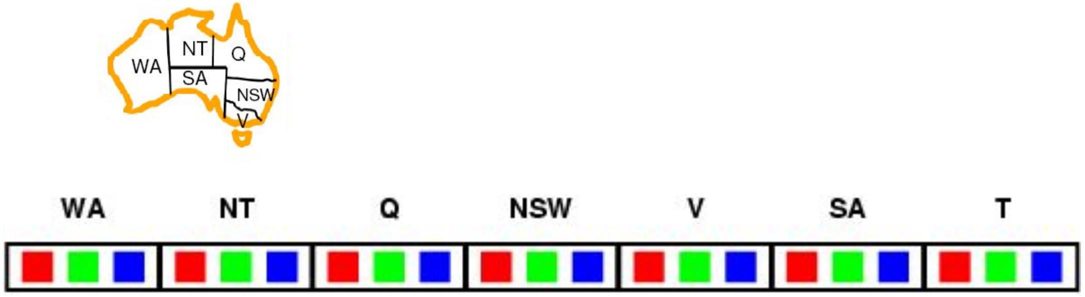

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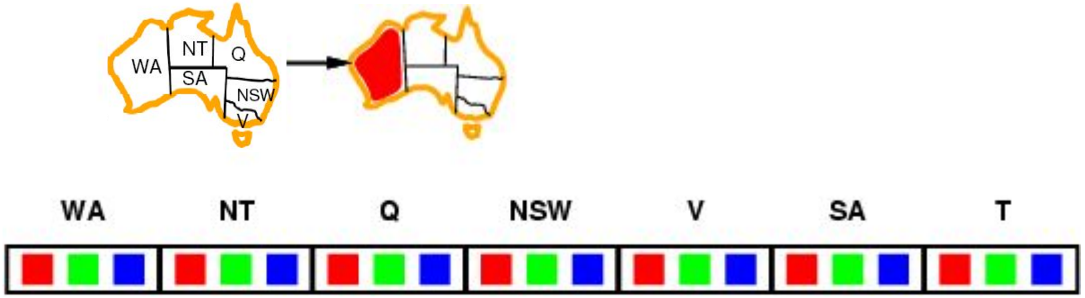

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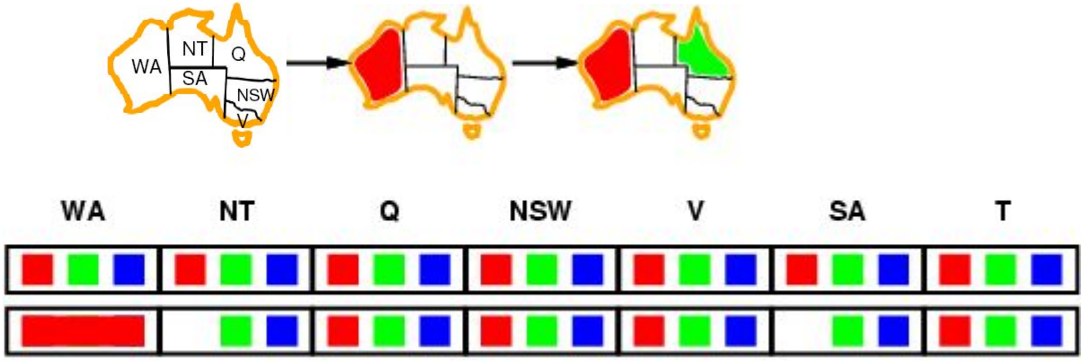

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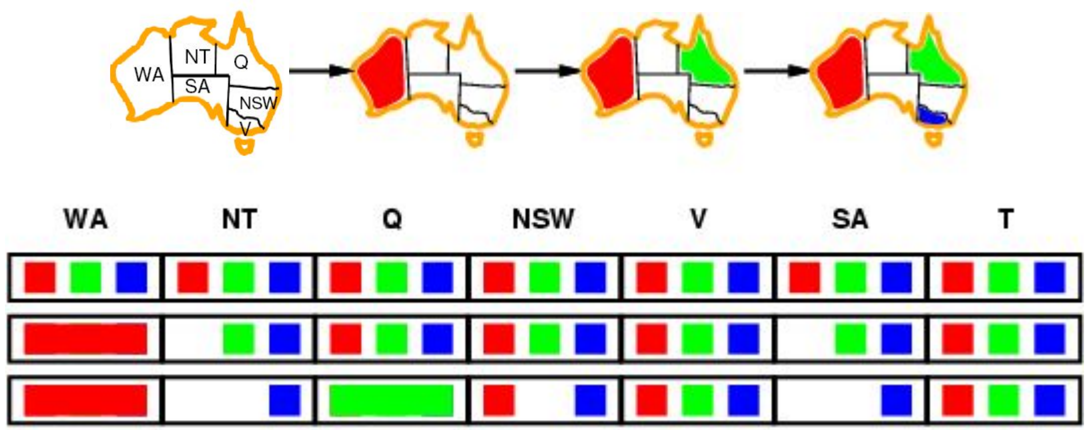
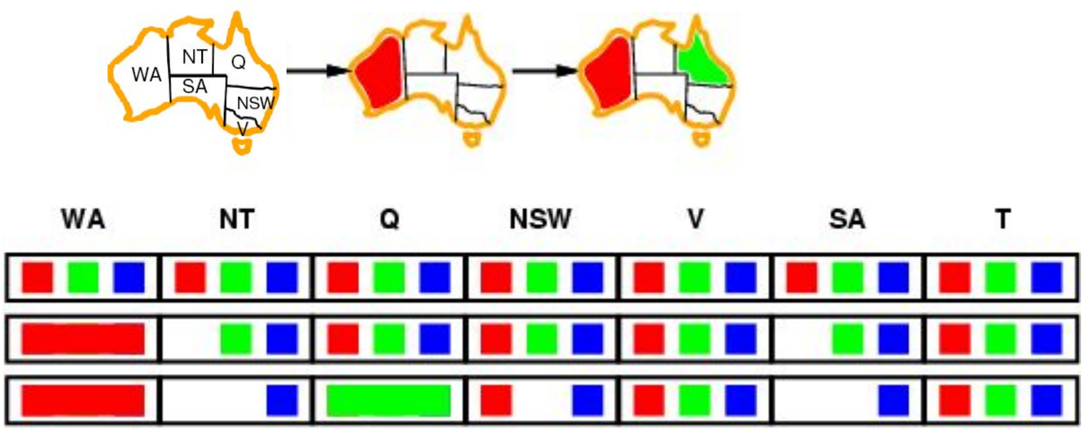


# Constraint propagation

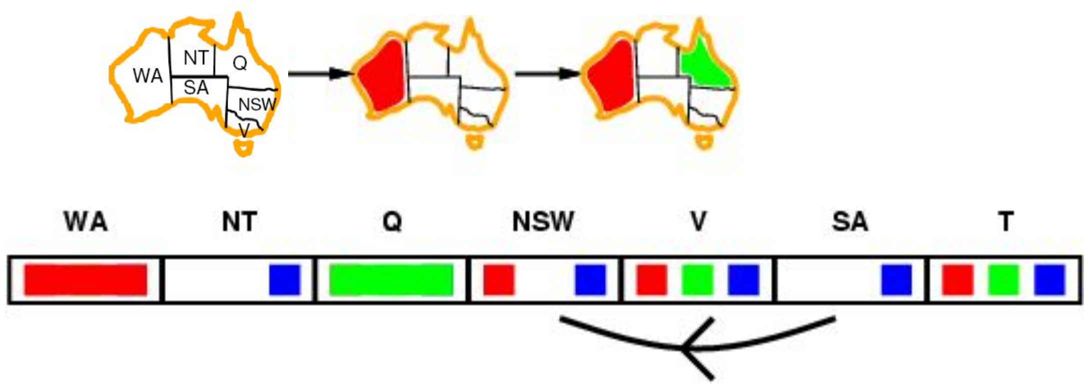
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures




- NT and SA cannot both be blue!
- **Constraint propagation** repeatedly enforces constraints *locally*

# Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* □ *Y* is consistent iff for every value of *X* there is some allowed value of *Y*




Consistent!

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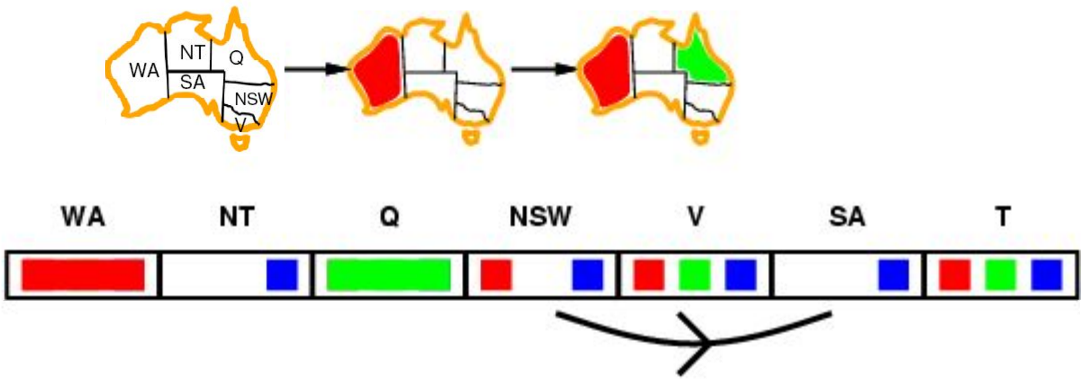

# Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
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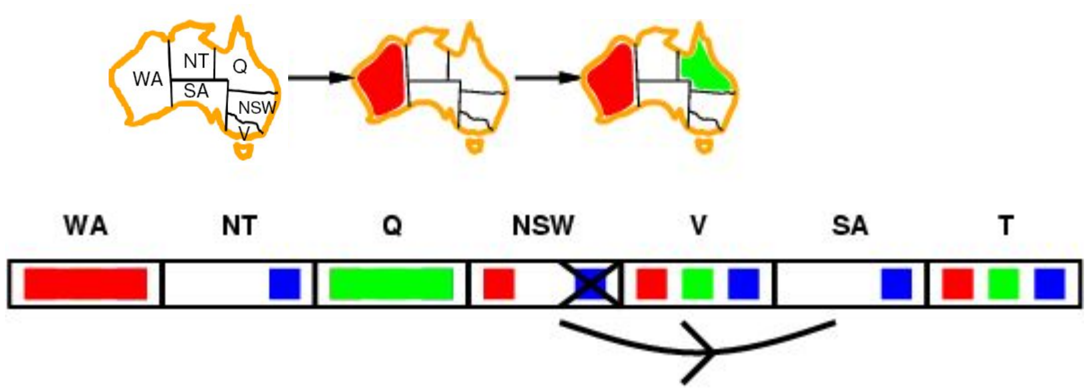
  - When checking *X* □*Y*, throw out any values of X for which there isn't an allowed value of Y

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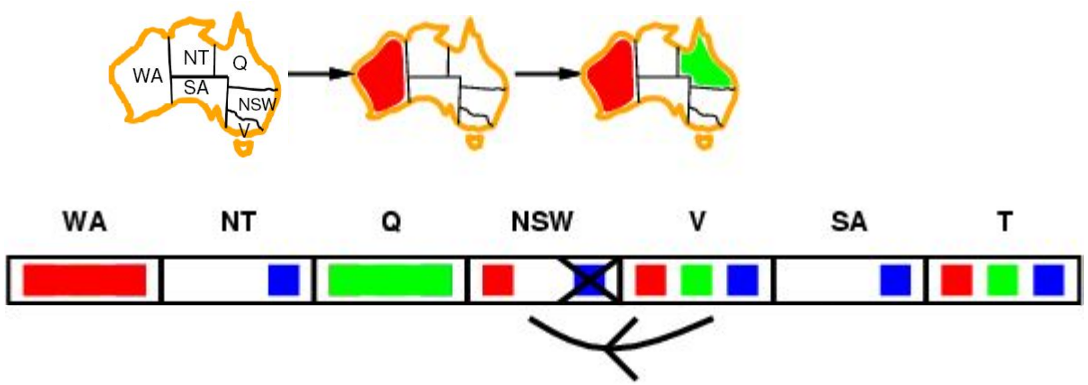


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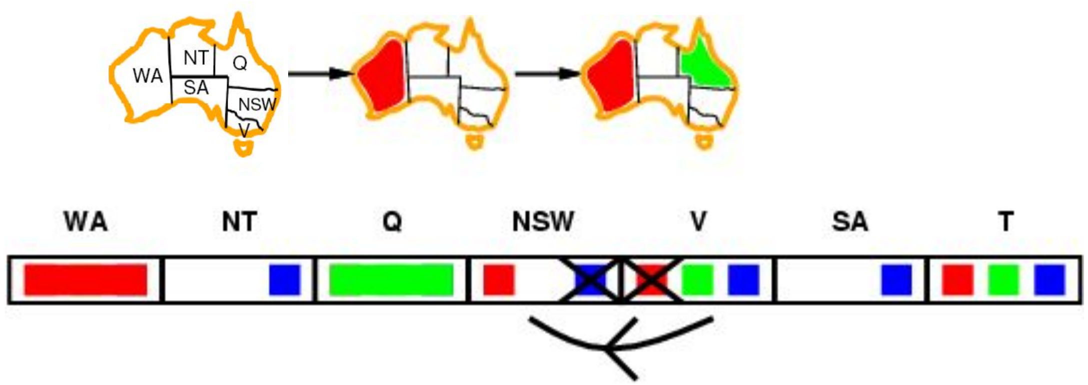


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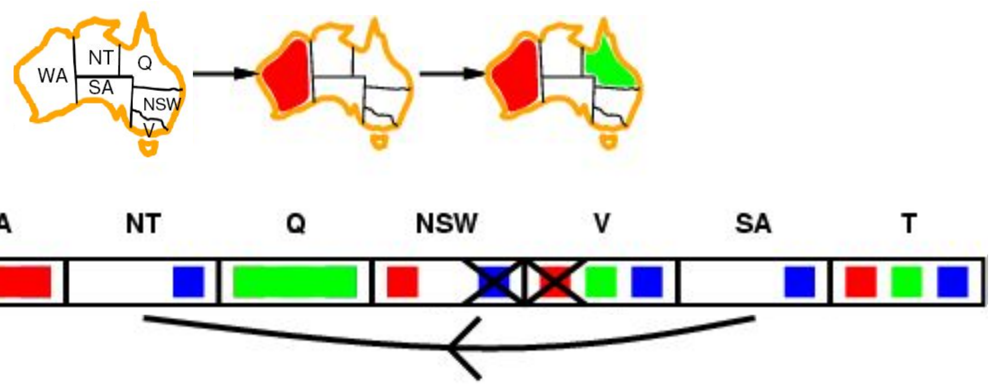

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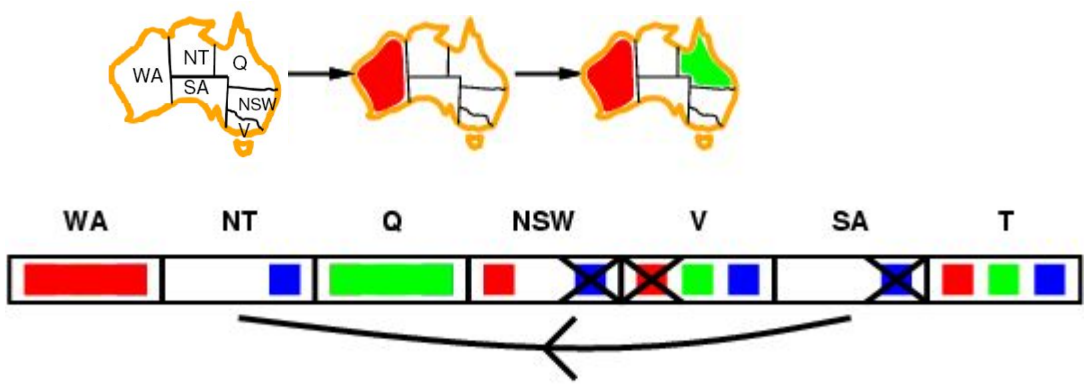



- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

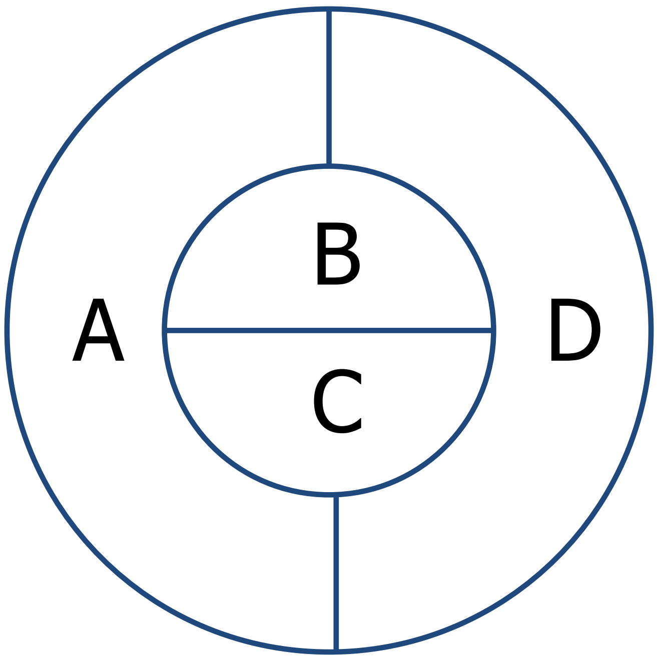
# Arc consistency algorithm AC-3

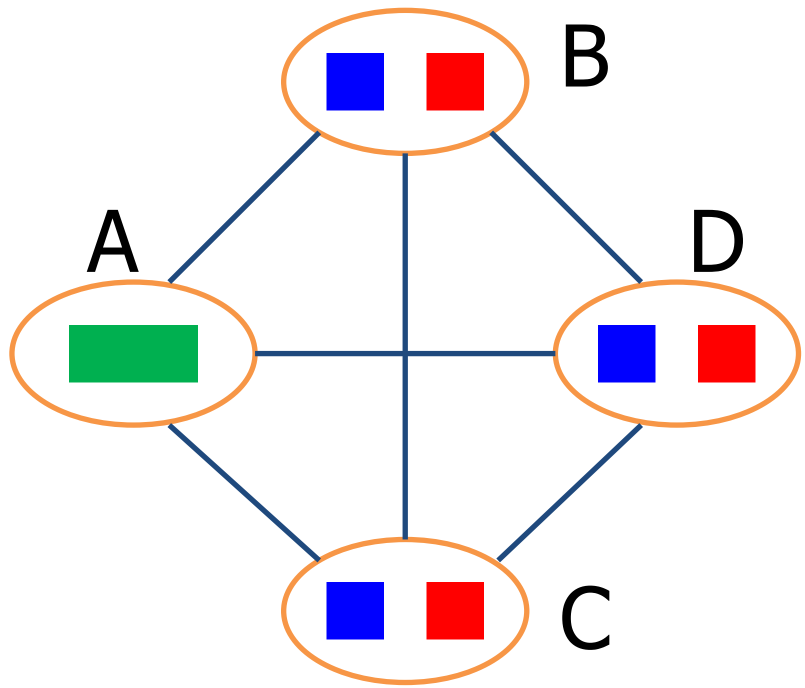
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

---

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi]
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

# Does arc consistency always detect the lack of a solution?



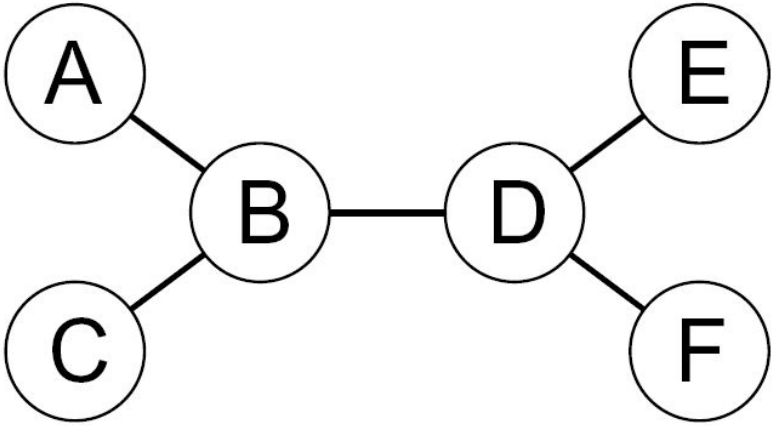




- There exist stronger notions of consistency (path consistency, k-consistency), but we won’t worry about them

# Tree-structured CSPs

- Certain kinds of CSPs can be solved without resorting to backtracking search!

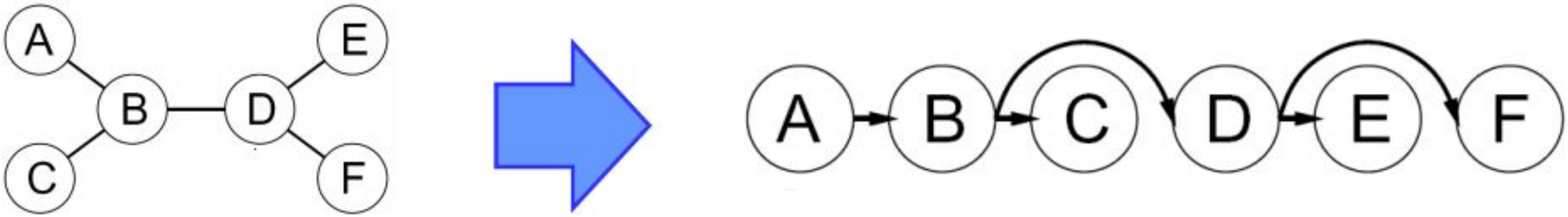
- *Tree-structured CSP*: constraint graph does not have any loops




Source: P. Abbeel, D. Klein, L. Zettlemoyer

# Algorithm for tree-structured CSPs

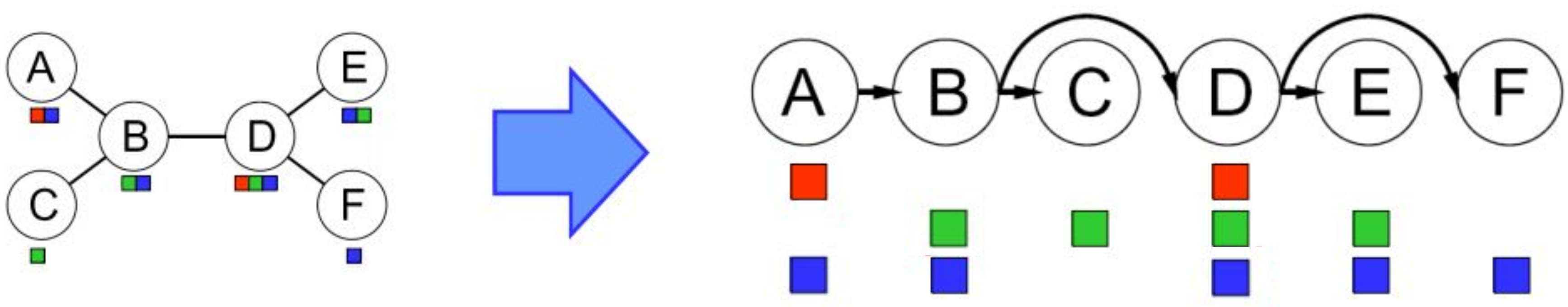
- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering




http://cs188ai.wikia.com/wiki/Tree_Structure_CSPs

# Algorithm for tree-structured CSPs

- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering
- Backward removal phase: check arc consistency starting from the rightmost node and going backwards

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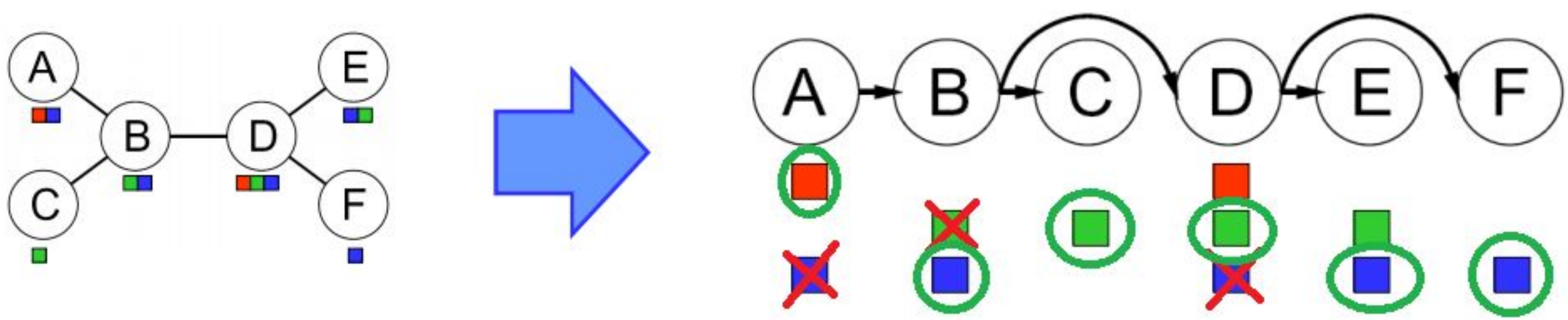
# Algorithm for tree-structured CSPs

- Choose one variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering
- Backward removal phase: check arc consistency starting from the rightmost node and going backwards
- Forward assignment phase: select an element from the domain of each variable going left to right. We are guaranteed that there will be a valid assignment because each arc is consistent

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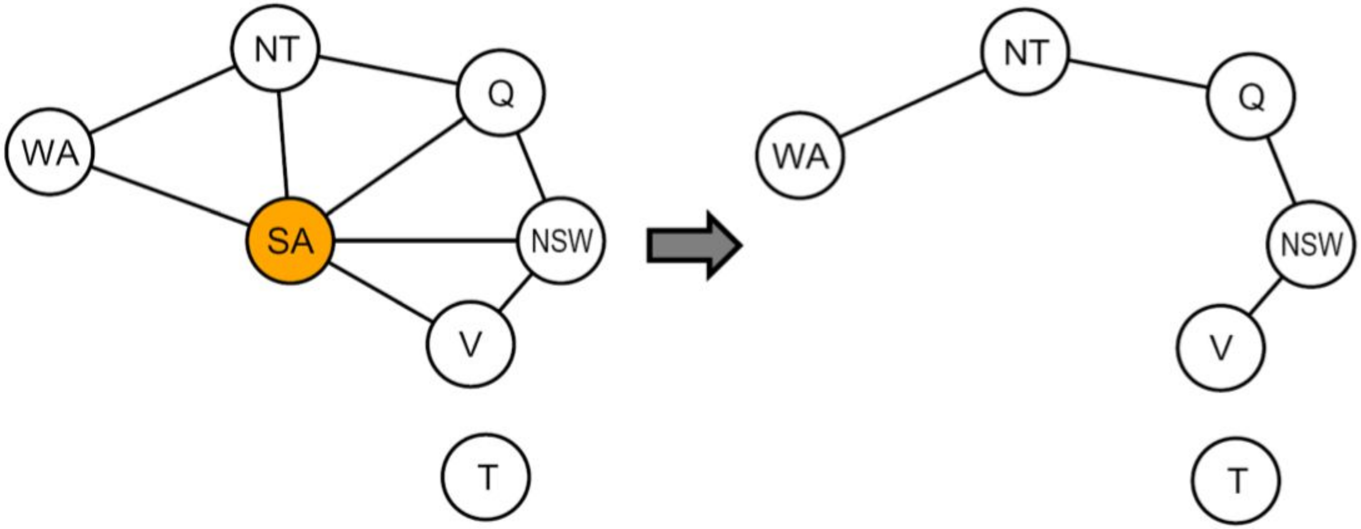
# Algorithm for tree-structured CSPs

- If n is the numebr of variables and m is the domain size, what is the running time of this algorithm?
  - O($nm^2$): we have to check arc consistency once for every node in the graph (every node has one parent), which involves looking at pairs of domain values

# Nearly tree-structured CSPs




- **Cutset conditioning:** find a subset of variables whose removal makes the graph a tree, instantiate that set in all possible ways, prune the domains of the remaining variables and try to solve the resulting tree-structured CSP
- Cutset size *c* gives runtime $O(m^c\ (n - c)m^2)$

Source: P. Abbeel, D. Klein, L. Zettlemoyer

# Algorithm for tree-structured CSPs

- Running time is $O(nm^2)$
  (n is the number of variables, m is the domain size)
  - We have to check arc consistency once for every node in the graph (every node has one parent), which involves looking at pairs of domain values
- What about backtracking search for general CSPs?
  - Worst case $O(m^n)$
- Can we do better?

# Computational complexity of CSPs

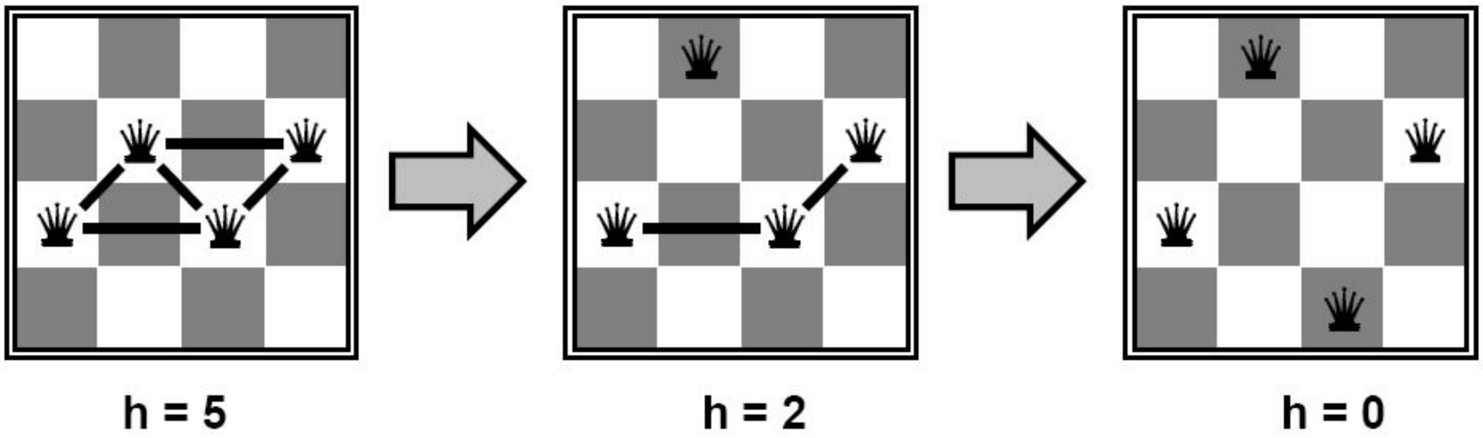
- The satisfiability (SAT) problem:
  - Given a Boolean formula, is there an assignment of the variables that makes it evaluate to true?

$$(X_1 \vee \overline{X}_7 \vee X_{13}) \wedge (\overline{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

- SAT is *NP-complete*
  - NP: class of decision problems for which the "yes" answer can be verified in polynomial time
  - An NP-complete problem is in NP and every other problem in NP can be efficiently reduced to it (Cook, 1971)
  - Other NP-complete problems: graph coloring, n-puzzle, generalized sudoku
  - It is not known whether P = NP, i.e., no efficient algorithms for solving SAT in general are known

# Local search for CSPs

- Start with "complete" states, i.e., all variables assigned
- Allow states with unsatisfied constraints
- Attempt to improve states by reassigning variable values
- Hill-climbing search:
  - In each iteration, randomly select any conflicted variable and choose value that violates the fewest constraints
  - I.e., attempt to greedily minimize total number of violated constraints




h = number of conflicts

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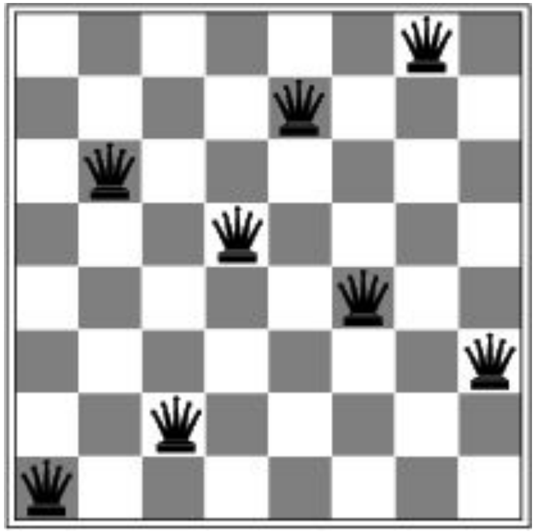

$h = 1$

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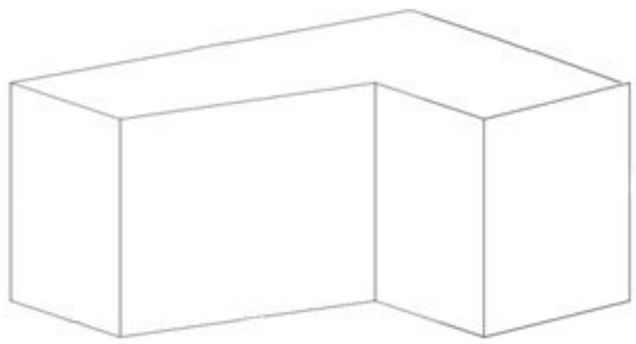
- For more on local search, see ch. 4

# CSP in computer vision: Line drawing interpretation

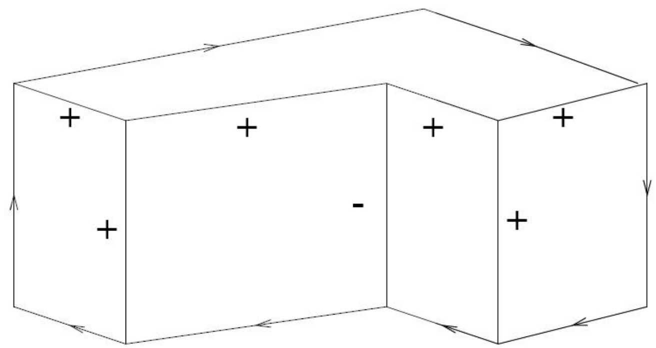
An example polyhedron:



**Variables:** edges

**Domains:** +, −, →, ←

Desired output:




David Waltz, 1975

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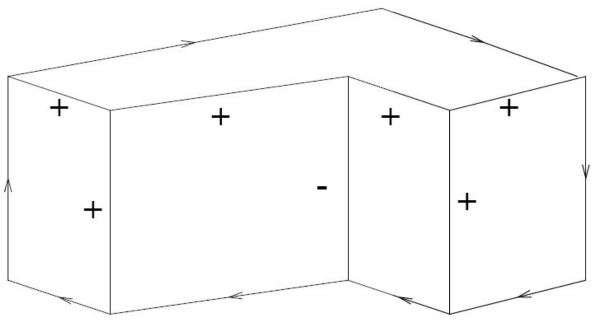
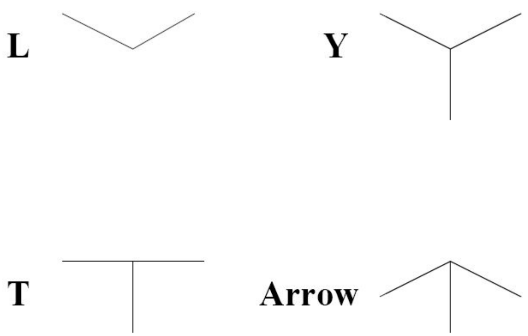
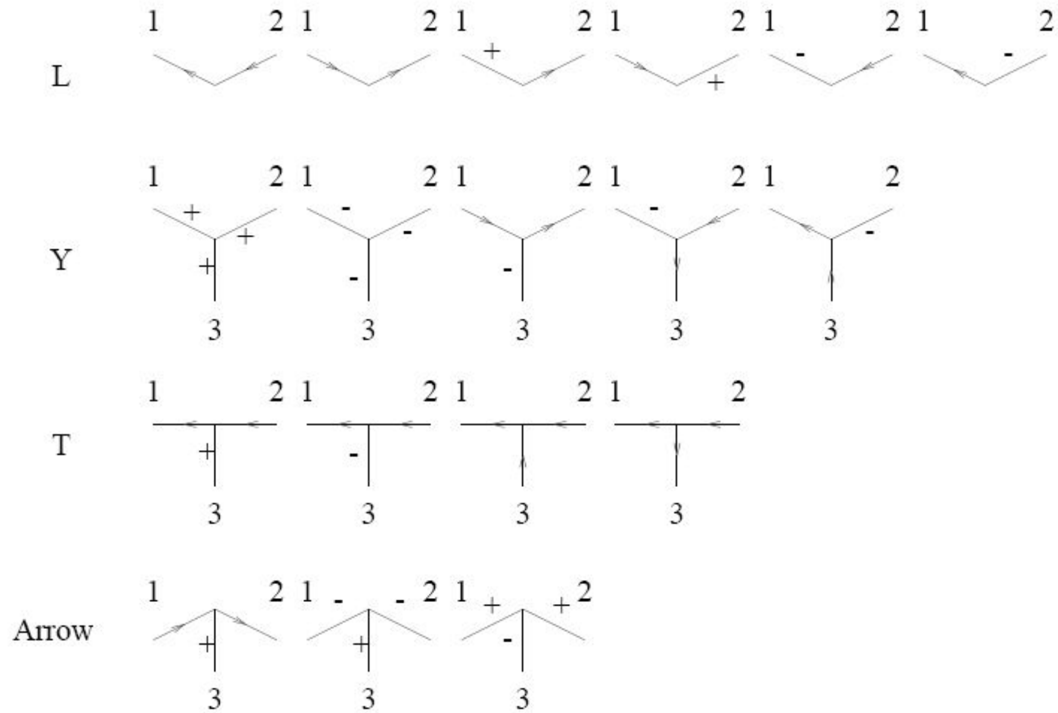


Four vertex types:




Constraints imposed by each vertex type:




David Waltz, 1975

# CSP in computer vision: 4D Cities

1. When was each photograph taken?
2. When did each building first appear?
3. When was each building removed?

**Set of Photographs**:

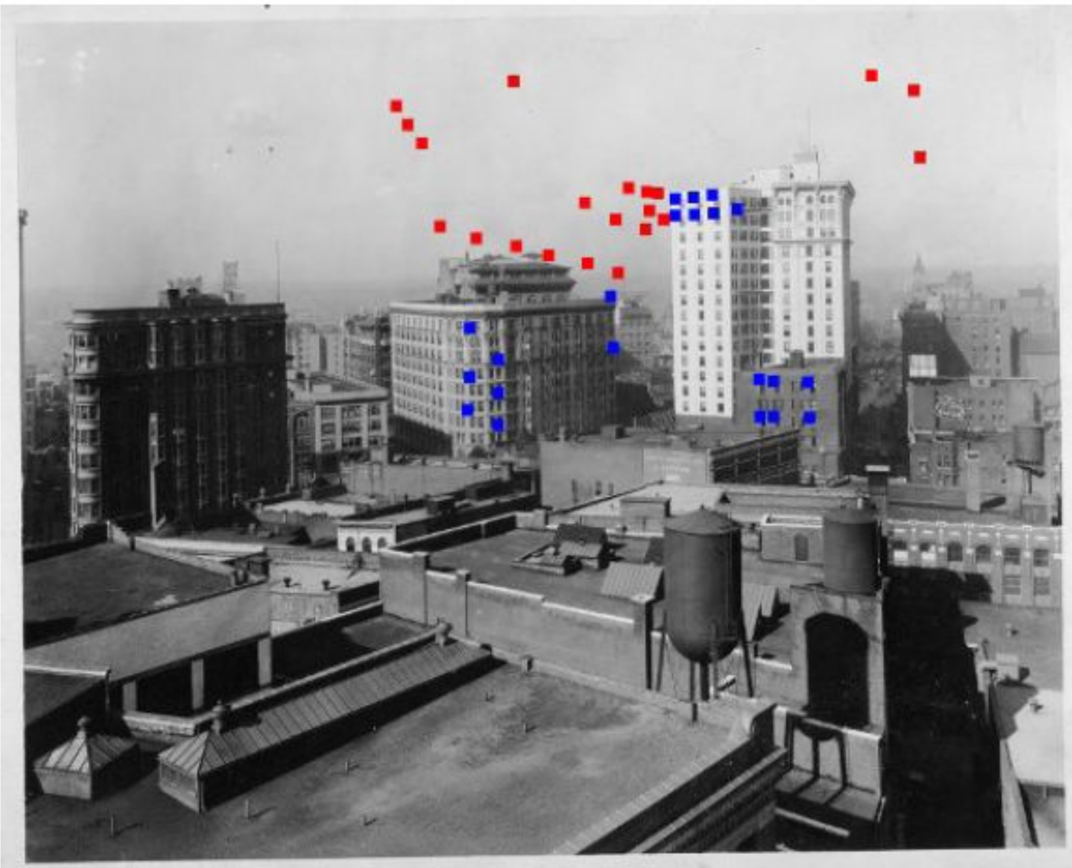
**Set of Objects: Buildings**

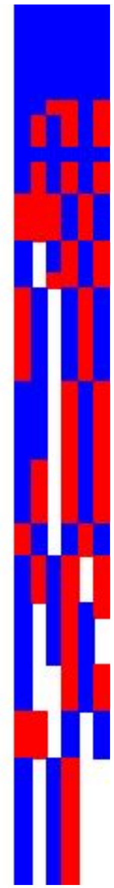
G. Schindler, F. Dellaert, and S.B. Kang, [Inferring Temporal Order of Images From 3D Structure](), IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR), 2007.

[http://www.cc.gatech.edu/~phlosoft/]()

# CSP in computer vision: 4D Cities

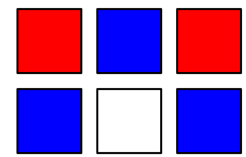
observed missing occluded



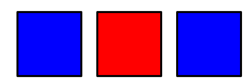


Columns: images
Rows: points

Satisfies constraints:



Violates constraints:

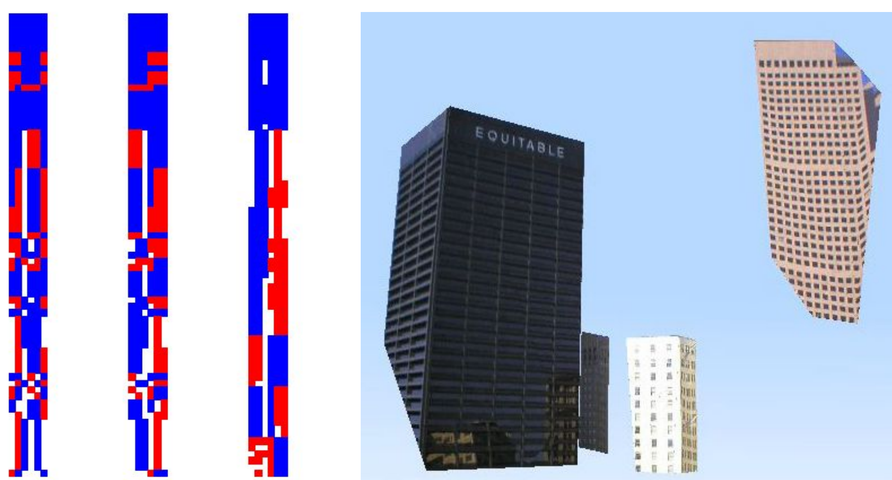


- Goal: reorder images (columns) to have as few violations as possible

# CSP in computer vision: 4D Cities

- **Goal:** reorder images (columns) to have as few violations as possible
- **Local search:** start with random ordering of columns, swap columns or groups of columns to reduce the number of conflicts

- Can also reorder the rows to group together points that appear and disappear at the same time – that gives you buildings



# Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- **Backtracking** = depth-first search where successor states are generated by considering assignments to a single variable
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure
  - **Constraint propagation** (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Complexity of CSPs
  - NP-complete in general (exponential worst-case running time)
  - Efficient solutions possible for special cases (e.g., tree-structured CSPs)
- Alternatives to backtracking search: local search