

# Lecture 11.

## Representation of Functions as Power Series.

## Representations of Functions as Power Series

In this section we learn how to represent some familiar functions as sums of power series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to evaluate functions on calculators and computers.)

### ■ Representations of Functions using Geometric Series

We will obtain power series representations for several functions by manipulating geometric series. We start with an equation that we have seen before.

1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

We first encountered this equation in Example 11.2.7, where we obtained it by observing that the series is a geometric series with  $a = 1$  and  $r = x$ . Here our point of view is different: we now regard Equation 1 as expressing the function  $f(x) = 1/(1-x)$  as a sum of a power series. We say that  $\sum_{n=0}^{\infty} x^n$ ,  $|x| < 1$ , is a *power series representation* of  $1/(1-x)$  on the interval  $(-1, 1)$ .

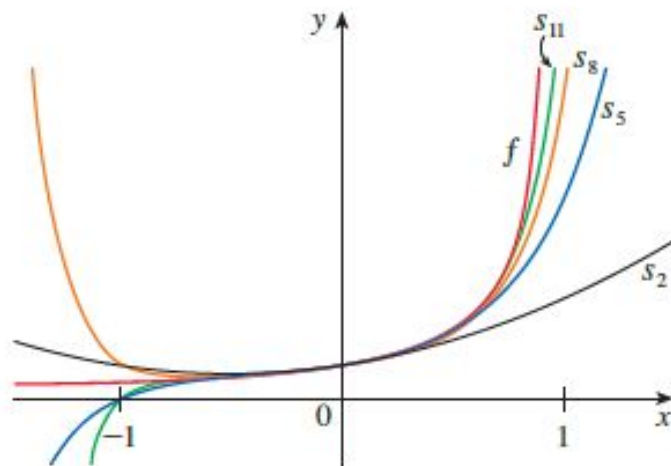
A geometric illustration of Equation 1 is shown in Figure 1. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n \rightarrow \infty} s_n(x)$$

where

$$s_n(x) = 1 + x + x^2 + \dots + x^n$$

is the  $n$ th partial sum. Notice that as  $n$  increases,  $s_n(x)$  becomes a better approximation to  $f(x)$  for  $-1 < x < 1$ .



The power series (1) that represents the function  $f(x) = 1/(1-x)$  can be used to obtain power series representations of many other functions, as we see in the following examples.

**EXAMPLE 1** Express  $1/(1 + x^2)$  as the sum of a power series and find the interval of convergence.

**SOLUTION** Replacing  $x$  by  $-x^2$  in Equation 1, we have

$$\begin{aligned}\frac{1}{1 + x^2} &= \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \dots\end{aligned}$$

Because this is a geometric series, it converges when  $|-x^2| < 1$ , that is,  $x^2 < 1$ , or  $|x| < 1$ . Therefore the interval of convergence is  $(-1, 1)$ . (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.) ■

**EXAMPLE 2** Find a power series representation for  $1/(x + 2)$ .

**SOLUTION** In order to put this function in the form of the left side of Equation 1, we first factor a 2 from the denominator:

$$\begin{aligned}\frac{1}{2 + x} &= \frac{1}{2\left(1 + \frac{x}{2}\right)} = \frac{1}{2\left[1 - \left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n\end{aligned}$$

This series converges when  $|-x/2| < 1$ , that is,  $|x| < 2$ . So the interval of convergence is  $(-2, 2)$ . ■

**EXAMPLE 3** Find a power series representation of  $x^3/(x + 2)$ .

**SOLUTION** Since this function is just  $x^3$  times the function in Example 2, all we have to do is to multiply that series by  $x^3$ :

$$\begin{aligned}\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \dots\end{aligned}$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

As in Example 2, the interval of convergence is  $(-2, 2)$ . ■

## ■ Differentiation and Integration of Power Series

The sum of a power series is a function  $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$  whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

**2 Theorem** If the power series  $\sum c_n(x - a)^n$  has radius of convergence  $R > 0$ , then the function  $f$  defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval  $(a - R, a + R)$  and

$$(i) f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

$$(ii) \int f(x) dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \dots \\ = C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both  $R$ .

**NOTE 1** Equations (i) and (ii) in Theorem 2 can be rewritten in the form

$$(iii) \quad \frac{d}{dx} \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n(x-a)^n]$$

$$(iv) \quad \int \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 44.)

**NOTE 2** Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 45.)



**EXAMPLE 4** Express  $1/(1 - x)^2$  as a power series by differentiating Equation 1. What is the radius of convergence?

**SOLUTION** We start with

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Differentiating each side, we get

$$\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$$

If we wish, we can replace  $n$  by  $n + 1$  and write the answer as

$$\frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} (n + 1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely,  $R = 1$ . ■

**EXAMPLE 5** Find a power series representation for  $\ln(1 + x)$  and its radius of convergence.

**SOLUTION** We notice that the derivative of this function is  $1/(1 + x)$ . From Equation 1 we have

$$\frac{1}{1 + x} = \frac{1}{1 - (-x)} = 1 - x + x^2 - x^3 + \dots \quad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned} \ln(1 + x) &= \int \frac{1}{1 + x} dx = \int (1 - x + x^2 - x^3 + \dots) dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \quad |x| < 1 \end{aligned}$$

To determine the value of  $C$  we put  $x = 0$  in this equation and obtain  $\ln(1 + 0) = C$ . Thus  $C = 0$  and

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x| < 1$$

The radius of convergence is the same as for the original series:  $R = 1$ . ■

**EXAMPLE 6** Find a power series representation for  $f(x) = \tan^{-1}x$ .

**SOLUTION** We observe that  $f'(x) = 1/(1 + x^2)$  and find the required series by integrating the power series for  $1/(1 + x^2)$  found in Example 1.

$$\begin{aligned}\tan^{-1}x &= \int \frac{1}{1 + x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

To find  $C$  we put  $x = 0$  and obtain  $C = \tan^{-1}0 = 0$ . Therefore

$$\begin{aligned}\tan^{-1}x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

Since the radius of convergence of the series for  $1/(1 + x^2)$  is 1, the radius of convergence of this series for  $\tan^{-1}x$  is also 1. ■

## ■ Functions Defined by Power Series

Some of the most important functions in the sciences are defined by power series and are not expressible in terms of elementary functions (as described in Section 7.5). Many of these arise naturally as solutions of differential equations. One such class of functions is the **Bessel functions**, named after the German astronomer Friedrich Bessel (1784–1846). These functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, Bessel functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead. Bessel functions appear in the next example as well as in Exercises 39 and 40. Other examples of functions defined by power series are given in Exercises 38 and 41.

**EXAMPLE 8** The Bessel function of order 0 is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

- (a) Find the domain of  $J_0$ .  
(b) Find the derivative of  $J_0$ .

**SOLUTION**

(a) Let  $a_n = (-1)^n x^{2n} / [2^{2n} (n!)^2]$ . Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right| \\ &= \frac{x^{2n+2}}{2^{2n+2} (n+1)^2 (n!)^2} \cdot \frac{2^{2n} (n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2} \rightarrow 0 < 1 \quad \text{for all } x \end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of  $x$ . In other words, the domain of the Bessel function  $J_0$  is  $(-\infty, \infty) = \mathbb{R}$ .

(b) By Theorem 2,  $J_0$  is differentiable for all  $x$  and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n-1}}{2^{2n} (n!)^2}$$



## 11.10 | Taylor and Maclaurin Series

In Section 11.9 we were able to find power series representations for a certain restricted class of functions, namely, those that can be obtained from geometric series. Here we investigate more general problems: Which functions have power series representations? How can we find such representations? We will see that some of the most important functions in calculus, such as  $e^x$  and  $\sin x$ , can be represented as power series.

### ■ Definitions of Taylor Series and Maclaurin Series

We start by supposing that  $f$  is a function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \quad |x - a| < R$$

Let's try to determine what the coefficients  $c_n$  must be in terms of  $f$ . To begin, notice that if we put  $x = a$  in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 11.9.2, we can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots \quad |x - a| < R$$

and substitution of  $x = a$  in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \dots \quad |x - a| < R$$

Again we put  $x = a$  in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \dots \quad |x - a| < R$$

and substitution of  $x = a$  in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute  $x = a$ , we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \dots \cdot nc_n = n!c_n$$

Solving this equation for the  $n$ th coefficient  $c_n$ , we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for  $n = 0$  if we adopt the conventions that  $0! = 1$  and  $f^{(0)} = f$ . Thus we have proved the following theorem.

**5 Theorem** If  $f$  has a power series representation (expansion) at  $a$ , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \quad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for  $c_n$  back into the series, we see that if  $f$  has a power series expansion at  $a$ , then it must be of the following form.

$$\begin{aligned} \mathbf{6} \quad f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots \end{aligned}$$

The series in Equation 6 is called the **Taylor series of the function  $f$  at  $a$**  (or about  $a$  or centered at  $a$ ). For the special case  $a = 0$  the Taylor series becomes



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$$\boxed{7} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

**NOTE 1** When we find a Taylor series for a function  $f$ , there is no guarantee that the sum of the Taylor series is equal to  $f$ . Theorem 5 says that *if*  $f$  has a power series representation about  $a$ , then that power series must be the Taylor series of  $f$ . There exist functions that are not equal to the sum of their Taylor series, such as the function given in Exercise 96.

**NOTE 2** The power series representation at  $a$  of a function is unique, regardless of how it is found, because Theorem 5 states that if  $f$  has a power series representation  $f(x) = \sum c_n(x - a)^n$ , then  $c_n$  must be  $f^{(n)}(a)/n!$ . Thus all the power series representations we developed in Section 11.9 are in fact the Taylor series of the functions they represent.



**EXAMPLE 1** We know from Equation 11.9.1 that the function  $f(x) = 1/(1 - x)$  has a power series representation

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad |x| < 1$$

According to Theorem 5, this series must be the Maclaurin series of  $f$  with coefficients  $c_n$  given by  $f^{(n)}(0)/n!$ . To confirm this, we compute

$$f(x) = \frac{1}{1 - x} \quad f(0) = 1$$

$$f'(x) = \frac{1}{(1 - x)^2} \quad f'(0) = 1$$

$$f''(x) = \frac{1 \cdot 2}{(1 - x)^3} \quad f''(0) = 1 \cdot 2$$

$$f'''(x) = \frac{1 \cdot 2 \cdot 3}{(1 - x)^4} \quad f'''(0) = 1 \cdot 2 \cdot 3$$

and, in general,

$$f^{(n)}(x) = \frac{n!}{(1 - x)^{n+1}} \quad f^{(n)}(0) = n!$$

Thus

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$$

and, from Equation 7,

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$



**EXAMPLE 2** For the function  $f(x) = e^x$ , find the Maclaurin series and its radius of convergence.

**SOLUTION** If  $f(x) = e^x$ , then  $f^{(n)}(x) = e^x$ , so  $f^{(n)}(0) = e^0 = 1$  for all  $n$ . Therefore the Taylor series for  $f$  at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

To find the radius of convergence we let  $a_n = x^n/n!$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 < 1$$

so, by the Ratio Test, the series converges for all  $x$  and the radius of convergence is  $R = \infty$ . ■

## ■ When Is a Function Represented by Its Taylor Series?

From Theorem 5 and Example 2 we can conclude that *if we know* that  $e^x$  has a power series representation at 0, then this power series must be its Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether  $e^x$  *does* have a power series representation?

Let's investigate the more general question: under what circumstances is a function equal to the sum of its Taylor series? In other words, if  $f$  has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

As with any convergent series, this means that  $f(x)$  is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n \end{aligned}$$

Notice that  $T_n$  is a polynomial of degree  $n$  called the  $n$ th-degree Taylor polynomial of  $f$  at  $a$ . For instance, for the exponential function  $f(x) = e^x$ , the result of Example 2 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with  $n = 1, 2$ , and 3 are

$$T_1(x) = 1 + x \quad T_2(x) = 1 + x + \frac{x^2}{2!} \quad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

In general,  $f(x)$  is the sum of its Taylor series if

$$f(x) = \lim_{n \rightarrow \infty} T_n(x)$$

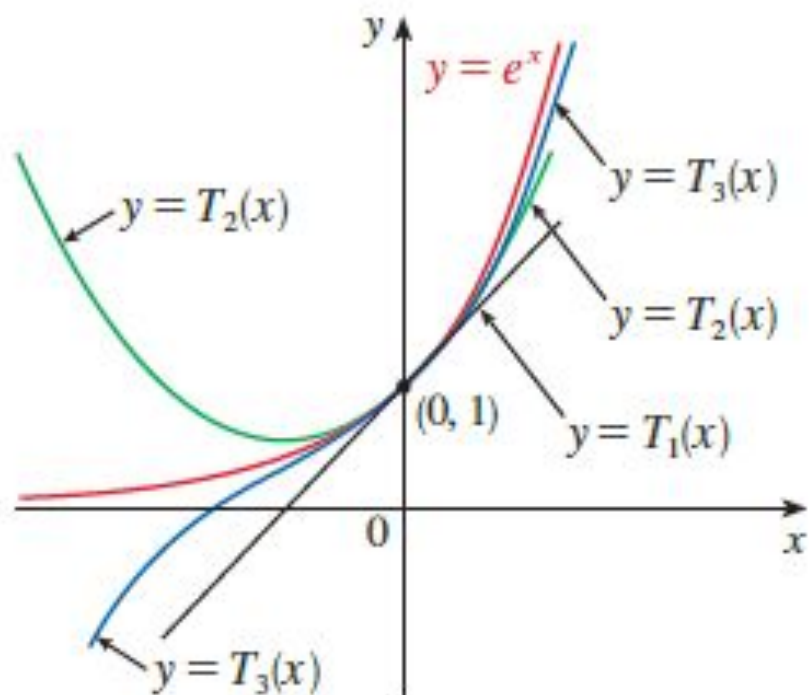
If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then  $R_n(x)$  is called the **remainder** of the Taylor series. If we can somehow show that  $\lim_{n \rightarrow \infty} R_n(x) = 0$ , then it follows that

$$\lim_{n \rightarrow \infty} T_n(x) = \lim_{n \rightarrow \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \rightarrow \infty} R_n(x) = f(x)$$

We have therefore proved the following theorem.



**FIGURE 1**

As  $n$  increases,  $T_n(x)$  appears to approach  $e^x$  in Figure 1. This suggests that  $e^x$  is equal to the sum of its Taylor series.

We have therefore proved the following theorem.

**8 Theorem** If  $f(x) = T_n(x) + R_n(x)$ , where  $T_n$  is the  $n$ th-degree Taylor polynomial of  $f$  at  $a$ , and if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for  $|x - a| < R$ , then  $f$  is equal to the sum of its Taylor series on the interval  $|x - a| < R$ .

In trying to show that  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for a specific function  $f$ , we usually use the following theorem.

**9 Taylor's Inequality** If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| \leq d$ , then the remainder  $R_n(x)$  of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$



**PROOF** We first prove Taylor's Inequality for  $n = 1$ . Assume that  $|f''(x)| \leq M$ . In particular, we have  $f''(x) \leq M$ , so for  $a \leq x \leq a + d$  we have

$$\int_a^x f''(t) dt \leq \int_a^x M dt$$

An antiderivative of  $f''$  is  $f'$ , so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leq M(x - a) \quad \text{or} \quad f'(x) \leq f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t) dt \leq \int_a^x [f'(a) + M(t - a)] dt$$

$$f(x) - f(a) \leq f'(a)(x - a) + M \frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leq \frac{M}{2} (x - a)^2$$

But  $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$ . So

$$R_1(x) \leq \frac{M}{2} (x - a)^2$$

A similar argument, using  $f''(x) \geq -M$ , shows that

$$R_1(x) \geq -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leq \frac{M}{2}|x - a|^2$$

Although we have assumed that  $x > a$ , similar calculations show that this inequality is also true for  $x < a$ .

This proves Taylor's Inequality for the case where  $n = 1$ . The result for any  $n$  is proved in a similar way by integrating  $n + 1$  times. (See Exercise 95 for the case  $n = 2$ .) ■

**NOTE** In Section 11.11 we will explore the use of Taylor's Inequality in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

When we apply Theorems 8 and 9 it is often helpful to make use of the following fact.

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$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

This is true because we know from Example 2 that the series  $\sum x^n/n!$  converges for all  $x$  and so its  $n$ th term approaches 0.

**EXAMPLE 3** Prove that  $e^x$  is equal to the sum of its Maclaurin series.

**SOLUTION** If  $f(x) = e^x$ , then  $f^{(n+1)}(x) = e^x$  for all  $n$ . If  $d$  is any positive number and  $|x| \leq d$ , then  $|f^{(n+1)}(x)| = e^x \leq e^d$ . So Taylor's Inequality, with  $a = 0$  and  $M = e^d$ , says that

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for } |x| \leq d$$

Notice that the same constant  $M = e^d$  works for every value of  $n$ . But, from Equation 10, we have

$$\lim_{n \rightarrow \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that  $\lim_{n \rightarrow \infty} |R_n(x)| = 0$  and therefore  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for all values of  $x$ . By Theorem 8,  $e^x$  is equal to the sum of its Maclaurin series, that is,

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$



In particular, if we put  $x = 1$  in Equation 11, we obtain the following expression for the number  $e$  as a sum of an infinite series:

**12**

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

**EXAMPLE 4** Find the Taylor series for  $f(x) = e^x$  at  $a = 2$ .

**SOLUTION** We have  $f^{(n)}(2) = e^2$  and so, putting  $a = 2$  in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x - 2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x - 2)^n$$

Again it can be verified, as in Example 2, that the radius of convergence is  $R = \infty$ . As in Example 3 we can verify that  $\lim_{n \rightarrow \infty} R_n(x) = 0$ , so

**13**

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x - 2)^n \quad \text{for all } x$$

We have two power series expansions for  $e^x$ , the Maclaurin series in Equation 11 and the Taylor series in Equation 13. The first is better if we are interested in values of  $x$  near 0 and the second is better if  $x$  is near 2.

## Independent Investigation Work!!

Find Taylor's series for:

- 1)  $y = \sin x$
- 2)  $y = \cos x$
- 3)  $y = \tan x$
- 4)  $y = (1+x)^n$ 
  - a)  $n = -1$
  - b)  $n = 1/2$
  - c)  $n = -1/2$