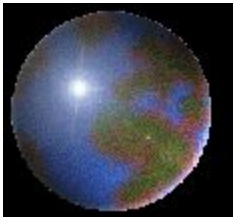


Linear Algebra



Chapter 1

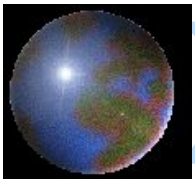
Linear Equations in Linear Algebra



1.1 Matrices and Systems of Linear Equations

Definition

- An equation such as $x+3y=9$ is called a *linear equation* (in *two* variables or unknowns).
- The graph of this equation is a straight line in the xy -plane.
- A pair of values of x and y that satisfy the equation is called a *solution*.



Definition

A *linear equation* in n variables $x_1, x_2, x_3, \dots, x_n$ has the form $a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = b$ where the coefficients $a_1, a_2, a_3, \dots, a_n$ and b are real numbers.



Solutions for system of linear equations

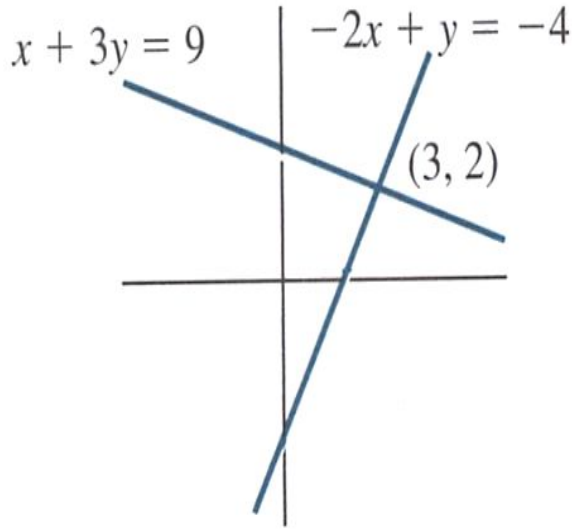


Figure 1.1
Unique solution

$x + 3y = 9$
 $-2x + y = -4$
 Lines intersect at $(3, 2)$
Unique solution:
 $x = 3, y = 2.$

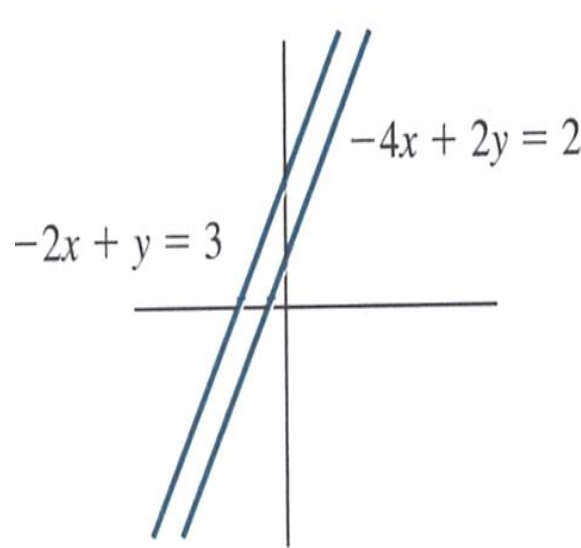


Figure 1.2
No solution

$-2x + y = 3$
 $-4x + 2y = 2$
 Lines are parallel.
 No point of intersection.
No solutions.

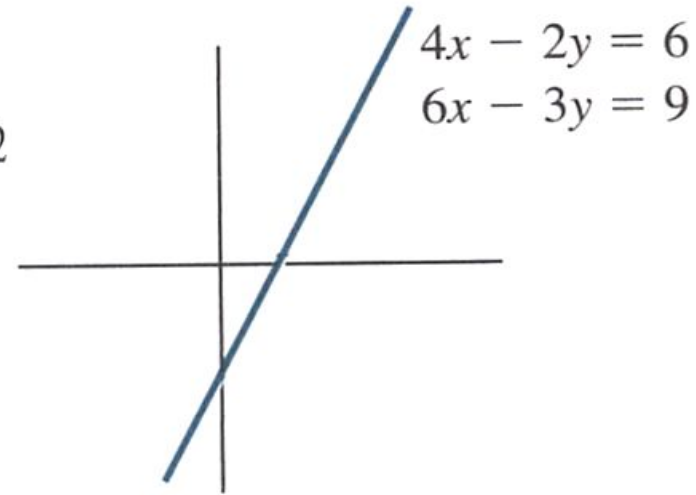


Figure 1.3
Many solution

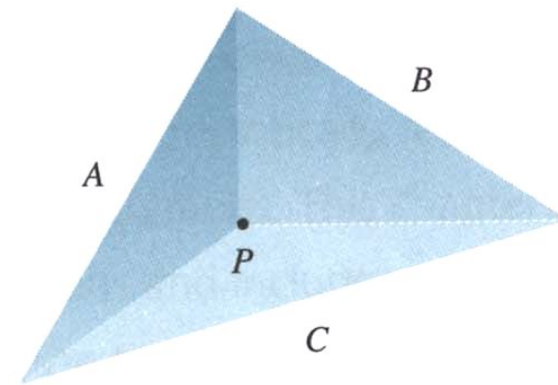
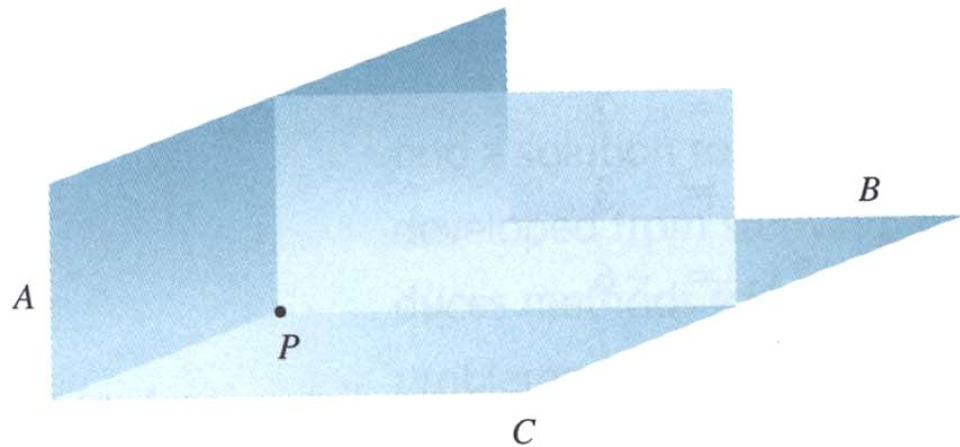
$4x - 2y = 6$
 $6x - 3y = 9$
 Both equations have the same graph. Any point on the graph is a solution.
Many solutions.

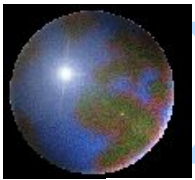


A linear equation in **three variables** corresponds to a plane in three-dimensional space.

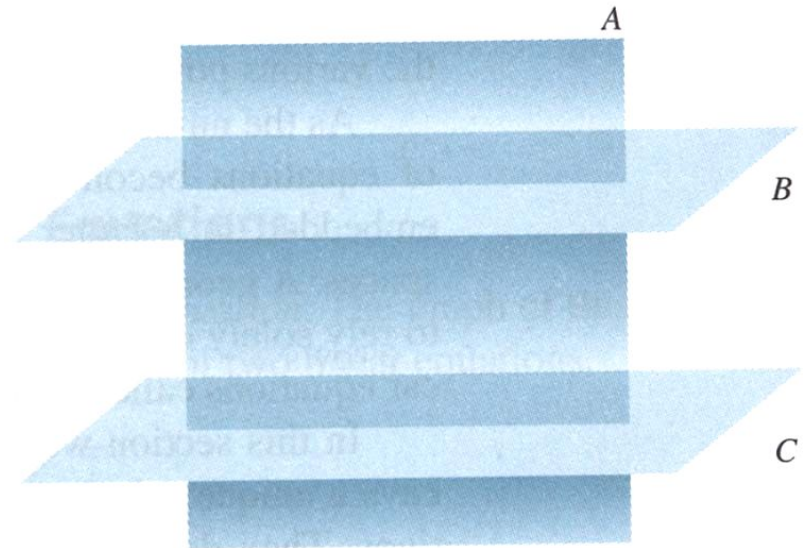
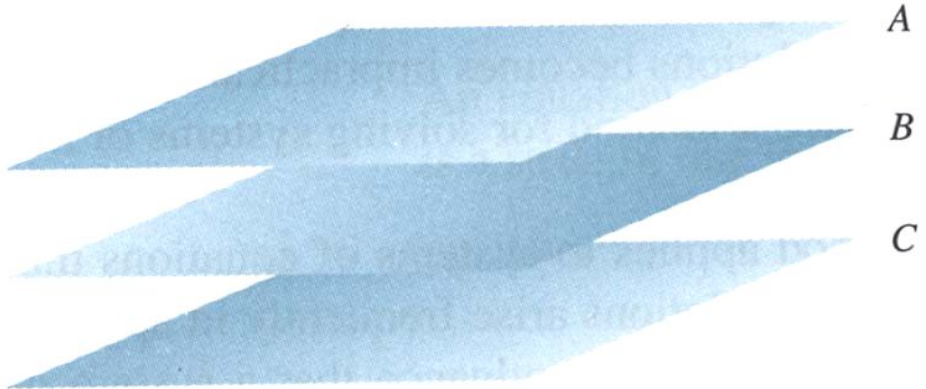
✂ Systems of three linear equations in three variables:

- *Unique solution*

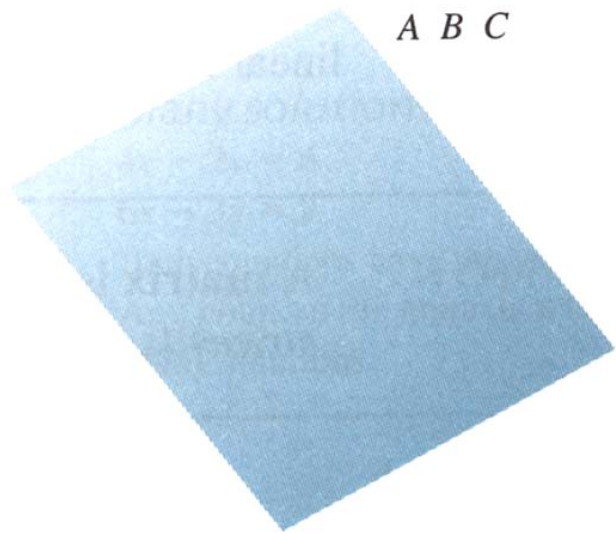
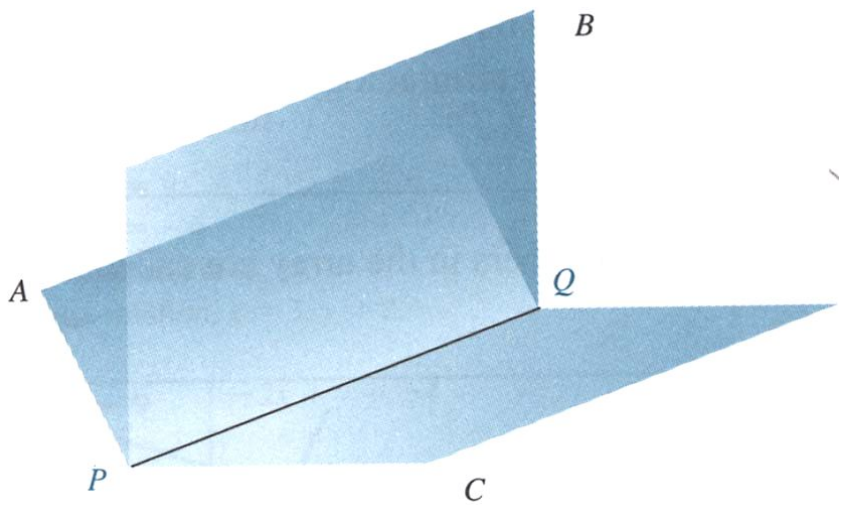


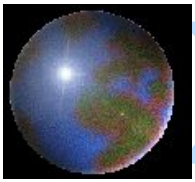


- No solutions*



- Many solutions*





A **solution** to a system of a three linear equations will be **points that lie on all three planes.**

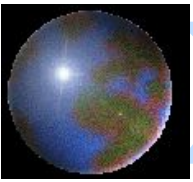
The following is an example of a system of three linear equations:

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + x_3 = 3$$

$$x_1 - x_2 - 2x_3 = -6$$

How to solve a system of linear equations? For this we introduce a method called **Gauss-Jordan elimination.**
(Section 1.2)



Definition

- A *matrix* is a rectangular array of numbers.
- The numbers in the array are called the *elements* of the matrix.

• Matrices

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 \\ 0 & 5 \\ -8 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 5 & 6 \\ 0 & -2 & 5 \\ 8 & 9 & 12 \end{bmatrix}$$



- **Row and Column**

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}$$

$$[2 \quad 3 \quad -4]$$

row 1

$$[7 \quad 5 \quad -1]$$

row 2

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

column 1

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

column 2

$$\begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

column 3

- **Submatrix**

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 3 & 0 \\ 5 & 1 & -2 \end{bmatrix}$$

matrix A

$$P = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$$

submatrices of A



• Size and Type

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \end{bmatrix}$$

Size : 2×3

$$\begin{bmatrix} 2 & 5 & 7 \\ -9 & 0 & 1 \\ -3 & 5 & 8 \end{bmatrix}$$

3×3 matrix

a square matrix

$$[4 \quad -3 \quad 8 \quad 5]$$

1×4 matrix

a row matrix

$$\begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}$$

3×1 matrix

a column matrix

• Location

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}$$

$$a_{13} = -4, a_{21} = 7$$

The element a_{ij} is in row i , column j

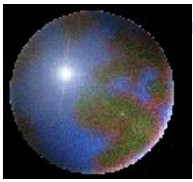
The element in location (1,3) is -4

• Identity Matrices

diagonal size

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Relations between system of linear equations and matrices

- **matrix of coefficients and augmented matrix**

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + x_3 = 3$$

$$x_1 - x_2 - 2x_3 = -6$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

matrix of coefficients

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

augmented matrix



Elementary Row Operations of Matrices

- Elementary Transformation
 - Elementary Row Operation
1. Interchange two equations.
 2. Multiply both sides of an equation by a nonzero constant.
 3. Add a multiple of one equation to another equation.
1. Interchange two rows of a matrix.
 2. Multiply the elements of a row by a nonzero constant.
 3. Add a multiple of the elements of one row to the corresponding elements of another row.



Example 1

Solving the following system of linear equation.

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 + 3x_2 + x_3 &= 3 \\x_1 - x_2 - 2x_3 &= -6\end{aligned}$$

\approx row equivalent

Solution

Equation Method

Initial system:

$$\begin{array}{l} \text{Eq2} + (-2)\text{Eq1} \\ \text{Eq3} + (-1)\text{Eq1} \end{array} \left\{ \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 - 2x_3 = -6 \end{array} \right.$$

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\x_2 - x_3 &= -1 \\-2x_2 - 3x_3 &= -8\end{aligned}$$

Analogous Matrix Method

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

\approx

$\text{R2} + (-2)\text{R1}$
 $\text{R3} + (-1)\text{R1}$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$



$$\begin{array}{l} \text{Eq1} + (-1)\text{Eq2} \rightarrow x_1 + x_2 + x_3 = 2 \\ \text{Eq3} + (2)\text{Eq2} \rightarrow -2x_2 - 3x_3 = -8 \end{array}$$

$$\begin{array}{l} x_1 + 2x_3 = 3 \\ x_2 - x_3 = -1 \\ (-1/5)\text{Eq3} \rightarrow -5x_3 = -10 \end{array}$$

$$\begin{array}{l} \text{Eq1} + (-2)\text{Eq3} \rightarrow x_1 + 2x_3 = 3 \\ \text{Eq2} + \text{Eq3} \rightarrow x_2 - x_3 = -1 \\ x_3 = 2 \end{array}$$

$$\begin{array}{l} x_1 = -1 \\ x_2 = 1 \\ x_3 = 2 \end{array}$$

The solution is

$$x_1 = -1, x_2 = 1, x_3 = 2.$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

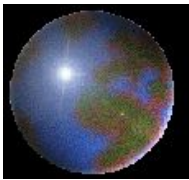
$$\begin{array}{l} \approx \\ \text{R1} + (-1)\text{R2} \\ \text{R3} + (2)\text{R2} \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

$$\begin{array}{l} \approx \\ (-1/5)\text{R3} \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{l} \approx \\ \text{R1} + (-2)\text{R3} \\ \text{R2} + \text{R3} \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is

$$x_1 = -1, x_2 = 1, x_3 = 2.$$



Example 2

Solving the following system of linear equation.

$$x_1 - 2x_2 + 4x_3 = 12$$

$$2x_1 - x_2 + 5x_3 = 18$$

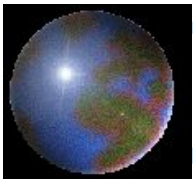
$$-x_1 + 3x_2 - 3x_3 = -8$$

Solution

$$\begin{bmatrix} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{bmatrix} \begin{array}{l} \\ R2 + (-2)R1 \\ R3 + R1 \end{array} \approx \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ \left(\frac{1}{3}\right)R2 \end{array} \approx \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ R1 + (2)R2 \\ R3 + (-1)R2 \end{array} \approx \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ \left(\frac{1}{2}\right)R3 \end{array} \approx \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} \\ R1 + (-2)R3 \\ R2 + R3 \end{array} \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{solution} \begin{cases} x_1 = 2 \\ x_2 = 1. \\ x_3 = 3 \end{cases}$$



Example 3

Solve the system

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - 8x_3 &= 32 \\ -2x_1 - x_2 &= -7 \end{aligned}$$

Solution

$$\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \xrightarrow{\left(\frac{1}{4}\right)R_1} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 + (-3)R_1 \\ R_3 + 2R_1 \end{array}}$$

$$\begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & -6 & 15 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\left(\frac{1}{3}\right)R_2} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 + (-2)R_2}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{l} R_1 + (-1)R_3 \\ R_2 + 2R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution is $x_1 = 2, x_2 = 3, x_3 = -1$.



Summary

$$\begin{aligned}
 4x_1 + 8x_2 - 12x_3 &= 44 \\
 3x_1 + 6x_2 - 8x_3 &= 32 \\
 -2x_1 - x_2 &= -7
 \end{aligned}
 \quad [A : B] = \begin{bmatrix} \boxed{4} & \boxed{8} & \boxed{-12} & \boxed{44} \\ \boxed{3} & \boxed{6} & \boxed{-8} & \boxed{32} \\ \boxed{-2} & \boxed{-1} & \boxed{0} & \boxed{-7} \end{bmatrix}$$

A
B

Use row operations to $[A : B]$:

$$\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \quad \text{i.e., } [A : B] \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \approx [I_n : X]$$

Def. $[I_n : X]$ is called the *reduced echelon form* of $[A : B]$.

- Note. 1.** If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution, then A is row equivalent to I_n ($A \approx I_n$).
- 2.** If $A \approx I_n$, then the system has unique solution.



Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned}
 x_1 - x_2 + 3x_3 &= b_1 \\
 2x_1 - x_2 + 4x_3 &= b_2 \\
 -x_1 + 2x_2 - 4x_3 &= b_3
 \end{aligned}
 \quad \text{for} \quad
 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \text{ in turn}$$

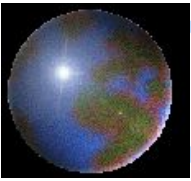
Solution

$$\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{bmatrix} \xrightarrow[\text{R3+R1}]{\text{R2+(-2)R1}} \begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow[\text{R3+(-1)R2}]{\text{R1+R2}} \begin{bmatrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow[\text{R2+2R3}]{\text{R1+(-1)R3}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$$

The solutions to the three systems are

$$\begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{cases}, \begin{cases} x_1 = 0 \\ x_2 = 3 \\ x_3 = 1 \end{cases}, \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 2 \end{cases} .$$



Homework

- Exercises will be given by the teachers of the practical classes.



1.2 Gauss-Jordan Elimination

Definition

A matrix is in *reduced echelon form* if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a *leading 1*.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.



- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

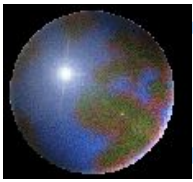
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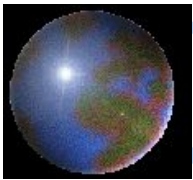
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- elementary row operations, reduced echelon form
- The reduced echelon form of a matrix is **unique**.



Gauss-Jordan Elimination

- System of linear equations
 - ⇒ augmented matrix
 - ⇒ reduced echelon form
 - ⇒ solution



Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot (leading 1)

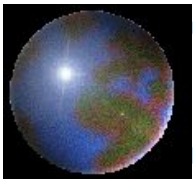
$$\begin{array}{l} \approx \\ R1 \leftrightarrow R2 \end{array} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \begin{array}{l} \\ \left(\frac{1}{3}\right)R1 \\ \left(\frac{1}{3}\right)R1 \end{array} \approx \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\begin{array}{l} \approx \\ R3 + (-4)R1 \end{array} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \begin{array}{l} \\ \left(\frac{1}{2}\right)R2 \\ \left(\frac{1}{2}\right)R2 \end{array} \approx \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\begin{array}{l} \approx \\ R1 + R2 \\ R3 + (-2)R2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \begin{array}{l} \\ R1 + (-2)R3 \\ R2 + R3 \end{array} \approx \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.



Example 2

Solve, if possible, the system of equations

$$3x_1 - 3x_2 + 3x_3 = 9$$

$$2x_1 - x_2 + 4x_3 = 7$$

$$3x_1 - 5x_2 - x_3 = 7$$

Solution

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \xrightarrow{\left(\frac{1}{3}\right)R1} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \xrightarrow{\substack{R2+(-2)R1 \\ R3+(-3)R1}} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

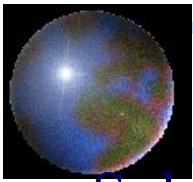
$$\begin{matrix} \approx \\ R1+R2 \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 3x_3 = 4 \\ x_2 + 2x_3 = 1 \end{matrix} \Rightarrow \begin{matrix} x_1 = -3x_3 + 4 \\ x_2 = -2x_3 + 1 \end{matrix}$$

The ^{R3+2R2}general solution to the system is

$$x_1 = -3r + 4$$

$$x_2 = -2r + 1$$

$$x_3 = r, \text{ where } r \text{ is real number (called a parameter).}$$



Example 3

Solve the system of equations

$$2x_1 - 4x_2 + 12x_3 - 10x_4 = 58$$

$$-x_1 + 2x_2 - 3x_3 + 2x_4 = -14$$

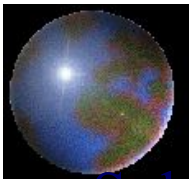
$$2x_1 - 4x_2 + 9x_3 - 6x_4 = 44$$

⇒ many
sol.

Solution

$$\begin{aligned} & \begin{bmatrix} 2 & -4 & 12 & -10 & 58 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix} \\ & \approx \begin{matrix} R_2+R_1 \\ R_3+(-2)R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix} \xrightarrow{\left(\frac{1}{3}\right)R_2} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix} \\ & \approx \begin{matrix} R_1+(-6)R_2 \\ R_3+3R_2 \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{matrix} R_1+(-1)R_3 \\ R_2+R_3 \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \begin{cases} x_1 - 2x_2 = -2 \\ x_3 = 6 \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2r - 2 \\ x_2 = r \\ x_3 = 6 \\ x_4 = 1 \end{cases}, \text{ for some } r.$$



Example 4

Solve the system of equations

$$x_1 + 2x_2 - x_3 + 3x_4 + x_5 = 2$$

$$2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 = 6$$

$$-x_1 - 2x_2 + x_3 - x_4 + 3x_5 = 4$$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 2 & 4 & -2 & 6 & 3 & 6 \\ -1 & -2 & 1 & -1 & 3 & 4 \end{bmatrix} \xrightarrow[\text{R3+R1}]{\text{R2+(-2)R1}} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{\text{R2} \leftrightarrow \text{R3}} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)\text{R2}} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{R1+(-3)R2}} \begin{bmatrix} 1 & 2 & -1 & 0 & -5 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{R2+(-2)R3}]{\text{R1+5R3}} \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x_1 = -2x_2 + x_3 + 3$$

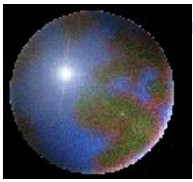
$$x_1 = -2r + s + 3$$

$$\Rightarrow x_4 = -1$$

$$\Rightarrow x_2 = r, x_3 = s, x_4 = -1, \text{ for some } r \text{ and } s.$$

$$x_5 = 2$$

$$x_5 = 2$$



Example 5

This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$

Solution

$$\begin{aligned} & \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow[\text{R2+(-2)R1}]{\text{R3+(-1)R1}} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\text{R2} \leftrightarrow \text{R3}} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\left(\frac{1}{3}\right)\text{R2}} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow[\text{R4+(-2)R2}]{\text{R1+R2}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix} \xrightarrow[\text{R4+(-4)R3}]{\text{R1+(-1)R3}, \text{R2+R3}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\left(\frac{1}{13}\right)\text{R4}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$0x_1 + 0x_2 + 0x_3 = 1$$

The system has no solution.



Homogeneous System of linear Equations

Definition

A system of linear equations is said to be *homogeneous* if all the constant terms are zeros.

Example:
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that $x_1 = 0, x_2 = 0, x_3 = 0$ is a solution.

Theorem 1.1

A system of homogeneous linear equations in n variables always has the solution $x_1 = 0, x_2 = 0, \dots, x_n = 0$. This solution is called the **trivial solution**.



Homogeneous System of linear Equations

Note. Non trivial solution

$$\text{Example: } \begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

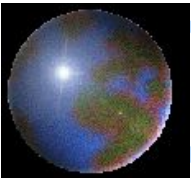
The system has other nontrivial solutions.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \boxtimes \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\therefore x_1 = -3r, \quad x_2 = 4r, \quad x_3 = r$$

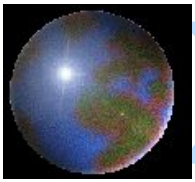
Theorem 1.2

A system of homogeneous linear equations that has more variables than equations has many solutions.



Homework

- Exercise will be given by the teachers of the practical classes.



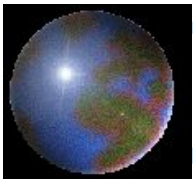
1.3 *Gaussian Elimination*

Definition

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)



Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

Solution

Starting with the augmented matrix, create zeros below the pivot in the first column.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \begin{matrix} \\ R2 + R1 \\ R3 + (-2)R1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

At this stage, we create a zero only below the pivot.

$$\begin{matrix} \\ \\ R3 + (-2)R2 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{2}R3 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We have arrived at the echelon form.

Echelon form



The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$x_3 + 3x_4 = 1$$

$$x_4 = 2$$

We get

$$x_3 + 3(2) = 1$$

$$x_3 = -5$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$

$$x_1 + 2x_2 = 10$$

$$x_1 = -2x_2 + 10$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$



Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

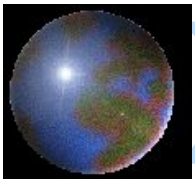
Solution

We arrive at the echelon form as in the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \boxtimes \approx \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Echelon form

This marks the end of the forward elimination of variables from equations. We now commence the **back substitution** using matrices.



$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \approx \\ R1 + (-2)R3 \\ R2 + (-3)R3 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\begin{array}{l} \approx \\ R1 + (-3)R2 \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$x_1 + 2x_2 = 10$$

$$x_3 = -5$$

$$x_4 = 2$$

Let $x_2 = r$. We get same solution as previously,

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$