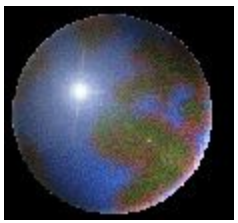
Linear Algebra



Chapter 1

Linear Equations in Linear Algebra

1.1 Matrices and Systems of Linear Equations

Definition

- An equation such as $x+3y=9$ is called a ***linear equation*** (in *two* variables or unknowns).
- The graph of this equation is a straight line in the xy-plane.
- A pair of values of x and y that satisfy the equation is called a ***solution***.

Definition

A ***linear equation*** in n variables $x_1, x_2, x_3, \ldots, x_n$ has the form $a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n = b$

where the coefficients $a_1, a_2, a_3, \ldots, a_n$ and b are real numbers.

Solutions for system of linear equations

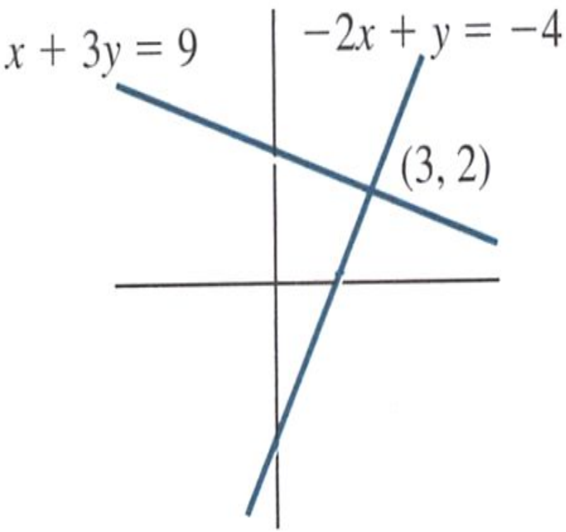


Figure 1.1

Unique solution

$$\begin{aligned} x + 3y &= 9 \\ -2x + y &= -4 \end{aligned}$$

Lines intersect at (3, 2)

<u>Unique solution</u>:

$x = 3, y = 2.$

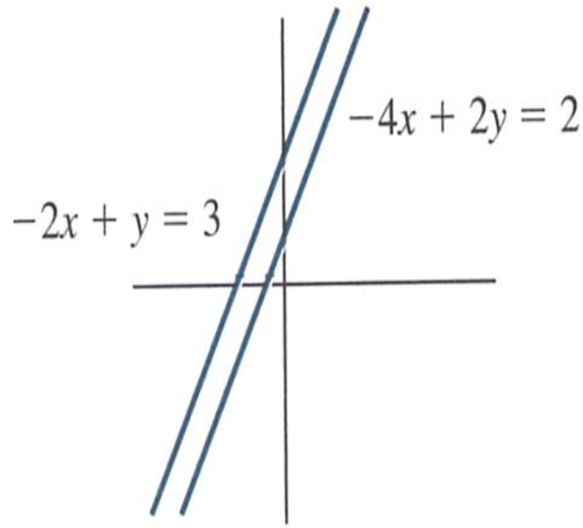


Figure 1.2

No solution

$$\begin{aligned} -2x + y &= 3 \\ -4x + 2y &= 2 \end{aligned}$$

Lines are parallel.

No point of intersection.

<u>No solutions</u>.

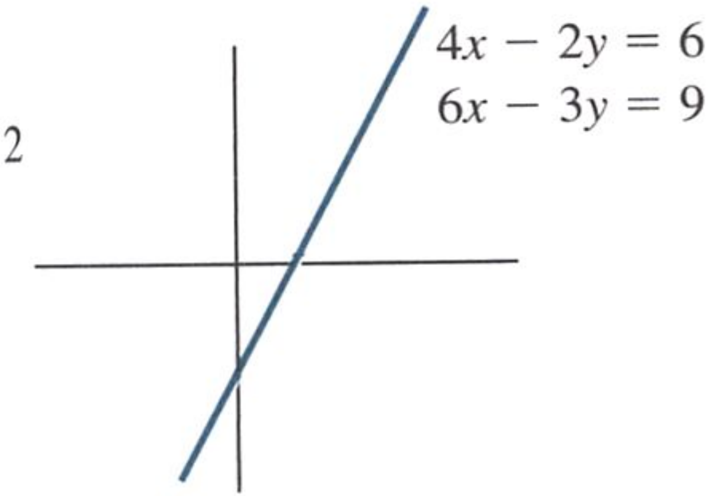


Figure 1.3

Many solution

$$\begin{aligned} 4x - 2y &= 6 \\ 6x - 3y &= 9 \end{aligned}$$

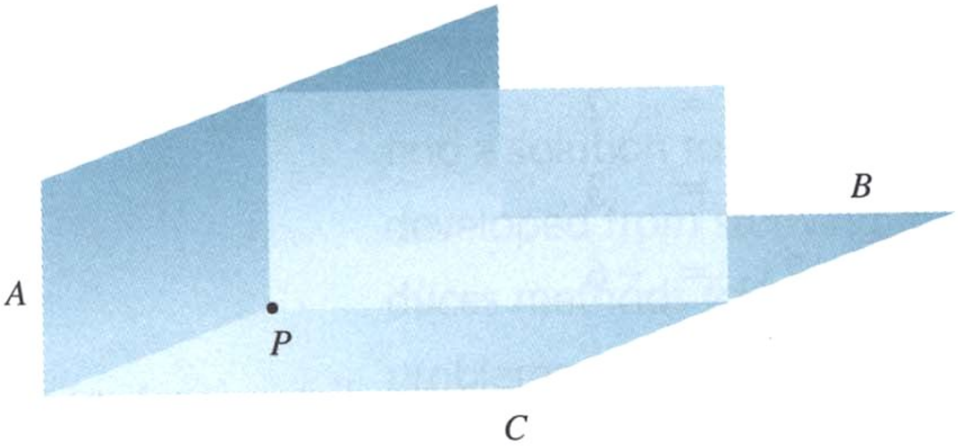
Both equations have the same graph. Any point on the graph is a solution.

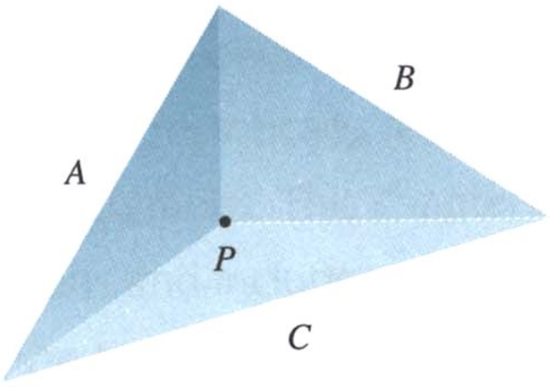
<u>Many solutions</u>.

A linear equation in three variables corresponds to a plane in three-dimensional space.

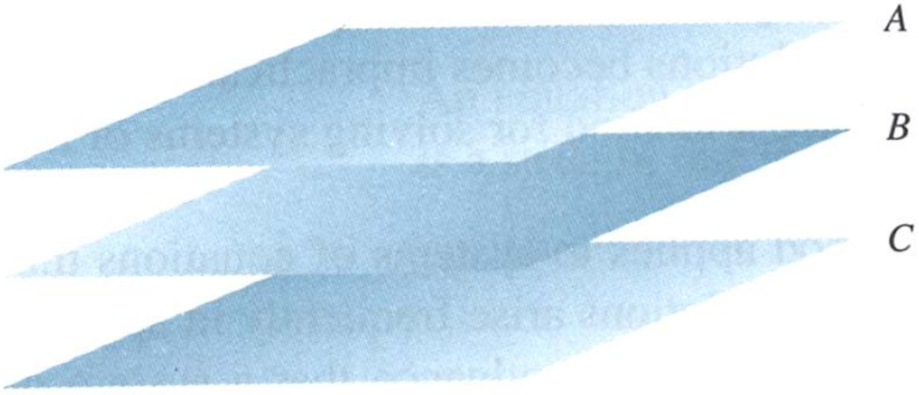
※ Systems of three linear equations in three variables:

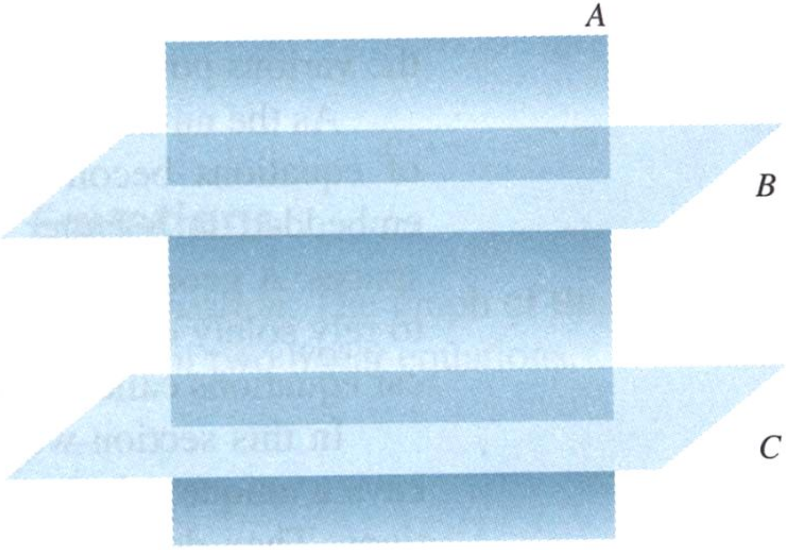
- *Unique solution*



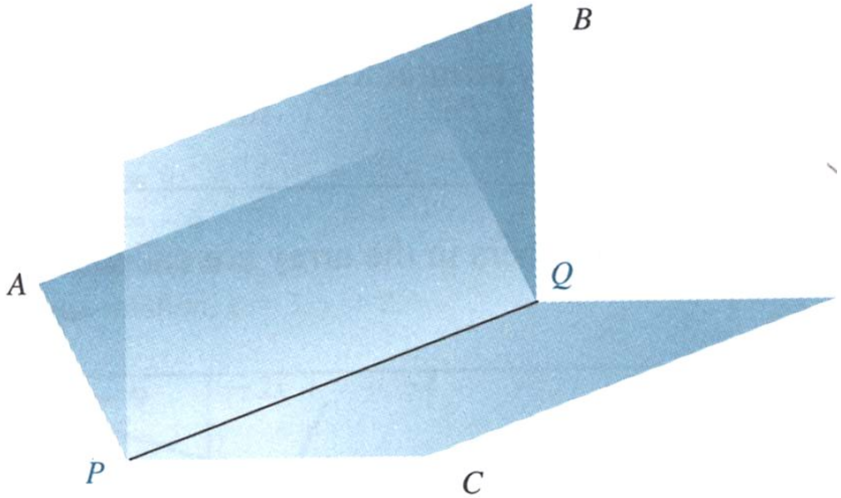


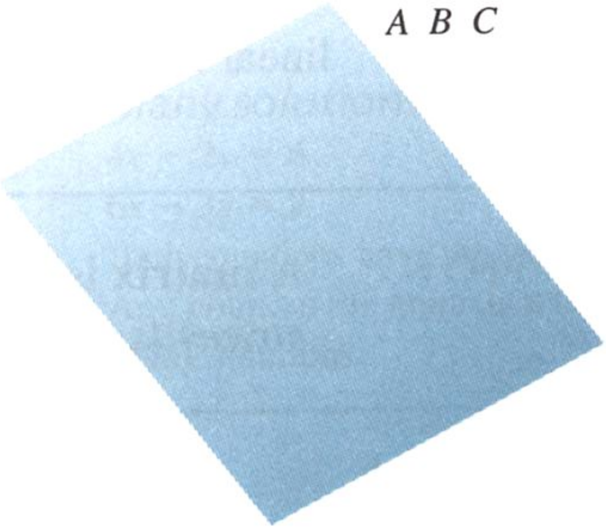
- *No solutions*





- *Many solutions*





A solution to a system of a three linear equations will be points that lie on all three planes.

The following is an example of a system of three linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

How to solve a system of linear equations? For this we introduce a method called Gauss-Jordan elimination. (Section1.2)

Definition

- A ***matrix*** is a rectangular array of numbers.
- The numbers in the array are called the ***elements*** of the matrix.

- **Matrices**

$$A=\begin{bmatrix}2 & 3 & -4\\ 7 & 5 & -1\end{bmatrix} \quad \mathrm{B}=\begin{bmatrix}7 & 1\\ 0 & 5\\ -8 & 3\end{bmatrix} \quad \mathrm{C}=\begin{bmatrix}3 & 5 & 6\\ 0 & -2 & 5\\ 8 & 9 & 12\end{bmatrix}$$

- **Row and Column**

$$A=\begin{bmatrix}2 & 3 & -4\\ 7 & 5 & -1\end{bmatrix}$$

$$\underset{\text{row 1}}{\begin{bmatrix}2 & 3 & -4\end{bmatrix}} \quad \underset{\text{row 2}}{\begin{bmatrix}7 & 5 & -1\end{bmatrix}} \quad \underset{\text{column 1}}{\begin{bmatrix}2\\ 7\end{bmatrix}} \quad \underset{\text{column 2}}{\begin{bmatrix}3\\ 5\end{bmatrix}} \quad \underset{\text{column 3}}{\begin{bmatrix}-4\\ -1\end{bmatrix}}$$

- **Submatrix**

$$A=\begin{bmatrix}1 & 7 & 4\\ 2 & 3 & 0\\ 5 & 1 & -2\end{bmatrix}$$

matrix A

$$P=\begin{bmatrix}1 & 7\\ 2 & 3\\ 5 & 1\end{bmatrix} \quad Q=\begin{bmatrix}7\\ 3\\ 1\end{bmatrix} \quad R=\begin{bmatrix}1 & 4\\ 5 & -2\end{bmatrix}$$

submatrices of A

- **Size and Type**

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 & 7 \\ -9 & 0 & 1 \\ -3 & 5 & 8 \end{bmatrix} \qquad \begin{bmatrix} 4 & -3 & 8 & 5 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}$$

Size : 2×3 | 3×3 matrix | 1×4 matrix | 3×1 matrix

a square matrix | a row matrix | a column matrix

- **Location**

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix} \qquad a_{13} = -4, \; a_{21} = 7$$

The element a_{ij} *is in* row i , column j

The element in location (1,3) is −4

- **Identity Matrices**

diagonal size

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Relations between system of linear equations and matrices

- **matrix of coefficients and augmented matrix**

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

matrix of coefficients   augmented matrix

Elementary Row Operations of Matrices

- Elementary Transformation

1. Interchange two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

- Elementary Row Operation

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Example 1

Solving the following system of linear equation.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$\approx$ row equivalent

Solution

Equation Method

Initial system:

Eq2+(–2)Eq1

Eq3+(–1)Eq1

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

Analogous Matrix Method

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

$$\underset{\substack{R2+(-2)R1 \\ R3+(-1)R1}}{\approx} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

Eq1+(–1)Eq2
Eq3+(2)Eq2

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

(–1/5)Eq3

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ -5x_3 &= -10 \end{aligned}$$

$\approx$
R1+(–1)R2
R3+(2)R2

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Eq1+(–2)Eq3
Eq2+Eq3

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ x_3 &= 2 \end{aligned}$$

$\approx$
(–1/5)R3

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

$\approx$
R1+(–2)R3
R2+R3

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is

$x_1 = -1, x_2 = 1, x_3 = 2.$

The solution is

$x_1 = -1, x_2 = 1, x_3 = 2.$

Example 2

Solving the following system of linear equation.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{1 \quad -2 \quad 4 \quad 12} \\ 2 \quad -1 \quad 5 \quad 18 \\ -1 \quad 3 \quad -3 \quad -8 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{3} & \boxed{-3} & \boxed{-6} \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{1} & \boxed{-1} & \boxed{-2} \\ 0 & 1 & 1 & 4 \end{bmatrix} \underset{\substack{R1+(2)R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{2} & \boxed{6} \end{bmatrix}$$

$$\underset{\left(\frac{1}{2}\right)R3}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{1} & \boxed{3} \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{solution} \begin{cases} x_1 = 2 \\ x_2 = 1. \\ x_3 = 3 \end{cases}$$

Example 3

Solve the system

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - \ \ 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\quad &= -7 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{4 \quad 8 \quad -12 \quad 44} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\left(\frac{1}{4}\right)\mathrm{R1}}{\approx} \begin{bmatrix} \boxed{1 \quad 2 \quad -3 \quad 11} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\substack{\mathrm{R2}+(-3)\mathrm{R1} \\ \mathrm{R3}+2\mathrm{R1}}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & \boxed{3 \quad -6 \quad 15} \end{bmatrix}$$

$$\underset{\mathrm{R2} \leftrightarrow \mathrm{R3}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{3 \quad -6 \quad 15} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\left(\frac{1}{3}\right)\mathrm{R2}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{1 \quad -2 \quad 5} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\mathrm{R1}+(-2)\mathrm{R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & \boxed{1 \quad -1} \end{bmatrix}$$

$$\underset{\substack{\mathrm{R1}+(-1)\mathrm{R3} \\ \mathrm{R2}+2\mathrm{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The solution is $x_1 = 2,\ x_2 = 3,\ x_3 = -1.$

Summary

$$
\begin{aligned}
4x_1 + 8x_2 - 12x_3 &= 44 \\
3x_1 + 6x_2 - \ \ 8x_3 &= 32 \\
-2x_1 - x_2 \qquad\quad &= -7
\end{aligned}
\qquad
[A:B] = \left[\begin{array}{ccc|c} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array}\right]
$$

$\underbrace{\qquad\qquad}_{A}$ $\underbrace{\quad}_{B}$

Use row operations to $[A: B]$:

$$
\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \quad \text{i.e., } [A:B] \approx \cdots \approx [I_n : X]
$$

Def. $[I_n : X]$ is called the ***reduced echelon form*** of $[A : B]$.

Note. 1. If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution, then A is row equivalent to I_n ($A \approx I_n$).

2. If $A \approx I_n$, then the system has unique solution.

Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_1 - x_2 + 4x_3 &= b_2 \\ -x_1 + 2x_2 - 4x_3 &= b_3 \end{aligned} \quad \text{for} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \text{ in turn}$$

Solution

$$\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{bmatrix} \underset{\substack{\text{R2+(–2)R1} \\ \text{R3+R1}}}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{bmatrix}$$

$$\underset{\substack{\text{R1+R2} \\ \text{R3+(–1)R2}}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} \underset{\substack{\text{R1+(–1)R3} \\ \text{R2+2R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & \boxed{1} & 0 & -2 \\ 0 & 1 & 0 & \boxed{-1} & 3 & 1 \\ 0 & 0 & 1 & \boxed{2} & 1 & 2 \end{bmatrix}$$

The solutions to the three systems are $\begin{cases} x_1 = 1 \\ x_2 = -1, \\ x_3 = 2 \end{cases} \begin{cases} x_1 = 0 \\ x_2 = 3, \\ x_3 = 1 \end{cases} \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 2 \end{cases}.$

Homework

- Exercises will be given by the teachers of the practical classes.

1.2 Gauss-Jordan Elimination

Definition

A matrix is in ***reduced echelon form*** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a ***leading* 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(✓) (□) (✓) (□)

- elementary row operations, reduced echelon form
- The reduced echelon form of a matrix is unique.

Gauss-Jordan Elimination

- System of linear equations
  $\Rightarrow$ augmented matrix
  $\Rightarrow$ reduced echelon form
  $\Rightarrow$ solution

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot (leading 1)

$$\underset{R1\leftrightarrow R2}{\approx}\begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx}\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\underset{R3+(-4)R1}{\approx}\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx}\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx}\begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx}\begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.

Example 2

Solve, if possible, the system of equations

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+2R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -3x_3 + 4 \\ x_2 &= -2x_3 + 1 \end{aligned}$$

The general solution to the system is

$$\begin{aligned} x_1 &= -3r + 4 \\ x_2 &= -2r + 1 \\ x_3 &= r \end{aligned}$$

, where r is real number (called a parameter).

Example 3

Solve the system of equations

$$\begin{aligned} 2x_1 - 4x_2 + 12x_3 - 10x_4 &= 58 \\ -x_1 + 2x_2 - 3x_3 + 2x_4 &= -14 \\ 2x_1 - 4x_2 + 9x_3 - 6x_4 &= 44 \end{aligned}$$

$\Rightarrow$ many sol.

Solution

$$\begin{bmatrix} 2 & -4 & 12 & -10 & 58 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R1}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix}$$

$$\underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix}$$

$$\underset{\substack{R1+(-6)R2 \\ R3+3R2}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 - 2x_2 &= -2 \\ x_3 &= 6 \\ x_4 &= 1 \end{aligned} \Rightarrow \begin{cases} x_1 = 2r - 2 \\ x_2 = r \\ x_3 = 6 \\ x_4 = 1 \end{cases}, \text{ for some } r.$$

Example 4

Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 + x_5 &= 2 \\ 2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 &= 6 \\ -x_1 - 2x_2 + x_3 - x_4 + 3x_5 &= 4 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 2 & 4 & -2 & 6 & 3 & 6 \\ -1 & -2 & 1 & -1 & 3 & 4 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\underset{R2\leftrightarrow R3}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R1+(-3)R2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & -5 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1+5R3 \\ R2+(-2)R3}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 &= -2x_2 + x_3 + 3 \\ x_4 &= -1 \\ x_5 &= 2 \end{aligned} \qquad \Rightarrow \begin{aligned} x_1 &= -2r + s + 3 \\ x_2 &= r,\ x_3 = s,\ x_4 = -1,\ \text{, for some } r \text{ and } s. \\ x_5 &= 2 \end{aligned}$$

Example 5

This example illustrates a system that has <u>no solution</u>. Let us try to solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{\text{R2}+(-2)\text{R1} \\ \text{R3}+(-1)\text{R1}}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\text{R2}\leftrightarrow\text{R3}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)\text{R2}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{\text{R1}+\text{R2} \\ \text{R4}+(-2)\text{R2}}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix} \underset{\substack{\text{R1}+(-1)\text{R3} \\ \text{R2}+\text{R3} \\ \text{R4}+(-4)\text{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\underset{\left(\frac{1}{13}\right)\text{R4}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{0x_1 + 0x_2 + 0x_3 = 1}$$

The system has no solution.

Homogeneous System of linear Equations

Definition

A system of linear equations is said to be ***homogeneous*** if all the constant terms are zeros.

Example:
$$\begin{cases} \quad x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ is a solution.

Theorem 1.1

A system of homogeneous linear equations in n variables always has the solution $x_1 = 0, x_2 = 0. \ldots, x_n = 0$. This solution is called the **trivial solution**.

Homogeneous System of linear Equations

Note. Non trivial solution

Example: $$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

The system has other nontrivial solutions.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\therefore x_1 = -3r,\ x_2 = 4r,\ x_3 = r$$

Theorem 1.2

A system of homogeneous linear equations that has more variables than equations has many solutions.

Homework

- Exercise will be given by the teachers of the practical classes.

1.3 Gaussian Elimination

Definition

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)

Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

Solution

Starting with the augmented matrix, create zeros below the pivot in the first column.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

At this stage, we create a zero only below the pivot.

$$\underset{R3+(-2)R2}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \underset{\frac{1}{2}R3}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Echelon form

We have arrived at the echelon form.

The corresponding system of equation is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ x_4 &= 2 \end{aligned}$$

We get

$$x_3 + 3(2) = 1$$

$$x_3 = -5$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$

$$x_1 + 2x_2 = 10$$

$$x_1 = -2x_2 + 10$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10,\ x_2 = r,\ x_3 = -5,\ x_4 = 2$$

Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\
-x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\
2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4
\end{aligned}
$$

Solution

We arrive at the echelon form as in the previous example.

$$
\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \square \approx \underbrace{\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{Echelon form}}
$$

This marks the end of the forward elimination of variables from equations. We now commence the back substitution using matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+(-3)R3}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R1+(-3)R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ x_3 &= -5 \\ x_4 &= 2 \end{aligned}$$

Let $x_2 = r$. We get same solution as previously,

$$x_1 = -2r + 10,\ x_2 = r,\ x_3 = -5,\ x_4 = 2$$